Chapter 1

A Quick Review of Trigonometry

In this Appendix we give a review of those concepts from Trigonometry that are essential for an understanding of the basics of Calculus. We assume that the reader has some acquaintance with Trigonometry at the High-School level, although the lack of such knowledge should not deter the student from reading this. For basic notions regarding lines, their equations, distance formulae, etc. we refer to the previous chapter on straight lines, Appendix B. You should also use this chapter in conjunction with Chapter 1 and Chapter 3.8. We also assume knowledge of the notions of an angle, and basic results from ordinary geometry dealing with triangles. If you don't remember any of this business just pick up any book on geometry from a used bookstore (or the university library) and do a few exercises to refresh your memory. We recall that *plane trigonometry* (or just trigonometry) derives from the Greek and means something like *the study of the measure of triangles* and this, on the plane, or in two dimensions, as opposed to *spherical trigonometry* which deals with the same topic but where triangles are drawn on a sphere (like the earth).

The quick way to review trigonometry is by relying on the *unit circle* whose equation is given by $x^2 + y^2 = 1$ in plane (Cartesian) coordinates. This means that any point (x, y) on the circle has the coordinates related by the fundamental relation $x^2 + y^2 = 1$. For example, $(\sqrt{2}/2, -\sqrt{2}/2)$ is such a point, as is $(\sqrt{3}/2, 1/2)$ or even $(1, 0)$. However, $(-1, 1)$ is not on this circle (why?). In this chapter, as in Calculus, all angles will be measured in **RADIANS** (not degrees).

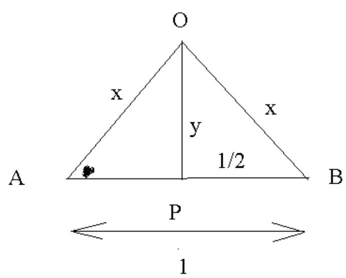
Don't forget that, in Calculus, we always assume that angles are described in **radians** and not degrees. The conversion is given by

$$Radians = \frac{(Degrees) \times (\pi)}{180}$$

For example, 45 degrees $= 45\ \pi/180 = \pi/4 \approx 0.7853981633974$ radians.

So, for example, 360 degrees amounts to 2π radians, while 45 degrees is $\pi/4$

radians. Radian measure is so useful in trigonometry (and in Calculus) that we basically have to forget that "degrees" ever existed! So, from now on we talk about angular measure using radians only! (At first, if you find this confusing go back to the box above and back substitute to get the measure in degrees). Okay, now let's review the properties of two really basic right-angled triangles, the **right-angled isosceles triangle** (that we refer to as RT45–abbreviation for a "right triangle with a 45 degree angle") because both its base angles must be equal to $\pi/4$ radians, and the right angled triangle one of whose angles measures $\pi/6$ radians (that we will refer to as RT30– why do you think we use the "30"?).

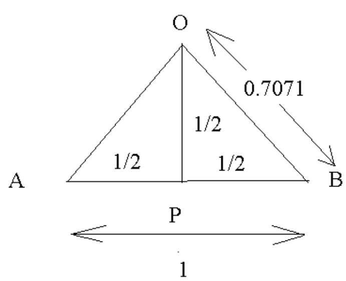


An RT45 Isosceles Triangle

Figure 1

1.1 The right-angled isosceles triangle (RT45)

This triangle, reproduced in the margin as Fig. 1 has two equal angles at its base ($\angle$OAP = $\angle$OBP) of measure equal to $\pi/4$ and another angle (namely $\angle$AOB) of measure $\pi/2$ (the "right-angle"). Let's find the measure x and y of the side OA and the perpendicular OP in this triangle so that we can remember once and for all the various relative measures of the sides of such a triangle. We note that the line segment AB has length 1, and the segments AP and PB each have length 1/2 (since OP must bisect AB for such a triangle). Using the theorem of Pythagoras on $\triangle$OAB we see that $1^2 = x^2 + x^2$ (since the triangle is isosceles) from which we get that $2x^2 = 1$ or $x = \pm\sqrt{2}/2$. But we choose $x = \sqrt{2}/2$ since we are dealing with side-lengths. Okay, now have x, what about y?

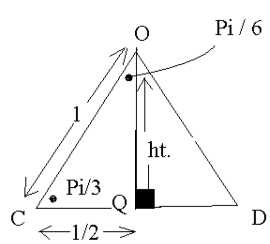


The final measures of the RT45 triangle

To get at y we apply Pythagoras to the triangle $\triangle$OPB with hypotenuse OB. In this case we see that $x^2 = (1/2)^2 + y^2$ But we know that $x^2 = 1/2$, so we can solve for y^2, that is $y^2 = 1/2 - 1/4 = 1/4$ from which we derive that $y = \pm 1/2$. Since we are dealing with side-lengths we get $y = 1/2$, that's all. Summarizing this we get Fig. 2, the RT45 triangle, as it will be called in later sections. Note that it has two equal base angles equal to $\pi/4$ radians, two equal sides of length $\sqrt{2}/2$ and the hypotenuse of length 1.

Summary of the RT45: Referring to Fig. 5, we see that our RT45 triangle has a hypotenuse of length 1, two sides of length $\sqrt{2}/2$, an altitude equal to 1/2, and two equal base angles of measure $\pi/4$ radians.

1.2 The RT30 triangle



The right-angled triangle: RT30

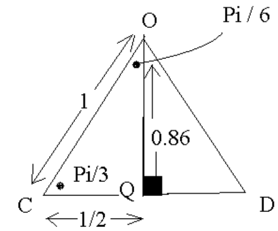
Figure 3

Now, this triangle, reproduced in the margin (see Fig. 3) as $\triangle$OCQ, derives from the equilateral triangle $\triangle$OCD all of whose sides have length 1. In this triangle $\angle$OCQ = $\angle$ODQ) have radian measure equal to $\pi/3$ while angle $\angle$COQ has measure $\pi/6$. Let's find the length of the altitude $h = OQ$, given that we know that OC has length 1, and CQ has length equal to 1/2. Using the theorem of Pythagoras on $\triangle$OCQ we see that $1^2 = (1/2)^2 + h^2$ from which we get that $h^2 = 3/4$ or $h = \pm\sqrt{3}/2$. But we choose $h = \sqrt{3}/2$ since we are dealing with side-lengths, just like before.

Summarizing this we get Fig. 4, the RT30 triangle. Note that it has two equal base angles equal to $\pi/4$ radians, two equal sides of length $\sqrt{2}/2$ and the hypotenuse of length 1.

Summary of the RT30: Referring to Fig. 4, we see that our RT30 triangle has a hypotenuse of length 1, one side of length $\sqrt{3}/2$, one side of length $1/2$, and a hypotenuse equal to 1 unit. Its angles are $\pi/6, \pi/3, \pi/2$ in radians (or 30-60-90 in degrees).

The final "mental images" should resemble Fig. 5 and Fig. 6.



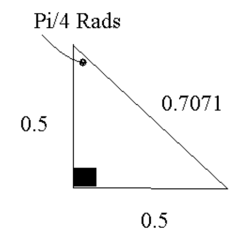
The final measures of the RT30 triangle

Figure 4

1.3 The basic trigonometric functions

Now we return to the unit circle whose equation that consists of all points $P(x, y)$ such that $x^2 + y^2 = 1$, see Fig. 7. The center (or origin) of our cartesian coordinate system is denoted by O and a point on the circle is denoted by P. The positive x-axis consists of the set of all points x such that $x > 0$. So, for example, the points $(1, 0), (0.342, 0), (6, 0)$ etc. are all on the positive x-axis. We now proceed to define the trigonometric functions of a given angle θ (read this as "thay-ta") whose measure is given in *radians*. The angle is placed in *standard position* as follows:

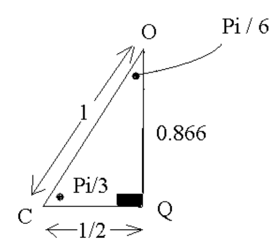
If $\theta > 0$, its vertex is placed at O and one of the legs of θ is placed along the positive x-axis. The other leg, the terminal side, is positioned **counterclockwise** along a ray OP until the desired measure is attained. For instance, the angle $\pi/2$ (or 90 degrees) is obtained by placing a leg along the positive x-axis and another along the y-axis. The angle is measured counterclockwise. The point P on the unit circle corresponding to this angle has coordinates $(0, 1)$.



The RT45 triangle

Figure 5

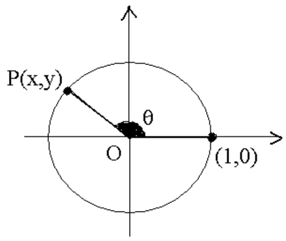
If $\theta < 0$, its vertex is placed at O and one of the legs of θ is placed along the positive x-axis. The other leg, the terminal side, is positioned **clockwise** along a ray OP until the desired measure is attained. For instance, the angle $-\pi/2$ (or -90 degrees) is obtained by placing a leg along the positive x-axis and another along the negative y-axis. The angle is measured clockwise. The point P on the unit circle corresponding to this angle has coordinates $(0, -1)$.



The final measures of the RT30 triangle

Figure 6

Given a right-angled triangle $\triangle$ PQO, see Fig. 8, we recall the definitions of "opposite" and "adjacent" sides: Okay, we all remember that the hypotenuse of $\triangle$ PQO is the side opposite to the right-angle, in this case, the side PO. The other two definitions depend on which vertex (or angle) that we distinguish. In the case of Fig. 8 the vertex O is distinguished. It coincides with the vertex of the angle $\angle$ QOP whose measure will be denoted by θ. The side OQ is called the *adjacent* side or simply the adjacent because it is *next* to O. The other remaining side is called the *opposite* because it is *opposite* to O (or not next to it anyhow). Now, **in trigonometry these three sides each have a length which, all except for the hypotenuse, can be either positive or negative** (or even zero, which occurs when the triangle collapses to a line segment). The hypotenuse, however, always has a positive length.



The unit circle

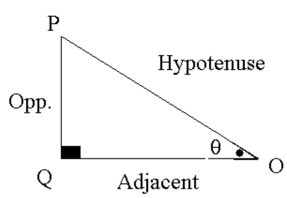
Figure 7

We are now in a position to define the basic trigonometric functions. Let's say we are given an angle (whose radian measure is) θ. We place this angle in standard position as in Fig. 7 and denote by P the point on the terminal side of this angle that intersects the unit circle (see Fig. 9). Referring to this same figure, at P we produce an altitude (or perpendicular) to the x-axis which meets this axis at Q, say. Then $\triangle$ PQO is a right-angled triangle with side PQ being the opposite side, and OQ being the adjacent side (of course, OP is the hypotenuse). The **trigonometric functions of this angle** θ are given as follows:

$$\sin\theta = \frac{opposite}{hypotenuse} \qquad \cos\theta = \frac{adjacent}{hypotenuse} \qquad \tan\theta = \frac{opposite}{adjacent} \tag{1.1}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}. \tag{1.2}$$

Example 1 Calculate the following trigonometric functions for the various given angles:



A typical right-angled triangle

Figure 8

1. $\sin(\pi/4)$

2. $\cos(2\pi/3)$

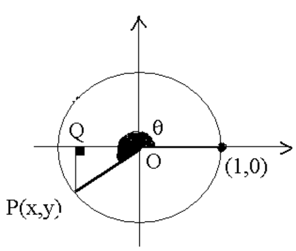
3. $\tan(-\pi/6)$

4. $\sec(-\pi/3)$

5. $\csc(5\pi/4)$

Solution 1) For this one use the RT45 triangle, Fig. 5. According to (1.1), $\sin(\pi/4) = (1/2)/(1/\sqrt{2}) = \sqrt{2}/2$.

2) For the next angle, namely $2\pi/3$ (or 120 degrees) it is best to draw a picture such as the one in Fig. 10. Note that this angle gives the internal angle $\angle$ POQ the value of $\pi - (2\pi/3) = \pi/3$ radians. So $\angle$POQ $= 2\pi/3$. But $\triangle$ PQO is a RT30 triangle (see Fig. 6). Comparing Fig. 6 with the present triangle PQO we see that $\sin(2\pi/3) =$ adj./hyp. $= (\sqrt{3}/2)/1 = \sqrt{3}/2$.



The unit circle

Figure 9

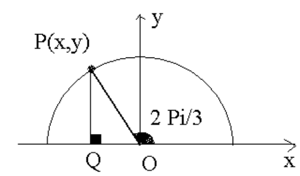
3) In this case, we need to remember that the negative sign means that the angle is measured in a *clockwise direction*, see Fig. 11. Note that the opposite side QP has a negative value (since it is below the x-axis). The resulting triangle $\triangle$ PQO is once again a RT30 triangle (see Fig. 6). As before we compare Fig. 6 with the present triangle PQO. Since $\tan(-\pi/6) =$ opp./adj. $= (-1/2)/(\sqrt{3}/2) = -1/\sqrt{3}$, since the opposite side has value $-1/2$.

4) First we note that $\sec(-\pi/3) = 1/\cos(-\pi/3)$ so we need only find $\cos(-\pi/3)$. Proceeding as in 3) above, the angle is drawn in a clockwise direction, starting from the positive x-axis, an amount equal to $\pi/3$ radians (or 60 degrees). This produces $\triangle$ PQO whose central angle $\angle$ POQ has a value $\pi/3$ radians (see Fig. 13). Note that in this case the opposite side QP is negative, having a value equal to $-\sqrt{3}/2$. The adjacent side, however, has a positive value equal to $1/2$.

Degs	0	30	45	60	90	120	135	150	180
Rads	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
sin	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
cos	0	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1
tan	0	$\sqrt{3}/3$	1	$\sqrt{3}$	**und.**	$-\sqrt{3}$	-1	$-\sqrt{3}/3$	0

Degs	210	225	240	270	300	315	330	360
Rads	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
sin	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1	$-\sqrt{3}/2$	$-\sqrt{2}/2$	$-1/2$	0
cos	$-\sqrt{3}/2$	$-\sqrt{2}/2$	$-1/2$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
tan	$\sqrt{3}/3$	1	$\sqrt{3}$	**und.**	$-\sqrt{3}$	-1	$-\sqrt{3}/3$	0

Table 1.1: Basic trigonometric functions and their values



A 120 degree angle in standard position

Figure 10

Since $\cos(-\pi/3)$ = adj./hyp. $= (1/2)/1 = 1/2$, we conclude that $\sec(-\pi/3) = 1/(1/2) = 2$.

5) In this last example, we note that $5\pi/4$ (or 225 degrees) falls in the 3rd quadrant (see Fig. 12) where the point $P(x, y)$ on the unit circle will have $x < 0, y < 0$. We just need to find out the $\sin(5\pi/4)$, since the cosecant is simply the reciprocal of the sine value. Note that central angle $\angle POQ = \pi/4$ so that $\triangle$ PQO is a RT45 triangle (cf., Fig. 5). But $\sin(5\pi/4)$ = opp./hyp. and since the opposite side has a negative value, we see that $\sin(5\pi/4) = (-1/2)/(1/\sqrt{2}) = -1/\sqrt{2}$.

Remark: Angles whose radian measure exceeds 2π radians (or more than 360 degrees) are handled by reducing the problem to one where the angle is less than 2π radians by removing an appropriate number of multiples of 2π. For example, the angle whose measure is $13\pi/6$ radians, when placed in standard position, will look like $2\pi + \pi/6$, or just like a $\pi/6$ angle (because we already have gone around the unit circle once). So, $\sin(13\pi/6) = \sin(\pi/6) = 1/2$.

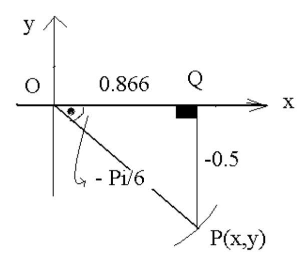


Figure 11

The charts in Table 1.1 should be memorized (it's sort of like a "multiplication table" but for trigonometry). The first row gives the angular measure in degrees while the second has the corresponding measure in radians. In some cases the values are *undefined*, (for example, $\tan(\pi/2)$) because the result involves division by zero (an invalid operation in the real numbers). In this case we denote the result by **und.**

1.4 Identities

This section involves mostly algebraic manipulations of symbols and not much geometry. We use the idea that the trigonometric functions are defined using the unit circle in order to derive the basic identities of trigonometry.

For example, we know that if θ is an an angle with vertex at the origin O of the plane, then the coordinates of the point $P(x, y)$ at its terminal side (where it meets the unit circle) must be given by $(\cos\theta, \sin\theta)$. Why? By definition .... Think about it!

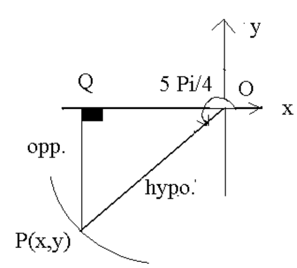


Figure 12

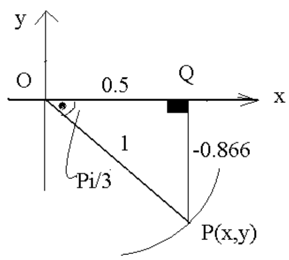


Figure 13

If you want to find the $\cos\theta$, you need to divide the adjacent by the hypotenuse, but this means that the adjacent is divided by the number 1 (which is the radius of the unit circle). But since this number is equal to $\cos\theta$ this means that the adjacent is equal to $\cos\theta$. But the "adjacent side length" is also equal to the x-coordinate of the point P. So, $\cos\theta = x$. A similar argument applies to the y-coordinate and so we get $y = \sin\theta$. So, the coordinates of P are given by $(\cos\theta, \sin\theta)$. But P is on the unit circle, and so $x^2 + y^2 = 1$. It follows that

$$\sin^2\theta + \cos^2\theta = 1. \tag{1.3}$$

for any angle θ in radians (positive or negative, small or large). Okay, now we divide both sides of equation (1.3) by the number $\cos^2\theta$, provided $\cos^2\theta \neq 0$.. Using the definitions of the basic trigonometric functions we find that $\tan^2\theta + 1 = \sec^2\theta$. From this we get the second fundamental identity, namely that

$$\sec^2\theta - \tan^2\theta = 1. \tag{1.4}$$

provided all the quantities are defined. The third fundamental identity is obtained similarly. We divide both sides of equation (1.3) by the number $\sin^2\theta$, provided $\sin^2\theta \neq 0$.. Using the definitions of the basic trigonometric functions again we find that $1 + \cot^2\theta = \csc^2\theta$. This gives the third fundamental identity, i.e.,

$$\csc^2\theta - \cot^2\theta = 1. \tag{1.5}$$

once again, provided all the quantities are defined.

Next, there are two basic "Laws" in this business of trigonometry, that is the **Law of Sines** and the **Law of Cosines**, each of which is very useful in applications of trigonometry to the real world.

1.4.1 The Law of Sines

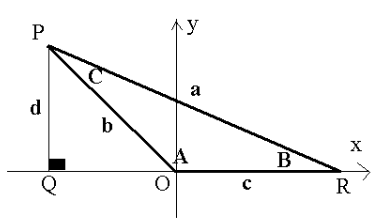


Figure 14

Before we proceed to recall this Law, remember that every angle in this book is to be measured in radians (and not degrees). This is particularly important for the Law of Sines where we will be relating the side length of a plane triangle with the angle opposite the side (when measured in radians). In order to set the scene for what follows we begin by referring to Fig. 15. Here we have a triangle OPR in standard position (and we can assume that R, P are on the unit circle with O at its center, since any other triangle would be similar to this one and so the sides would be proportional). Denote ∠ POR by A, ∠ORP by B and ∠ RPO by C, for brevity. Also, (cf., Fig. 15) denote the side lengths RP, PO, OR by a, b, c (all assumed positive in this result).

Now comes the proof of the sine law, given by equation (1.6) below. Referring to Fig. 15 once again and using the definition of the sine function, we see that

$$\frac{d}{b} = \sin A \implies d = b\sin A.$$

In addition, since PQR is a right-angled triangle,

$$\frac{d}{a} = \sin B \implies d = a\sin B.$$

Combining these last two equations and eliminating d we find that $b \sin A = a \sin B$ and so provided that we can divide both sides by the product $\sin A \sin B$ we get

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

Proceeding in exactly the same way for the other two angles we can deduce that $b \sin C = c \sin B$ from which we get the Sine Law:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \quad (1.6)$$

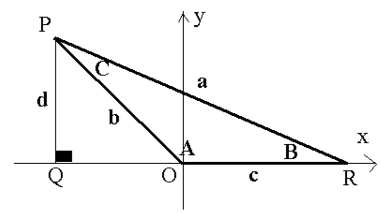


Figure 15

Example 2 *Estimating the height of a building without asking an engineer:* Okay, but I'll assume you have a protractor! So, you're standing 10 m. away from the base of a very tall building and you pull out your protractor and measure the angle subtended by the point where you are standing (just pick a point 10 m away from the door, say) and the highest point on the building (of course, you will look absolutely nuts when you pull this thing out and start measuring by holding this instrument before your eyes). So, you measure this angle to be 72 degrees. Anyhow, given this simple scenario, how do you find the height of the building?

Solution: See Fig. 16. You can guess that there is a right-angled triangle whose vertices are at your eyeball, the top point on the door and the top of the building. You also have two angles and one side of this triangle. So, the Sine Law tells you that you can always find the exact shape of the triangle (and so all the sides, including the height of the building). How? Well, first you need to convert 72 degrees to radians...so,

$$\frac{72 \times \pi}{180} = 1.2566 rads.$$

Then, we need to find the "third" angle which is given by $180 - 72 - 90 = 18$ degrees, one that we must also convert to radians ... In this case we get 0.314159 rads.We put all this info. together using the Sine Law to find that

$$\frac{height}{\sin 1.2566} = \frac{10}{\sin 0.314159}$$

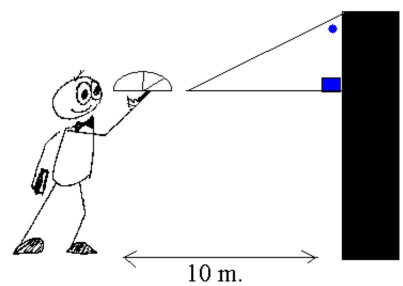


Figure 16

and solving for the (approximate) height we get $height \approx 30.8$ m. Of course, you need to add your approximate height of, say, 1.8m to this to get that the building height is approximately $30.8 + 1.8 = 32.6$ meters.

1.4.2 The Law of Cosines

You can think of this Law as a generalization of the Theorem of Pythagoras, the one you all know about, you know, about the square of the hypotenuse of a right-angled triangle etc. Such a "generalization" means that this result of Pythagoras is a *special case* of the Law of Cosines. So, once again a picture helps to set the scene. Look at Fig. 17. It is similar to Fig. 15 but there is additional information. Next, you will need the formula for the distance between two points on a plane (see Chapter B).

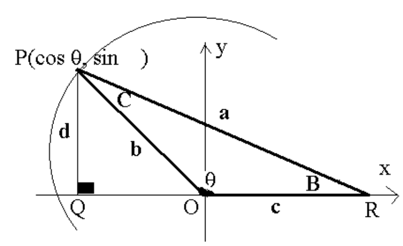


Figure 17

Referring to Fig. 17 we asume that our triangle has been placed in standard position with its central angle at O (This simplifies the discussion). **We want**

a relationship between the central angle $\theta = \angle$ POR and the sides a, b, c of the triangle so that when $\theta = \pi/2$ we get the classical Theorem of Pythagoras.

Now, by definition of the trigonometric functions we see that the point P has coordinates $P(b\cos\theta, b\sin\theta)$. The coordinates of R are easily seen to be $R(c, 0)$. By the *distance formula* we see that the square of the length of PR (or equivalently, the square of the distance between the points P and R) is given by

$$\begin{aligned} a^2 &= (b\cos\theta - c)^2 + (b\sin\theta - 0)^2, \\ &= b^2\cos^2\theta - 2bc\cos\theta + c^2 + b^2\sin^2\theta, \\ &= b^2(\cos^2\theta + \sin^2\theta) + c^2 - 2bc\cos\theta, \\ &= b^2 + c^2 - 2bc\cos\theta, \quad \text{by (1.3)}. \end{aligned}$$

This last expression is the Cosine Law. That is, for any triangle with side lengths a, b, c and contained angle θ, (i.e., θ is the angle at the vertex where the sides of length b and c meet), we have

$$a^2 = b^2 + c^2 - 2bc\cos\theta. \tag{1.7}$$

Note that when $\theta = \pi/2$, or the triangle is right-angled, then a^2 is simply the square of the hypotenuse (because $\cos(\pi/2) = 0$) and so we recover Pythagoras' theorem.

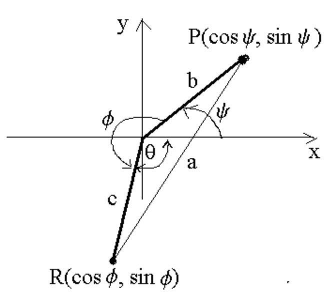


Figure 18

1.4.3 Identities for the sum and difference of angles

Now, we can use these two laws to derive some really neat identities that relate the cosine of the sum or difference of two given angles with various trigonometric functions of the individual angles. Let's see how this is done. Consider Fig. 18 where ψ (pronounced "p-see") and ϕ (pronounced "fee") are the two given angles and the points P and R are on the unit circle. Let's say **we want a formula for $\cos(\psi - \phi)$**. First we find the coordinates of P and R in the figure and see that, by definition, we must have $P(\cos\psi, \sin\psi)$. Similarly, the coordinates of R are given by $R(\cos\phi, \sin\phi)$.

Now, look at $\triangle$ OPR in Fig. 18. By the Cosine Law (1.7), we have that $a^2 = b^2 + c^2 - 2bc\cos\theta$, where the central angle θ is related to the given angles via $\theta + \phi - \psi = 2\pi$ radians. Furthermore, $b = c = 1$ here because P and R are on the unit circle. Solving for θ in the previous equation we get

$$\theta = 2\pi - \phi + \psi.$$

But a^2 is just the square of the distance between P and R, b^2 is just the square of the distance between O and P and finally, c^2 is just the square of the distance between O and R. Using the distance formula applied to each of the lengths a, b, c above, we find that

$$\begin{aligned} (\cos\psi - \cos\phi)^2 + (\sin\psi - \sin\phi)^2 &= (\cos^2\psi + \sin^2\psi) + \\ &\quad (\cos^2\phi + \sin^2\phi) - 2\cos(\theta), \\ 2 - 2\cos\psi\cos\phi - 2\sin\psi\sin\phi &= 2 - 2\cos(2\pi + \psi - \phi), \end{aligned}$$

where we have used (1.3) repeatedly with the angle θ there replaced by ψ and ϕ, respectively. Now note that $\cos(2\pi + \psi - \phi) = \cos(\psi - \phi)$. Simplifying the last display gives the identity,

$$\cos(\psi - \phi) \quad = \quad \cos\psi\cos\phi + \sin\psi\sin\phi. \tag{1.8}$$

valid for any angles ψ, ϕ whatsoever. As a consequence, we can replace ψ in (1.8) by $\psi = \pi/2$, leaving ϕ as arbitrary. Since $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$, (1.8) gives us a new relation,

$$\cos\left(\frac{\pi}{2} - \phi\right) \quad = \quad \sin\phi. \tag{1.9}$$

But ϕ is arbitrary, so we can replace ϕ in (1.9) by $\pi/2 - \phi$ and get another new identity, that is,

$$\sin\left(\frac{\pi}{2} - \phi\right) \quad = \quad \cos\phi. \tag{1.10}$$

also valid for any angle ϕ whatsoever.

Now let ϕ, ψ be arbitrary angles once again. Replacing ϕ by $\psi + \phi$ in (1.9) we get

$$\sin(\psi + \phi) = \cos\left((\frac{\pi}{2} - \psi) - \phi)\right).$$

Using the *cos-difference-formula* (1.8) and combining this with (1.9) and (1.10) we obtain,

$$\begin{aligned}
\sin(\psi + \phi) &= \cos\left((\frac{\pi}{2} - \psi) - \phi)\right) \\
&= \cos\left(\frac{\pi}{2} - \psi\right)\cos\phi + \sin\left(\frac{\pi}{2} - \psi\right)\sin\phi \\
&= \sin\psi\cos\phi + \cos\psi\sin\phi.
\end{aligned}$$

We distinguish this formula for future use as the "sin-sum-formula" given by

$$\sin(\psi + \phi) \quad = \quad \sin\psi\cos\phi + \cos\psi\sin\phi, \tag{1.11}$$

and valid for any angles ψ, ϕ as usual. Now, in (1.9) we replace ϕ by $-\psi$ and rearrange terms to find:

$$\begin{aligned}
\sin(-\psi) &= \cos\left(\psi + \frac{\pi}{2}\right) \\
&= \cos\left(\psi - \left(-\frac{\pi}{2}\right)\right) \\
&= \cos(\psi)\cos(-\pi/2) + \sin(\psi)\sin(-\pi/2) \\
&= -\sin(\psi),
\end{aligned}$$

since $\cos(-\pi/2) = 0$ and $\sin(-\pi/2) = -1$. This gives us the following identity,

$$\sin(-\psi) = -\sin(\psi) \tag{1.12}$$

valid for any angle ψ.

In addition, this identity (1.8) is really interesting because the angles ϕ, ψ can really be anything at all! For example, if we set $\psi = 0$ and note that $\sin 0 = 0$, $\cos 0 = 1$, we get another important identity, similar to the one above, namely that

$$\cos(-\phi) = \cos(\phi) \tag{1.13}$$

for any angle ϕ. We sometimes call (1.8) a "cos-angle-difference" identity. To get a "'cos-angle-sum" identity we write $\psi + \phi$ as $\psi + \phi = \psi - (-\phi)$ and then apply (1.8) once again with ϕ replaced by $-\phi$. This gives

$$\begin{aligned} \cos(\psi + \phi) &= \cos\psi\cos(-\phi) + \sin\psi\sin(-\phi). \\ &= \cos\psi\cos\phi - \sin\psi\sin\phi. \end{aligned}$$

where we used (1.13) and (1.12) respectively to eliminate the minus signs. We display this last identity as

$$\boxed{\cos(\psi + \phi) = \cos\psi\cos\phi - \sin\psi\sin\phi.} \tag{1.14}$$

The final "sin-angle-difference" identity should come as no surprise. We replace ϕ by $-\phi$ in (1.11), then use (1.13) and (1.12) with ψ replaced by ϕ. This gives

$$\boxed{\sin(\psi - \phi) = \sin\psi\cos\phi - \cos\psi\sin\phi,} \tag{1.15}$$

Now we can derive a whole bunch of other identities! For example, the identity

$$\boxed{\cos(2\phi) = \cos^2\phi - \sin^2\phi,} \tag{1.16}$$

is obtained by setting $\phi = \psi$ in (1.14). Similarly, setting $\phi = \psi$ in (1.11) gives the new identity

$$\boxed{\sin(2\phi) = 2\sin\phi\cos\phi,} \tag{1.17}$$

Returning to (1.16) and combining this with (1.3) we find that

$$\begin{aligned} \cos(2\phi) &= \cos^2\phi - \sin^2\phi \\ &= \cos^2\phi - (1 - \cos^2\phi) \\ &= 2\cos^2\phi - 1. \end{aligned}$$

Isolating the square-term in the preceding formula we get a very important identity, namely,

$$\cos^2(\phi) = \frac{1+\cos(2\phi)}{2}. \tag{1.18}$$

On the other hand, combining (1.16) with (1.3) we again find that

$$\begin{aligned} \cos(2\phi) &= \cos^2\phi - \sin^2\phi \\ &= (1-\sin^2\phi) - \sin^2\phi \\ &= 1 - 2\sin^2\phi. \end{aligned}$$

Isolating the square-term in the preceding formula just like before we get the complementary identity to (1.17)

$$\sin^2(\phi) = \frac{1-\cos(2\phi)}{2}. \tag{1.19}$$

The next example shows that you don't really have to know the value of an angle, but just the value of one of the trigonometric functions of that angle, in order to determine the other trigonometric functions.

Example 3 *Given that θ is an acute angle such that $\sin\theta = \sqrt{3}/2$, find $\cos\theta$ and $\cot\theta$.*

Solution: With problems like this it is best to draw a picture, see Fig. 19. The neat thing about trigonometry is you don't always have to put your triangles inside the unit circle, it helps, but you don't have to. This is one example where it is better if you don't! For instance, note that $\sin\theta = \sqrt{3}/2$ means we can choose the side PQ to have length $\sqrt{3}$ and the hypotenuse OP to have length 2. So, using the Theorem of Pythagoras we get that the length of OQ is 1 unit. We still don't know what θ is, right? But we DO know all the sides of this triangle, and so we can determine all the other trig. functions of this angle, θ. For example, a glance at Fig. 19 shows that $\cos\theta = 1/2$ and so $\cot\theta = 1/(\tan\theta) = 1/\sqrt{3}$.

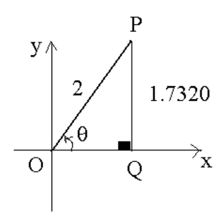


Figure 19

N.B.: There is something curious here, isn't there? We drew our picture, Fig. 19 so that θ is an acute angle, because we were asked to do so! What if the original angle were obtuse? Would we get the same answers?

Answer: No. For example, the obtuse angle $\theta = 2\pi/3$ also has the property that $\sin\theta = \sqrt{3}/2$ (Check this!) However, you can verify that $\cos\theta = -1/2$ and $\cot\theta = -1/\sqrt{3}$. The moral is, the more information you have, the better. If we weren't given that θ was acute to begin with, we wouldn't have been able to calculate the other quantities uniquely.

Example 4 *If θ is an obtuse angle such that $\cot\theta = 0.2543$, find $\cos\theta$ and $\csc\theta$.*

Solution: Once again we draw a picture, see Fig. 20. This time we make the angle obtuse and put it in a quadrant where the cotangent is positive! Note that $\cot\theta = 0.2543 > 0$ and θ obtuse means that θ is in Quadrant III (cf., Fig. 21). So, this means we can choose the side OQ to have length -0.2543 and the opposite side to have length -1. Using the Theorem of Pythagoras once again we get that the length of the hypotenuse OP is 1.0318 units. Just as before , we DO now know all the sides of this triangle, and so we can determine all the other trig. functions of this angle, θ.

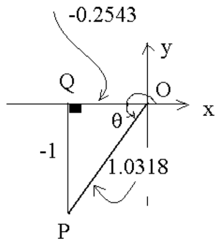


Figure 20

So, a glance at Fig. 20 shows that $\cos\theta = -0.2543/1.0318 = -0.2465$ and $\csc\theta = 1/(\sin\theta) = -1.0318$.

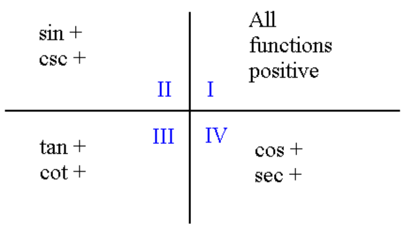


Figure 21

Example 5 *Given that θ is an angle in Quadrant II such that $\cos^2\theta = \cos\theta + 1$, find the value of $\sin\theta$.*

Solution: No picture is required here, but it's okay if you draw one. Note that $\cos\theta$ is not given explicitly at the outset so you have to find it buried in the information provided. Observe that if we write $x = \cos\theta$ then we are really given that $x^2 = x + 1$ which is a quadratic equation. Solving for x using the *quadratic formula* we get $\cos\theta = (1 \pm \sqrt{5})/2$. However, one of these roots is greater than 1 and so it cannot be the cosine of an angle. It follows that the other root, whose value is $(1 - \sqrt{5})/2 = -0.618034$, is the one we need to use.

Thus, we are actually given that $\cos\theta = -0.618034$, θ is in Quadrant II, and we need to find $\sin\theta$. So, now we can proceed as in the examples above. Note that in Quadrant II, $\sin\theta > 0$. In addition, if we do decide to draw a picture it would look like Fig. 22.

Since $\cos\theta = -0.618034$, we can set the adjacent side to have the value -0.618034 and the hypotenuse the value 1. From Pythagoras, we get the opposite side with a value of 0.78615. It follows that $\sin\theta = 0.78615/1 = 0.78615$.

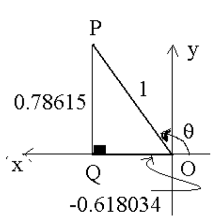


Figure 22

Example 6 *Prove the following identity by transforming the expression on the left into the one on the right using the identities (1.3-1.5) above and the definitions of the various trig. functions used:*

$$\sin^2 x + \frac{1 - \tan^2 x}{\sec^2 x} = \cos^2 x.$$

Solution: We leave the first term alone and split the fraction in the middle so that it looks like

$$\begin{aligned}
\sin^2 x + \frac{1 - \tan^2 x}{\sec^2 x} &= \sin^2 x + \frac{1}{\sec^2 x} - \frac{\tan^2 x}{\sec^2 x}, \\
&= \sin^2 x + \cos^2 x - \frac{\sin^2 x}{\cos^2 x} \cdot \frac{cos^2 x}{1}, \quad \text{(by definition)} \\
&= \sin^2 x + \cos^2 x - \sin^2 x, \\
&= \cos^2 x,
\end{aligned}$$

which is what we needed to show.

All of the above identities (1.3-1.5) and (1.8-1.19) are used in this book (and in Calculus, in general) and you should strive to **remember all the boxed**

ones, at the very least. Remembering how to get from one to another is also very useful, because it helps you to remember the actual identity by deriving it!

Exercise Set 1

Evaluate the following trigomometric functions at the indicated angles using any of the methods or identities in this chapter (use your calculator only to CHECK your answers). Convert degrees to radians where necessary.

1. $\cos\frac{\pi}{3}$	6. $\sin\frac{5\pi}{4}$	11. $\cos\frac{3\pi}{2}$	16. $\cos\frac{-7\pi}{4}$	21. $\cos 225^o$
2. $\sin\frac{2\pi}{3}$	7. $\cos\frac{7\pi}{6}$	12. $\sin\frac{3\pi}{2}$	17. $\cos\frac{17\pi}{4}$	22. $\cos 405^o$
3. $\tan\frac{\pi}{6}$	8. $\sin\frac{-3\pi}{4}$	13. $\tan\frac{3\pi}{2}$	18. $\cos\frac{5\pi}{2}$	23. $\cos 960^o$
4. $\cos\frac{-5\pi}{4}$	9. $\cos\frac{3\pi}{4}$	14. $\tan\frac{7\pi}{4}$	19. $\cos\frac{11\pi}{6}$	24. $\sin(-210^o)$
5. $\cos\frac{7\pi}{4}$	10. $\sin\frac{5\pi}{3}$	15. $\sin\frac{7\pi}{6}$	20. $\cos\frac{-13\pi}{6}$	25. $\tan(-1125^o)$

26. If $\cot\phi = 3/4$ and ϕ is an acute angle, find $\sin\phi$ and $\sec\phi$.

27. If $\cos u = -1/4$ and u is in Quadrant II, find $\csc u$ and $\tan u$.

28. If $\sin\phi = 1/3$ and ϕ is an acute angle, find $\cos\phi$ and $\tan\phi$.

29. If $\tan v = -3/4$ and v is in Quadrant IV, find $\sin v$ and $\cos v$.

30. If $\sec\phi = 2$ and ϕ is an acute angle, find $\sin\phi$ and $\tan\phi$.

31. If $\csc w = -3$ and w is in Quadrant III, find $\cos w$ and $\cot w$.

Prove the following identities using the basic identities in the text by converting the left hand side into the right hand side.

33. $(\tan x + \cot x)^2 = \sec^2 x \csc^2 x$.

34. $\sin\theta + \cot\theta\cos\theta = \csc\theta$.

35. $\dfrac{\cos x}{1+\sin x} + \tan x = \sec x$.

36. $\tan^2 y - \sin^2 y = \tan^2 y \sin^2 y$.

37. $\dfrac{1+\cot x}{1+\tan x} = \cot x$.

38. $\dfrac{1}{\tan\phi + \cot\phi} = \sin\phi\cos\phi$.

39. $\sin^2 x\cot^2 x + \cos^2 x\tan^2 x = 1$.

40. $\dfrac{1}{\sin^2 x} + \dfrac{1}{\cos^2 x} = \dfrac{1}{\sin^2 x - \sin^4 x}$.

41. $\sin^4 \phi - \cos^4 \phi = 2\sin^2 \phi \, - \, 1$.

Prove the following relations using the angle-sum/difference formulae, (1.8-1.15) and/or the others.

42. $\cos(\dfrac{\pi}{2} + \theta) = \cos\theta$.

43. $\sin(\pi + x) = -\sin x$.

44. $\cos(\dfrac{3\pi}{2} + \theta) = \sin\theta$.

45. $\sin(\pi - x) = \sin x$.

46. $\cos(\pi - x) = -\cos x$.

47. $\sin(\dfrac{3\pi}{2} + \theta) = -\cos\theta$.

48. $\cos(\dfrac{3\pi}{2} - \theta) = -\sin\theta$.

49. $\tan(\pi + x) = \tan x$.

50. $\tan(\pi - x) = -\tan x$.

□□