Rationalizing Substitutions

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In this chapter we look at a few more substitutions that can be used effectively to transform some types of integrals to those involving rational functions. In this way we may be able to integrate the original functions by referring to the method of Partial Fractions from Chapter 8. Before we start though we need to remind the reader of a notion called the *least common multiple* of two given positive integers. As the phrase suggests, the least common multiple (abbr. lcm) of two numbers x, y (assumed integers) is the smallest number that is a multiple of each one of x and y. For example, the $\text{lcm}\{2, 4\} = 4$, since 4 is the smallest number that is a multiple of both 2 and itself. Other examples include, the $\text{lcm}\{2, 3, 4\} = 12$, $\text{lcm}\{2, 3\} = 6$, $\text{lcm}\{2, 5\} = 10$, $\text{lcm}\{2, 4, 6\} = 12$ etc. Thus, given two fractions, say 1/2 and 1/3, the least common multiple of their denominators is 6.

As a typical example we're going to try to get rid of those crazy looking roots in integrands so as to make the new expression look like a rational function.

Example 1 *Evaluate the integral* $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx$.

Solution: Well, in order to eliminate the "square root" here it would be nice to try out the substitution $x = z^2$, $dx = 2z\,dz$. This is because

$$\begin{aligned}
\int \frac{\sqrt{x}}{1+x}\,dx &= \int \frac{z}{1+z^2}\,2z\,dz \\
&= \int \frac{2z^2}{1+z^2}\,dz \\
&= 2\int \left(1 - \frac{1}{1+z^2}\right)dz \\
&= 2z - 2\,\text{Arctan}\,(z) + C \\
&= 2\sqrt{x} - 2\,\text{Arctan}\,(\sqrt{x}) + C,
\end{aligned}$$

where C is the usual constant of integration. Note that the guessed substitution gave us a rational function in z which, coupled with the method of partial fractions, allowed for an easy integration.

Okay, but what if the original intergand involves many different roots or fractional roots? The general method involves the notion of a least common multiple introduced above.

Example 2 *Evaluate the integral* $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx.$

Solution: Here we have two different powers of x, namely $1/2$ and $1/3$ (these two fractions have been simplified so that their numerators and denominators have no common factors). Then we let n *be the lcm of their denominators*; $n = \mathrm{lcm}\{2,3\} = 6$ and then use the substitution $x = z^6$, $dx = 6z^5\,dz$. Looks weird, right? But it works, because then the roots of x become *powers* of z ...

$$\begin{aligned}
\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx &= \int \frac{1}{z^3+z^2}\,(6z^5\,dz) \\
&= \int \frac{6z^3}{z+1}\,dz \\
&= 6\int \left(z^2 - z + 1 - \frac{1}{z+1}\right)\,dz \\
&= 2z^3 - 3z^2 + 6z - 6\log|z+1| + C \\
&= 2(x^{1/6})^3 - 3(x^{1/6})^2 + 3z^2 - 6\log|x^{1/6}+1| \\
&= 2\sqrt{x} - 3\sqrt[3]{x} + 3z^2 - 6\log(x^{1/6}+1) + C,
\end{aligned}$$

where C is a constant. Note that $x \geq 0$ up here is necessary for the last line to hold or else we get complex numbers.

The idea on how to proceed seems clearer, no?

Now we describe the general method: Let' say we have an integrand with lots of roots (i.e., many fractional powers of the variable of integration), say,

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \ldots, \frac{p_m}{q_m},$$

where the p_i, q_i are integers and the fractions are each written in their lowest form (i.e., the numerators and denominators have no common factors). Let

$$n = \mathrm{lcm}\,\{q_1, q_2, q_3, \ldots, q_m\}.$$

Then the substitution $x = z^n$ may be tried in order to reduce the integral to a rational function in z (so that the method of partial fractions can be tried ...).

When can we use this device? Well, **if the integrand is a quotient of linear combinations of fractional powers of the variable of integration** then we can use it to simplify the integral. Examples of such expressions are

$$\frac{2+\sqrt[3]{x}-5x^{2/3}}{1-6x^2+\sqrt{x}}, \quad \frac{t^3-\sqrt{t}}{t^{1/4}-8}, \quad \frac{-2}{\sqrt[3]{x}-2x^{3/4}}.$$

Example 3 *Evaluate the integral* $\displaystyle\int \frac{1}{x+x^{2/3}}\,dx.$

Solution: The only powers of x here are 1 and $2/3$. So, the lcm of the denominators is 3. Thus, we let $x = z^3$, $dx = 3z^2\,dz$. Then

$$\int \frac{1}{x+x^{2/3}}\,dx = \int \frac{1}{z^3+z^2}(3z^2\,dz)$$

$$
\begin{aligned}
&= \int \frac{3}{z+1}\,dz \\
&= 3\log|z+1| + C \\
&= 3\log|\sqrt[3]{x}+1| + C,
\end{aligned}
$$

and C is a constant.

Example 4 *Evaluate* $\displaystyle\int \frac{\sqrt{t}}{2+\sqrt[4]{t}}\,dt.$

Solution: The powers of t here are $1/4$ and $1/2$ and the lcm of their denominators is 4. Thus, we let $t = z^4$, $dt = 4z^3\,dz$. Then

$$
\begin{aligned}
\int \frac{\sqrt{t}}{2+\sqrt[4]{t}}\,dt &= \int \frac{z^2}{2+z}(4z^3\,dz) \\
&= 4\int \frac{z^5}{z+2}\,dz \\
&= 4\int \left(z^4 - 2z^3 + 4z^2 - 8z + 16 - \frac{32}{z+2}\right)dz \\
&= \frac{4}{5}z^5 - 2z^4 + \frac{16}{3}z^3 - 16z^2 + 64z - 128\log|z+2| \\
&= \frac{4}{5}(t^{1/4})^5 - 2(t^{1/4})^4 + \frac{16}{3}(t^{1/4})^3 - 16(t^{1/4})^2 + 64t^{1/4} - 128\log|t^{1/4}+2| \\
&= \frac{4}{5}t^{4/5} - 2t + \frac{16}{3}t^{3/4} - 16t^{1/2} + 64t^{1/4} - 128\log|t^{1/4}+2| + C,
\end{aligned}
$$

where C is a constant.

Example 5 *Evaluate the integral* $\displaystyle\int_0^1 \frac{3\sqrt[4]{x}}{2\sqrt{x}+5\sqrt[3]{x}}\,dx.$

Solution: Now we have three different powers of x, namely $1/2$, $1/3$ and $1/4$ with n, the lcm of their denominators, given by $n = \text{lcm}\{2,3,4\} = 12$. The substitution to try is then $x = z^{12}$, $dx = 12z^{11}\,dz$. Using this we get,

$$
\begin{aligned}
\int \frac{3\sqrt[4]{x}}{2\sqrt{x}+5\sqrt[3]{x}}\,dx &= \int \frac{3z^3}{2z^6+5z^4}\,(12z^{11}\,dz) \\
&= 36\int \frac{z^{14}}{z^4(2z^2+5)}\,dz \\
&= 36\int \frac{z^{10}}{2z^2+5}\,dz \\
&= 36\int \left\{\frac{1}{2}z^8 - \frac{5}{4}z^6 + \frac{25}{8}z^4 - \frac{125}{16}z^2 + \frac{625}{32} - \frac{3125/32}{2z^2+5}\right\}dz \quad \text{(by long division)} \\
&= 36\cdot\left\{\frac{1}{18}z^9 - \frac{5}{28}z^7 + \frac{5}{8}z^5 - \frac{125}{48}z^3 + \frac{625}{32}z - \frac{625}{64}\sqrt{10}\ \text{Arctan}\left(\frac{1}{5}\sqrt{10}\,z\right)\right\} \\
&= 2z^9 - \frac{45}{7}z^7 + \frac{45}{2}z^5 - \frac{375}{4}z^3 + \\
&\quad + \frac{5625}{8}z - \frac{5625}{16}\sqrt{10}\,\text{Arctan}\left(\frac{1}{5}\sqrt{10}\,z\right),
\end{aligned}
$$

$$\begin{aligned}
&= 2\,(x^{1/12})^9 - \frac{45}{7}\,(x^{1/12})^7 + \frac{45}{2}\,(x^{1/12})^5 - \frac{375}{4}\,(x^{1/12})^3 + \\
&\quad + \frac{5625}{8}\,(x^{1/12}) - \frac{5625}{16}\,\sqrt{10}\,\mathrm{Arctan}\left(\frac{1}{5}\sqrt{10}\,(x^{1/12})\right) \\
&= 2\,x^{3/4} - \frac{45}{7}\,x^{7/12} + \frac{45}{2}\,x^{5/12} - \frac{375}{4}\,x^{1/4} + \\
&\quad + \frac{5625}{8}\,x^{1/12} - \frac{5625}{16}\,\sqrt{10}\,\mathrm{Arctan}\left(\frac{1}{5}\sqrt{10}\,(x^{1/12})\right) + C
\end{aligned}$$

where C is a constant. This is an antiderivative, so the required definite integral is easily evaluated. We see that

$$\begin{aligned}
\int_0^1 \frac{3\sqrt[4]{x}}{2\sqrt{x}+5\sqrt[3]{x}}\,dx &= (2\,x^{3/4} - \frac{45}{7}\,x^{7/12} + \frac{45}{2}\,x^{5/12} - \frac{375}{4}\,x^{1/4} \\
&\quad + \frac{5625}{8}\,x^{1/12} - \frac{5625}{16}\,\sqrt{10}\,\mathrm{Arctan}\left(\frac{1}{5}\sqrt{10}\,(x^{1/12})\right))\Big|_0^1 \\
&= 2 - \frac{45}{7} + \frac{45}{2} - \frac{375}{4} + \frac{5625}{8} - \frac{5625}{16}\,\sqrt{10}\,\mathrm{Arctan}\left(\frac{1}{5}\sqrt{10}\right), \\
&\approx 0.4898298573.
\end{aligned}$$

Note: The integrand is undefined at $x = 0$ even in the limiting sense as $x \to 0^+$, so this is really an *improper integral*. But still, the integral exists, as you can see. The situation is akin to the one experienced when we integrate the function $x^{-1/4}$ from $x = 0$ to $x = 1$. Even though the integrand is undefined at $x = 0$, there is a *finite* area under the graph of the integrand between those limits.

0.0.1 Integrating rational functions of trig. expressions

We recall that the method described in Chapter 7.5 is to be used when the integrand is a sum of powers of trigonometric expressions such as $\sin^m x, \cos^n x$ or even $\sec^p x, \tan^q x$. But what does one do if the integrand is a *rational* function in these quantities? In other words how do we integrate an expression of the form

$$\frac{2+\sin^2 x - \cos^3 x}{\sin x + \cos x}?$$

This problem was taken up a long time ago and the mehtod described in what follows is sometimes called the *Weierstrass substitution*. It is based on the fact that trig. identities (see Appendix C and the text) can be used to simplify such rational expressions once we make a preliminary substitution. The general statement is something to the effect that

> Any rational function of $\sin x$ and $\cos x$ can be integrated using the substitution
>
> $$z = \tan\left(\frac{x}{2}\right)$$
>
> followed by the method of partial fractions (see Chapter 7.4).

The whole procedure can be quite lengthy but the end product is that we can find an antiderivative for such expressions! So, why does this curious looking change of variable work? Here's why.

We need to write out every term involving a sine function or a cosine function in terms of the new variable z, right? In fact, we will show that any power of either $\sin x$ or $\cos x$ will be transformed into a power of z. In this way we can see that any rational expression in these trig. functions will become ordinary rational functions so that the method of partial fractions can be used.

To begin with let's look at what happens to $\sin x$ when we perform the substitution $z = \tan(x/2)$. In other words let's write $\sin x$ in terms of z. To do this we first note that

$$z = \tan\left(\frac{x}{2}\right) \quad \Longrightarrow \quad x = 2\mathrm{Arctan}\, z.$$

So, we really have to find $\sin x = \sin(2\mathrm{Arctan}\, z)$ in terms of z and thus we set up a triangle of reference, see Figure 1. Writing $\theta = \mathrm{Arctan}\, z$ and using the identity $\sin 2\theta = 2\sin\theta\cos\theta$ we have to find $\sin\theta$ and $\cos\theta$.

Now from Figure 1 we have that

$$\sin\theta = \frac{z}{\sqrt{1+z^2}}, \quad \cos\theta = \frac{1}{\sqrt{1+z^2}},$$

so that

$$\sin x = 2\sin\theta\cos\theta = 2\cdot\frac{z}{\sqrt{1+z^2}}\cdot\frac{1}{\sqrt{1+z^2}} = \frac{2z}{1+z^2}.$$

Of course, we obtained $\cos\theta$ as well using this calculation, that is, we found that since $\theta = x/2$,

$$\cos(x/2) = \frac{1}{\sqrt{1+z^2}}.$$

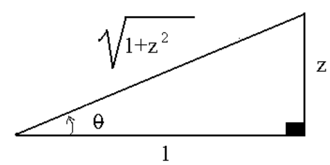

Figure 1

But the trigonometric identity $\cos 2\phi = 2\cos^2\phi - 1$, valid for any angle ϕ, means that we can set $\phi = x/2$ so that

$$\cos x = 2\cos^2(x/2) - 1 = 2\left(\frac{1}{\sqrt{1+z^2}}\right)^2 - 1 = \frac{1-z^2}{1+z^2}.$$

Finally, we need to determine the new "dx" term. This is not difficult since $z = \tan(x/2)$ implies that $dz = (1/2)\sec^2(x/2)\, dx$. But the trig. identity $\sec^2\phi - \tan^2\phi = 1$ valid for any angle ϕ means that we can set $\phi = x/2$ as before. This then gives us

$$\frac{dz}{dx} = (1/2)\sec^2(x/2) = (1/2)(1+\tan^2(x/2)) = (1/2)(1+z^2).$$

From this and the Chain Rule we also get that

$$\frac{dx}{dz} = \frac{2}{1+z^2}.$$

The preceding discussion can all be summarized in Table 1, for reference purposes.

Example 6 *Evaluate the integral* $I \equiv \displaystyle\int \frac{1}{4\cos x - 3\sin x}\, dx.$

Any rational function of $\sin x$ and $\cos x$ can be integrated using the substitution

$$z = \tan\left(\frac{x}{2}\right)$$

. In this case the preceding substitution demands that

$$\sin x = \frac{2z}{1+z^2}, \qquad \cos x = \frac{1-z^2}{1+z^2},$$

and

$$\frac{dx}{dz} = \frac{2}{1+z^2},$$

so that the "dx" term can be replaced formally by

$$dx = \frac{2\,dz}{1+z^2},$$

so that we can effect the integration.

Table 1: Rationalizing subtitutions for certain quotients of trigonometric functions

Solution: The integrand is a rational function of the sine and cosine function so we can use the substitution in Table 1. Thus,

$$\begin{aligned}
I &= \int \frac{1}{4\left(\frac{1-z^2}{1+z^2}\right) - 3\left(\frac{2z}{1+z^2}\right)} \left(\frac{2\,dz}{1+z^2}\right), \\
&= 2\int \frac{1}{4(1-z^2) - 3(2z)}\,dz, \\
&= \int \frac{1-2z}{z+2}\,dz \\
&= \int \left\{\frac{2/5}{1-2z} + \frac{1/5}{z+2}\right\} dz, \quad \text{(using partial fractions)} \\
&= \int \left\{-\frac{4/5}{z-(1/2)} + \frac{1/5}{z+2}\right\} dz, \\
&= -\frac{4}{5}\log|z - \frac{1}{2}| + \frac{1}{5}\log|z+2| + C, \\
&= -\frac{4}{5}\log|\tan(x/2) - \frac{1}{2}| + \frac{1}{5}\log|\tan(x/2)+2| + C.
\end{aligned}$$

Example 7 *Evaluate* $\displaystyle\int_0^2 \frac{1}{2+\sin x}\,dx.$

Solution: Using the substitution $z = \tan(x/2)$, etc. we find that an antiderivative is given by evaluating

$$\begin{aligned}
\int \frac{1}{2+\sin x}\,dx &= \int \frac{1}{2+\left(\frac{2z}{1+z^2}\right)}\left(\frac{2\,dz}{1+z^2}\right), \\
&= \int \frac{1}{z^2+z+1}\,dz
\end{aligned}$$

$$\begin{aligned}
&= \int \frac{1}{\left(z+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dz \\
&= \frac{2\sqrt{3}}{3}\text{Arctan}\left(\frac{\sqrt{3}}{3}(2z+1)\right) \\
&= \frac{2\sqrt{3}}{3}\text{Arctan}\left(\frac{\sqrt{3}}{3}(2\tan(x/2)+1)\right).
\end{aligned}$$

So the value of the given definite integral is given by

$$\begin{aligned}
\int_0^2 \frac{1}{2+\sin x}\, dx &= \frac{2\sqrt{3}}{3}\text{Arctan}\left(\frac{\sqrt{3}}{3}(2\tan(x/2)+1)\right)\Bigg|_{x=0}^{x=2} \\
&= \frac{2\sqrt{3}}{3}\left(\text{Arctan}\left(\frac{\sqrt{3}}{3}(2\tan(1)+1)\right) - \text{Arctan}\left(\frac{\sqrt{3}}{3}\right)\right) \\
&\approx .7491454107
\end{aligned}$$

Example 8 *Evaluate the following integral:* $I \equiv \int \frac{\sin x}{1+\sin x}\, dx$.

Solution: This integrand is a rational function in $\sin x$. So, as before we let $\sin x = \frac{2z}{1+z^2}$, $\cos x = \frac{1-z^2}{1+z^2}$, and $\frac{dx}{dz} = \frac{2}{1+z^2}$. Then

$$\begin{aligned}
I &= \int \frac{\frac{2z}{1+z^2}}{1+\frac{2z}{1+z^2}} \left(\frac{2\,dz}{1+z^2}\right), \\
&= \int \frac{4z}{(1+z)^2(1+z^2)}\, dz \\
&= \int \left\{\frac{2}{1+z^2} - \frac{2}{(1+z)^2}\right\} dz \quad \text{(using partial fractions + 3 coffees!)} \\
&= 2\,\text{Arctan}\, z + \frac{2}{1+z} \\
&= 2\,\text{Arctan}\,(\tan(x/2)) + \frac{2}{1+\tan(x/2)} \\
&= x + \frac{2}{1+\tan(x/2)} + C,
\end{aligned}$$

where C is a constant.

There is another way of handling this problem though, but it is tricky, and one would have to have had a revelation of sorts to see it ... Here goes. Note that

$$\begin{aligned}
\frac{\sin x}{1+\sin x} &= \frac{\sin x(1-\sin x)}{(1+\sin x)(1-\sin x)} \\
&= \frac{\sin x(1-\sin x)}{\cos^2 x} \\
&= \tan x \sec x - \tan^2 x, \\
&= \tan x \sec x - \sec^2 x + 1.
\end{aligned}$$

and so, using this newly derived identity, we can easily integrate the desired function. Why? Well,

$$\int \frac{\sin x}{1+\sin x}\, dx = \int (\tan x \sec x - \sec^2 x + 1)\, dx$$

$$\begin{aligned} &= \int \tan x \sec x\, dx - \int \sec^2 x\, dx + x + C^* \\ &= \sec x - \tan x + x + C^*, \end{aligned}$$

where C^* is a constant of integration. Now note that the two answers aren't quite the same so they MUST differ by a constant (from the theory of the integral). Let's see. Comparing our "two answers" we only need to show that the functions

$$\frac{2}{1+\tan(x/2)}, \quad \text{and} \quad \sec x - \tan x$$

differ by a constant. Why is this true? Using the important identities

$$\cos^2 \phi = \frac{1+\cos 2\phi}{2}, \quad \sin^2 \phi = \frac{1-\cos 2\phi}{2},$$

we find that (after we set $\phi = x/2$),

$$\tan\left(\frac{x}{2}\right) = \sqrt{\frac{1-\cos x}{1+\cos x}}.$$

This, in turn, gives us

$$\begin{aligned} \frac{2}{1+\tan(x/2)} &= \frac{2}{1+\sqrt{\frac{1-\cos x}{1+\cos x}}} \\ &= \frac{2\sqrt{1+\cos x}}{\sqrt{1+\cos x}+\sqrt{1-\cos x}} \\ &= \frac{2\sqrt{1+\cos x}(\sqrt{1+\cos x}-\sqrt{1-\cos x})}{(1+\cos x)-(1-\cos x)} \\ &= \frac{2(1+\cos x)-2\sqrt{1-\cos^2 x}}{2\cos x} \\ &= \sec x + 1 - 2\frac{\sqrt{\sin^2 x}}{2\cos x} \\ &= \sec x + 1 - \tan x. \end{aligned}$$

Thus, the two functions differ by the constant 1 (as we wanted to show).

Example 9 *Evaluate the following integral using two different methods:* $I \equiv \int \frac{\sin x}{2+\cos^2 x}\, dx$.

Solution: Since we have a rational function of sine and cosine we can use the substitution in Table 1. It follows that

$$\begin{aligned} I &= \int \frac{\left(\frac{2z}{1+z^2}\right)}{2+\left(\frac{1-z^2}{1+z^2}\right)^2} \frac{2\,dz}{1+z^2} \\ &= \int \frac{4z}{2(1+z^2)^2+(1-z^2)^2}\, dz \\ &= \int \frac{4z}{3z^4+2z^2+3}\, dz, \end{aligned}$$

which can then be integrated using the method of partial fractions (but it will be tedious!). Nevertheless, another option lies in the change of variable $\cos x = u$, $du = -\sin x\, dx$ which gives us that

$$\begin{aligned} I &= -\int \frac{1}{2+u^2}\, du \\ &= -\frac{1}{2}\int \frac{1}{1+(\frac{u}{\sqrt{2}})^2}\, du \\ &= -\frac{1}{2}\sqrt{2}\,\mathrm{Arctan}\left(\frac{1}{2}\sqrt{2}\, u\right), \quad (\text{using } u = \sqrt{2}\tan\theta, \text{ etc}) \\ &= -\frac{1}{2}\sqrt{2}\,\mathrm{Arctan}\left(\frac{1}{2}\sqrt{2}\cos x\right) + C. \end{aligned}$$

Exercise Set 1

Evaluate the following integrals using any method.

1. $\displaystyle\int \frac{\cos t}{3+\sin t}\, dt$
2. $\displaystyle\int \frac{\sqrt{x}}{1+2\sqrt{x}}\, dx$
3. $\displaystyle\int \frac{t^{2/3}}{1+t}\, dt$
4. $\displaystyle\int_0^{\pi/2} \frac{\sin 2t}{2+\cos t}\, dt$
5. $\displaystyle\int \frac{1}{2+3\sqrt{x}}\, dx$
6. $\displaystyle\int \frac{\sin x}{\tan x+\cos x}\, dx$
7. $\displaystyle\int \frac{2\sec t}{3\tan t+\cot t}\, dt$
8. $\displaystyle\int \frac{2-\sqrt{x}}{2+\sqrt{x}}\, dx$
9. $\displaystyle\int \sqrt{\frac{1+u}{1-u}}\, du$ (Hard)
10. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$
11. $\displaystyle\int \frac{1}{2x(1-\sqrt[4]{x})}\, dx$
12. $\displaystyle\int \frac{1}{x^2(1+\sqrt[3]{x})}\, dx$
13. $\displaystyle\int \sqrt{\frac{a+u}{b-u}}\, du$ (a, b are constants)

14. $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$

15. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{x}-\sqrt[3]{x}}\,dx$

16. $\displaystyle\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx$

17. $\displaystyle\int \frac{1}{\sqrt[3]{x}-\sqrt[4]{x}}\,dx$

18. $\displaystyle\int_0^\infty \frac{1}{1+\sqrt{x}}\,dx$

19. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}+1}\,dx$ (Hard)

20. $\displaystyle\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx$

21. $\displaystyle\int_0^\pi \frac{1}{\cos x+2\sin x}\,dx$

22. $\displaystyle\int_0^\infty \frac{1}{1+\sqrt[4]{x}}\,dx$

23. $\displaystyle\int_0^\pi \frac{1}{\cos 2x+\sin 2x}\,dx$

24. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}\,dx$

25. $\displaystyle\int_0^\infty \frac{1}{1+\sin^2 x}\,dx$

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