Rédei permutations with the same cycle structure

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Rédei functions

- Let $\mathbb{P}^1(\mathbb{F}_q) := \mathbb{F}_q \cup \{\infty\}$.
- Write $(x + \sqrt{y})^m$ as $N(x, y) + D(x, y)\sqrt{y}$.
- For $m \in \mathbb{N}$ and $a \in \mathbb{F}_q$, the *Rédei function* is $R_{m,a} : \mathbb{P}^1(\mathbb{F}_q) \to \mathbb{P}^1(\mathbb{F}_q)$ where

$$R_{m,a}(x) = \begin{cases} \dfrac{N(x,a)}{D(x,a)} & \text{if } D(x,a) \neq 0, x \neq \infty \\ \infty & \text{otherwise.} \end{cases}$$

- When $a \neq 0$ and q is odd, Carlitz showed that

$$R_{m,a}(x) = \sqrt{a}\frac{(x + \sqrt{a})^m + (x - \sqrt{a})^m}{(x + \sqrt{a})^m - (x - \sqrt{a})^m}.$$

Rédei functions

$$R_{1,a}(x) = x$$
$$R_{2,a}(x) = \frac{x^2 + a}{2x}$$
$$R_{3,a}(x) = \frac{x^3 + 3ax}{3x^2 + a}$$
$$R_{4,a}(x) = \frac{x^4 + 6ax^2 + a^2}{4x^3 + 4ax}$$
$$R_{5,a}(x) = \frac{x^5 + 10ax^3 + 5a^2x}{5x^4 + 10ax^2 + a^2}$$
$$R_{6,a}(x) = \frac{x^6 + 15ax^4 + 15a^2x^2 + a^3}{6x^5 + 20ax^3 + 6a^2x}$$
$$R_{7,a}(x) = \frac{x^7 + 21ax^5 + 35a^2x^3 + 7a^3x}{7x^6 + 35ax^4 + 21a^2x^2 + a^3}$$

Rédei functions

- From now on, q is odd.
- Let $\chi(a)$ be the quadratic character of $a \in \mathbb{F}_q^*$, that is,

$$\chi(a) = \begin{cases} 1 & \text{if } a \text{ is a square in } \mathbb{F}_q^* \\ -1 & \text{otherwise.} \end{cases}$$

- $R_{m,a}$ and $R_{n,a}$ induce the same function if and only if $m \equiv n \pmod{q - \chi(a)}$.

Rédei permutations

- For $a \neq 0$, $R_{m,a}$ induces a permutation of $\mathbb{P}^1(\mathbb{F}_q)$ if and only if $\gcd(m, q - \chi(a)) = 1$.
- If q is odd, then m is odd.

The cycle structure of a Rédei permutation

Proposition (Qureshi and Panario, 2015)

(a) *The decomposition of the Rédei permutation* $R_{m,a}$ *into cycles is*

$$\bigoplus_{d \mid q-\chi(a)} \left\{ \frac{\varphi(d)}{o_d(m)} \times Cyc(o_d(m)) \right\} \oplus (\chi(a)+1) \times \{\bullet\}$$

where $Cyc(c)$ *denotes a* c*-cycle.*

(b) *The number of fixed points of* $R_{m,a}$ *is* $\gcd(m-1, q-\chi(a)) + \chi(a) + 1$.

Main question

When do $R_{m,a}$ and $R_{n,b}$ have the same cycle structure?

$\star$ Recall: The decomposition of the Rédei permutation $R_{m,a}$ into cycles is

$$\bigoplus_{d \mid q-\chi(a)} \left\{ \frac{\varphi(d)}{o_d(m)} \times Cyc(o_d(m)) \right\} \oplus (\chi(a)+1) \times \{\bullet\}$$

where $Cyc(c)$ denotes a c-cycle.

A general criterion

Proposition (Deng, 2013)

Let X_1 and X_2 be finite sets, and $f_1 \colon X_1 \to X_1$ and $f_2 \colon X_2 \to X_2$ be permutations. Then f_1 and f_2 have the same cycle structure if and only if f_1^r and f_2^r have the same number of fixed points for every positive integer r.

The number of fixed points of $R_{m,a}^r$

⋆ Recall: $R_{m,a}$ has $\gcd(m-1, q-\chi(a))+\chi(a)+1$ fixed points.

- $R_{m,a} \circ R_{n,a} = R_{mn,a}$
- $\underbrace{R_{m,a} \circ \cdots \circ R_{m,a}}_{r \text{ times}} = R_{m^r,a}$
- The number of fixed points in the r^{th} iterate of $R_{m,a}$ is

$$\gcd(m^r-1, q-\chi(a))+\chi(a)+1.$$

A criterion for Rédei permutations

Proposition

The Rédei permutations $R_{m,a}$ and $R_{n,b}$ have the same cycle structure if and only if

$$\gcd(m^r - 1, q - \chi(a)) + \chi(a) = \gcd(n^r - 1, q - \chi(b)) + \chi(b)$$

for all positive integers r.

- In this talk, we focus on the case $\chi(a) = \chi(b) = \chi$. We need

$$\gcd(m^r - 1, q - \chi) = \gcd(n^r - 1, q - \chi)$$

  for all positive integers r.
- Work in progress: the case $\chi(a) \neq \chi(b)$.

Example: $\mathbb{P}^1(\mathbb{F}_{49})$

[3, 13, 17, 23, 27, 33, 37, 47]
[7, 43]
[9, 19, 29, 39]
[11, 21, 31, 41]

(a) $\chi = -1$

[5, 29]
[7, 31]
[11, 35]
[13, 37]
[19, 43]
[23, 47]

(b) $\chi = 1$

Lists with values of m and n for which $R_{m,a}$ and $R_{n,b}$ have the same cycle structure when $\chi(a) = \chi(b) = \chi$ over $\mathbb{P}^1(\mathbb{F}_{49})$.

- $\chi = -1$; Pattern: $+10$, $+10$, $+10$, $+10$

[**3**, 13, 17, 23, 27, 33, 37, 47]	numbers in symmetric positions add up to 50
[**7**, 43]	numbers in symmetric positions add up to 50
[**9**, 19, 29, 39]	numbers in symmetric positions add up to 48
[**11**, 21, 31, 41]	numbers in symmetric positions add up to 52, the above list $+2$

- $\chi = 1$; Pattern: $+24$, $+24$, $+24$, $+24$, $+24$, $+24$

[**5**, 29]	
[**7**, 31]	the above list $+2$
[**11**, 35]	the above list $+4$
[**13**, 37]	the above list $+2$
[**19**, 43]	the above list $+6$
[**23**, 47]	the above list $+4$

Question 1

Can we find families of Rédei permutations with the same cycle structure?

Families of Rédei permutations with the same cycle structure

q	m	n	Conditions
p^k	p^{ℓ_1}	p^{ℓ_2}	$1 \leq \ell_1, \ell_2 < k$, $\gcd(\ell_1, k) = \gcd(\ell_2, k)$ If $\chi = -1$, then $\nu_2(\ell_1), \nu_2(\ell_2)$ is either $>$ or $\leq \nu_2(k)$.
p^{2k}	p	$p^{2k} - p + 1$	$\chi = -1$
$\chi \pmod 8$	$\dfrac{q-\chi}{4} + 1$	$\dfrac{3(q-\chi)}{4} + 1$	None
$\chi \pm 2 \pmod 8$	$\dfrac{q-\chi \pm 2}{4}$	$\dfrac{q-\chi \pm 4}{2}$	None

The cycle structure of the families

χ	q	m	n	Cycle Structure
-1	p^k k odd prime	p^{ℓ_1} $1 \leq \ell_1 < k$ ℓ_1 odd	p^{ℓ_2} $1 \leq \ell_2 < k$ ℓ_2 odd	$(2 \times \{\bullet\}) \oplus \left(\frac{p-1}{2} \times \mathrm{Cyc}(2)\right) \oplus \left(\frac{p^k - p}{2k} \times \mathrm{Cyc}(2k)\right)$
-1	p^k k odd prime	p^{ℓ_1} $1 \leq \ell_1 < k$ ℓ_1 even	p^{ℓ_2} $1 \leq \ell_2 < k$ ℓ_2 even	$((p+1) \times \{\bullet\}) \oplus \left(\frac{p^k - p}{k} \times \mathrm{Cyc}(k)\right)$
1	p^k k prime	p^{ℓ_1} $1 \leq \ell_1 < k$	p^{ℓ_2} $1 \leq \ell_2 < k$	$((p+1) \times \{\bullet\}) \oplus \left(\frac{p^k - p}{k} \times \mathrm{Cyc}(k)\right)$
-1	p^{2k}	p	$p^{2k} - p + 1$	$(2 \times \{\bullet\}) \oplus \bigoplus_{\substack{d \mid 2k \\ \nu_2(d) = \nu_2(2k)}} (N_{2d} \times \mathrm{Cyc}(2d))$ where $2dN_{2d} = p^d + 1 - \sum_{\substack{s \mid 2d \\ \nu_2(s) = \nu_2(2d)}} 2sN_{2s} - 2$
χ	$\chi \pmod 8$	$\frac{q-\chi}{4} + 1$	$\frac{3(q-\chi)}{4} + 1$	$\begin{cases} \left(\left(\frac{q-\chi}{4} + \chi + 1\right) \times \{\bullet\}\right) \oplus \left(\frac{3(q-\chi)}{8} \times \mathrm{Cyc}(2)\right) & \text{if } \frac{q-\chi}{8} \text{ is odd} \\ \left(\left(\frac{q-\chi}{4} + \chi + 1\right) \times \{\bullet\}\right) \oplus \left(\frac{q-\chi}{8} \times \mathrm{Cyc}(2)\right) \oplus \left(\frac{q-\chi}{8} \times \mathrm{Cyc}(4)\right) & \text{if } \frac{q-\chi}{8} \text{ is even} \end{cases}$

Example: $\mathbb{P}^1(\mathbb{F}_{3^{60}})$

$R_{3,a}$ and $R_{3^{60}-2,b}$ have the same cycle structure on $\mathbb{P}^1\left(\mathbb{F}_{3^{60}}\right)$ when $\chi(a) = \chi(b) = -1$. To obtain the number of cycles and their corresponding lengths, we consider every positive divisor d of $2k = 60$ with $\nu_2(d) = \nu_2(60) = 2$.

- For $d = 4$, we get $N_8 = \left(3^4 + 1 - 2\right)/8$, so $N_8 = 10$.
- For $d = 12$, we get $N_{24} = \left(3^{12} + 1 - 8N_8 - 2\right)/24$, so $N_{24} = 22,140$
- For $d = 20$, we get $N_{40} = \left(3^{20} + 1 - 8N_8 - 2\right)/40$, so $N_{40} = 87,169,608$.
- For $d = 60$, we get $N_{120} = \left(3^{60} + 1 - 8N_8 - 24N_{24} - 40N_{40} - 2\right)/120$, so

$$N_{120} = 353,259,652,293,468,362,590,059,312.$$

Hence the cycle structure is

$$\begin{aligned} & (2 \times \{\bullet\}) \oplus (10 \times \mathrm{Cyc}(8)) \oplus (22,140 \times \mathrm{Cyc}(24)) \oplus (87,169,608 \times \mathrm{Cyc}(40)) \\ \oplus \quad & (353,259,652,293,468,362,590,059,312 \times \mathrm{Cyc}(120)). \end{aligned}$$

Example: $\mathbb{P}^1(\mathbb{F}_{49})$

[3, 13, 17, 23, 27, 33, 37, 47]
[7, 43]
[9, 19, 29, 39]
[11, 21, 31, 41]

initial families

(a) $\chi = -1$

[5, 29]
[7, 31]
[11, 35]
[13, 37]
[19, 43]
[23, 47]

initial families

(b) $\chi = 1$

Lists with values of m and n for which $R_{m,a}$ and $R_{n,b}$ have the same cycle structure when $\chi(a) = \chi(b) = \chi$ over $\mathbb{P}^1(\mathbb{F}_{49})$.

χ	q	m	n	Cycle Structure
-1	p^k k odd prime	p^{ℓ_1} $1 \leq \ell_1 < k$ ℓ_1 odd	p^{ℓ_2} $1 \leq \ell_2 < k$ ℓ_2 odd	$(2 \times \{\bullet\}) \oplus \left(\frac{p-1}{2} \times \mathrm{Cyc}(2)\right) \oplus \left(\frac{p^k - p}{2k} \times \mathrm{Cyc}(2k)\right)$
-1	p^k k odd prime	p^{ℓ_1} $1 \leq \ell_1 < k$ ℓ_1 even	p^{ℓ_2} $1 \leq \ell_2 < k$ ℓ_2 even	$((p+1) \times \{\bullet\}) \oplus \left(\frac{p^k - p}{k} \times \mathrm{Cyc}(k)\right)$
1	p^k k prime	p^{ℓ_1} $1 \leq \ell_1 < k$	p^{ℓ_2} $1 \leq \ell_2 < k$	$((p+1) \times \{\bullet\}) \oplus \left(\frac{p^k - p}{k} \times \mathrm{Cyc}(k)\right)$
-1	p^{2k}	p	$p^{2k} - p + 1$	$(2 \times \{\bullet\}) \oplus \bigoplus_{\substack{d \mid 2k \\ \nu_2(d) = \nu_2(2k)}} (N_{2d} \times \mathrm{Cyc}(2d))$ where $2dN_{2d} = p^d + 1 - \sum_{\substack{s \mid 2d \\ \nu_2(s) = \nu_2(2d)}} 2sN_{2s} - 2$
χ	$\chi \pmod 8$	$\frac{q-\chi}{4} + 1$	$\frac{3(q-\chi)}{4} + 1$	$\begin{cases} \left(\left(\frac{q-\chi}{4} + \chi + 1\right) \times \{\bullet\}\right) \oplus \left(\frac{3(q-\chi)}{8} \times \mathrm{Cyc}(2)\right) & \text{if } \frac{q-\chi}{8} \text{ is odd} \\ \left(\left(\frac{q-\chi}{4} + \chi + 1\right) \times \{\bullet\}\right) \oplus \left(\frac{q-\chi}{8} \times \mathrm{Cyc}(2)\right) \oplus \left(\frac{q-\chi}{8} \times \mathrm{Cyc}(4)\right) & \text{if } \frac{q-\chi}{8} \text{ is even} \end{cases}$

Question 2

Can we describe the Rédei permutations that decompose into 1- and j-cycles?

1- and j-cycles

⋆ Recall: The j^{th} iterate of $R_{m,a}$ has $\gcd(m^j - 1, q - \chi(a)) + \chi(a) + 1$ fixed points.

- If $R_{m,a}$ decomposes into 1- and j-cycles, then

$$\gcd(m^j - 1, q - \chi(a)) + \chi(a) + 1 = q + 1.$$

(q, χ, j)-admissible integers

⋆ Recall: $R_{m,a}$ has $\gcd(m-1, q-\chi(a)) + \chi(a) + 1$ fixed points.

Definition

An integer d is (q, χ, j)-admissible if there exists an $R_{m,a}$ that decomposes into 1- and j-cycles with $d + \chi + 1$ fixed points and $\chi(a) = \chi$.

- $d = \gcd(m-1, q-\chi)$

When $j = p$ is prime

Proposition

Let p be a prime, $q - \chi = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $d = p_1^{\beta_1} \cdots p_r^{\beta_r}$ with $0 \leq \beta_i \leq \alpha_i$. Then d is (q, χ, p)-admissible if and only if

$$\beta_i = \begin{cases} \alpha_i - 1 \text{ or } \alpha_i & \text{if } p_i = p > 2 \text{ and } \alpha_i \geq 2 \\ 1,\ \alpha_i - 1 \text{ or } \alpha_i & \text{if } p_i = p = 2 \text{ and } \alpha_i \geq 2 \\ 0 \text{ or } \alpha_i & \text{if } p_i \equiv 1 \pmod{p} \\ \alpha_i & \text{otherwise} \end{cases}$$

for each $i \in \{1, \ldots, r\}$.

An existence condition

- Case $p = 2$: Rédei involutions always exist.

Corollary

Let p be an odd prime. There exists a Rédei permutation with 1- and p-cycles if and only if $q - 1$ or $q + 1$ has a prime factor of the form $pk + 1$ or is divisible by p^2.

A characterization of Rédei permutations with 1- and p-cycles

Theorem

Let p be a prime and d be a (q, χ, p)-admissible integer. The Rédei permutation $R_{m,a}$ has $d + \chi + 1$ fixed points and p-cycles if and only if

(i) *m is a solution to*

$$\begin{cases} m \equiv 1 & \pmod{d} \\ m^{p-1} + m^{p-2} + \cdots + m + 1 \equiv 0 & \pmod{(q-\chi)/d} \end{cases}$$

(ii) $p^{\nu_p(q-\chi)} \nmid m - 1$, *if* $\nu_p(d) = \nu_p(q - \chi) - 1$.

Counting

Proposition

Let p be an odd prime and d be (q, χ, p)-admissible. Let M_d be the number of Rédei permutations with fixed parameter a, $d + \chi + 1$ fixed points, and p-cycles. Then

$$M_d = \begin{cases} (p-1)^u & \text{if } \nu_p(d) = \nu_p(q - \chi) \\ (p-1)^{u+1} & \text{if } \nu_p(d) = \nu_p(q - \chi) - 1, \end{cases}$$

where $u = |\{p' \text{ prime} \colon p' \equiv 1 \pmod{p}, p' \mid q - \chi \text{ and } p' \nmid d\}|$.

Rédei involutions

Theorem

Let $q-\chi = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the prime factorization of $q-\chi$, and $d = 2^{\beta_0} p_1^{\beta_1} \cdots p_r^{\beta_r}$ be a proper divisor of $q-\chi$. Then there exists a Rédei involution $R_{m,a}$ with $d+\chi+1$ fixed points over $\mathbb{P}^1(\mathbb{F}_q)$ if and only if $\beta_i \in \{0, \alpha_i\}$ for $1 \le i \le r$ and one of the following situations occurs:

(i) $\beta_0 \in \{\alpha_0 - 1, \alpha_0\}$ *and* $\beta_0 \ge 1$. *In this case, $R_{m,a}$ has a unique cycle structure and* $m \equiv k(q-\chi)/d - 1 \pmod{q-\chi}$ *for*

$$k = \begin{cases} \left(\dfrac{q-\chi}{2d}\right)^{\varphi(d)-1} + \dfrac{d}{2} & \text{if } \beta_0 = \alpha_0 - 1 \\ 2\left(\dfrac{q-\chi}{d}\right)^{\varphi(d)-1} & \text{if } \beta_0 = \alpha_0 \end{cases}$$

with k reduced modulo d.

(ii) $\alpha_0 \geq 3$ and $\beta_0 = 1$. In this case, $R_{m,a}$ and $R_{n,b}$ have the same cycle structure, where m or $n \equiv k(q-\chi)/d - 1 \pmod{q-\chi}$ for

$$k = \left(\frac{q-\chi}{2d}\right)^{\varphi(d)-1}$$

with k reduced modulo d, and $m \equiv n + (q-\chi)/2 \pmod{q-\chi}$.

Example: $\mathbb{P}^1(\mathbb{F}_{125})$

- $\chi(a) = 1$: $q - 1 = 2^2 \cdot 31$

j	prime $jk+1$, $jk+1 \mid q-1$?	$j^2 \mid q-1$?	d	M_d	m	# fixed points	# j-cycles
2	N/A		2	1	123	4	61
			4	1	61	6	60
			62	1	63	64	31
3	yes, 31	no	4	2	$5, 25$	6	40
4	no	N/A					
5	yes, 31	no	4	4	$33, 97, 101, 109$	6	24
prime ≥ 7	no	no					

Example: $\mathbb{P}^1(\mathbb{F}_{125})$

- $\chi(a) = -1$: $q + 1 = 2 \cdot 3^2 \cdot 7$

j	prime $jk+1$, $jk+1 \mid q+1$?	$j^2 \mid q+1$?	d	M_d	m	# fixed points	# j-cycles
2	N/A		2	1	125	2	62
			14	1	71	14	56
			18	1	55	18	54
3	yes, 7	yes	6	4	$25, 67, 79, 121$	6	40
			18	2	$37, 109$	18	36
			42	2	$43, 85$	42	28
4	no	N/A					
prime ≥ 5	no	no					

Example: $\mathbb{P}^1(\mathbb{F}_{49})$

[3, 13, 17, 23, 27, 33, 37, 47]	
[7, 43]	initial families
[9, 19, 29, 39]	
[11, 21, 31, 41]	1-and 5-cycles

(a) $\chi = -1$

[5, 29]	
[7, 31]	
[11, 35]	
[13, 37]	initial families
[19, 43]	
[23, 47]	involutions

(b) $\chi = 1$

Lists with values of m and n for which $R_{m,a}$ and $R_{n,b}$ have the same cycle structure when $\chi(a) = \chi(b) = \chi$ over $\mathbb{P}^1(\mathbb{F}_{49})$.

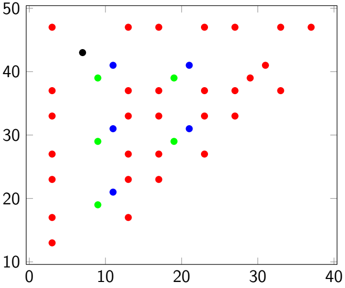
Back to the main question

When do $R_{m,a}$ and $R_{n,b}$ have the same cycle structure?

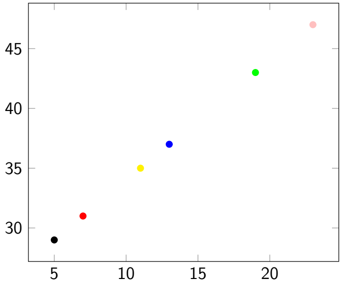
The set S_χ^q

- $\mathcal{S}_\chi^q = \{(m,n) \in \mathbb{N}^2 \colon R_{m,a}$ and $R_{n,b}$ are Rédei permutations with the same cycle structure for some $a, b \in \mathbb{F}_q$ with $\chi(a) = \chi(b) = \chi\}$.
- Clearly, $(m,n) \in S_\chi^q$ if and only if $(n,m) \in S_\chi^q$.

Plotting the points in S_χ^{49}



(a) $\chi = -1$



(b) $\chi = 1$

Pairs $(m, n) \in S_\chi^{49}$ with $1 < m < n < 49 - \chi$, color-coded by their cycle structures.

The distribution of the points in S_{-1}^{49} over lines

Equation of the line	Pairs (m, n) lying on the line
$y = x + 4$	$(13, 17)$, $(23, 27)$, $(33, 37)$
$y = x + 6$	$(17, 23)$, $(27, 33)$
$y = x + 10$	$(3, 13)$, $(9, 19)$, $(11, 21)$, $(13, 23)$, $(17, 27)$, $(19, 29)$, $(21, 31)$, $(23, 33)$, $(27, 37)$, $(29, 39)$, $(31, 41)$, $(37, 47)$
$y = x + 14$	$(3, 17)$, $(13, 27)$, $(23, 37)$, $(33, 47)$
$y = x + 16$	$(17, 33)$
$y = x + 20$	$(3, 23)$, $(9, 29)$, $(11, 31)$, $(13, 33)$, $(17, 37)$, $(19, 39)$, $(21, 41)$, $(27, 47)$
$y = x + 24$	$(3, 27)$, $(13, 37)$, $(23, 47)$
$y = x + 30$	$(3, 33)$, $(9, 39)$, $(11, 41)$, $(17, 47)$
$y = x + 34$	$(3, 37)$, $(13, 47)$
$y = x + 36$	$(7, 43)$
$y = x + 44$	$(3, 47)$

The distribution of the pairs $(m, n) \in S_{-1}^{49}$ over eleven lines.

A complete characterization of S_χ^q

Theorem

Suppose $q-\chi = p_1^{\alpha_1}\cdots p_t^{\alpha_t}$ *is the prime factorization of* $q-\chi$, *m is coprime with* $q-\chi$, *and* $\theta_i = o_{p_i}(m)$. *Then* $(m,n)\in S_\chi^q$ *if and only if* $n = m + k(q-\chi)/d$, *where d is a proper divisor of* $q-\chi$ *and k is an integer, and for each* p_i *that divides d the following conditions hold:*

(i) $p_i \nmid n$ *and* $o_{p_i}(n) = \theta_i$,

(ii) $\gcd(m^{\theta_i}-1, p_i^{\alpha_i}) = \gcd(n^{\theta_i}-1, p_i^{\alpha_i})$.

(iii) *If* $p_i = 2$, $\alpha_i > 1$ *and* $\nu_2(m-1) = 1$, *then* $\gcd(m^2-1, 2^{\alpha_i}) = \gcd(n^2-1, 2^{\alpha_i})$.

Idea of the proof

- Recall: $(m,n) \in S_\chi^q$ if and only if $\gcd(m^r-1, q-\chi) = \gcd(n^r-1, q-\chi)$ for all positive integers r.
- $q-\chi = p_1^{\alpha_1}\cdots p_t^{\alpha_t} \quad \Longrightarrow \quad \gcd(m^r-1, q-\chi) = \prod_{i=1}^{t} \gcd(m^r-1, p_i^{\alpha_i})$
- If p is odd and $\theta = o_p(m)$, then $\nu_p(m^r-1) = \begin{cases} 0 & \text{if } \theta \nmid r \\ \nu_p(m^\theta-1)+\nu_p(t) & \text{if } r = t\theta \end{cases}$
- If $p=2$, then $\nu_2(m^r-1) = \begin{cases} \nu_2(m-1) & \text{if } r \text{ is odd} \\ \nu_2(m^2-1)+\nu_2(r)-1 & \text{if } r \text{ is even} \end{cases}$

Symmetries in S_χ^q

Proposition

Suppose $q-\chi = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ *is the prime factorization of* $q-\chi$ *and* d *is a proper divisor of* $q-\chi$*. If* $(m,n)\in S_\chi^q$*, then* $(m+k(q-\chi)/d, n+k(q-\chi)/d)\in S_\chi^q$ *if and only if for each* p_i *that divides* d *the following conditions hold:*

(i) $p_i \nmid m+k(q-\chi)/d, n+k(q-\chi)/d$ *and* $\theta_i := o_{p_i}(m+k(q-\chi)/d) = o_{p_i}(n+k(q-\chi)/d)$,

(ii) $\gcd((m+k(q-\chi)/d)^{\theta_i}-1, p_i^{\alpha_i}) = \gcd((n+k(q-\chi)/d)^{\theta_i}-1, p_i^{\alpha_i})$.

(iii) *If* $p_i=2$, $\alpha_i>1$ *and* $\nu_2((m+k(q-\chi)/d)-1)=1$*, then* $\gcd((m+k(q-\chi)/d)^2-1, 2^{\alpha_i}) = \gcd((n+k(q-\chi)/d)^2-1, 2^{\alpha_i})$.

Equation of the line	Generator (m, n)	d	$(m + k \cdot 50/d, n + k \cdot 50/d)$, $(n + k \cdot 50/d, m + k \cdot 50/d)$ in $\mathbb{Z}_{50}^2$
$y = x + 4$	$(13, 17)$	5	$(23, 27)$, $(33, 37)$
$y = x + 44$ $y = x + 6$	$(3, 47)$	5	$(17, 23)$, $(27, 33)$
$y = x + 10$	$(3, 13)$	25	$(9, 19)$, $(11, 21)$, $(13, 23)$, $(17, 27)$, $(19, 29)$, $(21, 31)$, $(23, 33)$, $(27, 37)$, $(29, 39)$, $(31, 41)$, $(37, 47)$
$y = x + 14$ $y = x + 36$	$(3, 17)$	5	$(13, 27)$, $(23, 37)$, $(33, 47)$ $(7, 43)$
$y = x + 34$ $y = x + 16$	$(3, 37)$	5	$(13, 47)$ $(17, 33)$
$y = x + 20$	$(3, 23)$	25	$(9, 29)$, $(11, 31)$, $(13, 33)$, $(17, 37)$, $(19, 39)$, $(21, 41)$, $(27, 47)$
$y = x + 24$	$(3,27)$	5	$(13, 37)$, $(23, 47)$
$y = x + 30$	$(3, 33)$	25	$(9, 39)$, $(11, 41)$, $(17, 47)$

The distribution of the points over eleven lines and their corresponding generators.

All Rédei permutations $R_{m,a}$ over $\mathbb{P}^1(\mathbb{F}_{49})$, with $1 \leq m < 49 - \chi(a)$

$\chi(a)$	m	Cycle Structure
-1	1	$50 \times \{\bullet\}$
-1	$3, 13, 17, 23, 27, 33, 37, 47$	$(2 \times \{\bullet\}) \oplus (2 \times \mathrm{Cyc}(4)) \oplus (2 \times \mathrm{Cyc}(20))$
-1	$7, 43$	$(2 \times \{\bullet\}) \oplus (12 \times \mathrm{Cyc}(4))$
-1	$9, 19, 29, 39$	$(2 \times \{\bullet\}) \oplus (4 \times \mathrm{Cyc}(2)) \oplus (4 \times \mathrm{Cyc}(10))$
-1	$11, 21, 31, 41$	$(10 \times \{\bullet\}) \oplus (8 \times \mathrm{Cyc}(5))$
-1	49	$(2 \times \{\bullet\}) \oplus (24 \times \mathrm{Cyc}(2))$
1	1	$50 \times \{\bullet\}$
1	$5, 29$	$(6 \times \{\bullet\}) \oplus (10 \times \mathrm{Cyc}(2)) \oplus (6 \times \mathrm{Cyc}(4))$
1	$7, 31$	$(8 \times \{\bullet\}) \oplus (21 \times \mathrm{Cyc}(2))$
1	$11, 35$	$(4 \times \{\bullet\}) \oplus (11 \times \mathrm{Cyc}(2)) \oplus (6 \times \mathrm{Cyc}(4))$
1	$13, 37$	$(14 \times \{\bullet\}) \oplus (6 \times \mathrm{Cyc}(2)) \oplus (6 \times \mathrm{Cyc}(4))$
1	17	$(18 \times \{\bullet\}) \oplus (16 \times \mathrm{Cyc}(2))$
1	$19, 43$	$(8 \times \{\bullet\}) \oplus (9 \times \mathrm{Cyc}(2)) \oplus (6 \times \mathrm{Cyc}(4))$
1	$23, 47$	$(4 \times \{\bullet\}) \oplus (23 \times \mathrm{Cyc}(2))$
1	25	$(26 \times \{\bullet\}) \oplus (12 \times \mathrm{Cyc}(2))$
1	41	$(10 \times \{\bullet\}) \oplus (20 \times \mathrm{Cyc}(2))$

Theorem

The only isolated Rédei permutations are the isolated Rédei involutions.

- Idea of the proof:
  If $R_{m,a}$ is not an involution, then $\rho := o_{q-\chi}(m) > 2$. In this case, the pair $(m, m^{\rho-1})$ is in S_χ^q.

Final remark

The mappings

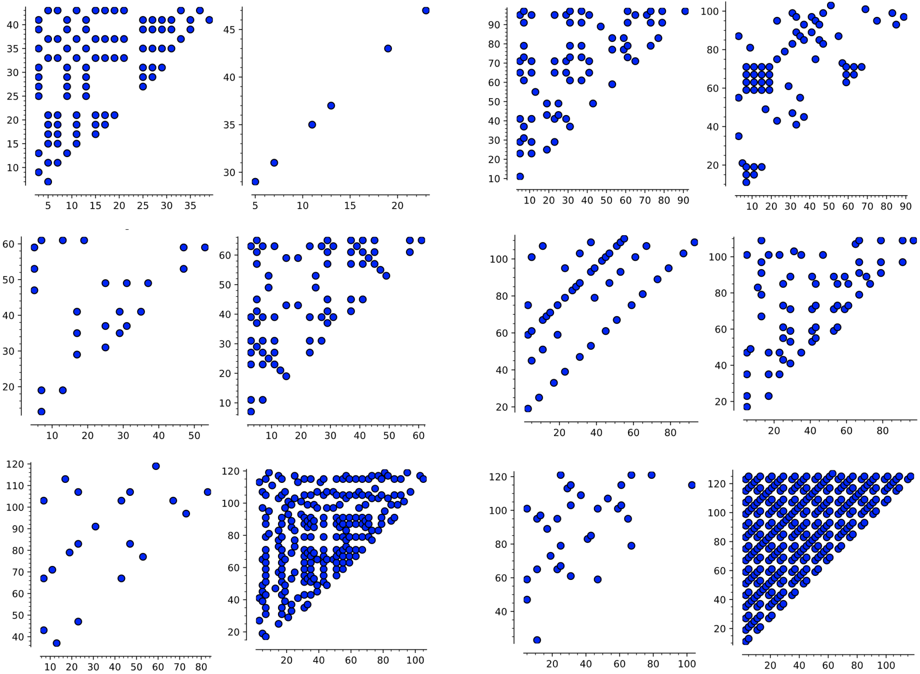
$$
\begin{aligned}
&R_{m,a} \colon \mathbb{D}_q \to \mathbb{D}_q \\
&f_m \colon \mathbb{Z}_{q-\chi} \to \mathbb{Z}_{q-\chi}, \quad x \mapsto mx \\
&g_m \colon \mathbb{C}_q \to \mathbb{C}_q, \qquad\quad x \mapsto x^m
\end{aligned}
$$

have the same cycle structure, where

$$
\mathbb{D}_q = \begin{cases} \mathbb{P}^1(\mathbb{F}_q) & \text{if } \chi(a) = -1 \\ \mathbb{P}^1(\mathbb{F}_q) \setminus \{\pm\sqrt{a}\} & \text{if } \chi(a) = 1, \end{cases} \qquad \mathbb{C}_q = \begin{cases} U_{q+1} & \text{if } \chi(a) = -1 \\ \mathbb{F}_q^* & \text{if } \chi(a) = 1, \end{cases}
$$

and U_{q+1} is the subgroup of order $q+1$ in $\mathbb{F}_{q^2}$.

Thank you!



40 35 30 25 20 15 10
5 10 15 20 25 30 35
45 40 35 30
5 10 15 20
90 80 70 60 50 40 30 20 10
10 20 30 40 50 60 70 80 90
100 80 60 40 20
10 20 30 40 50 60 70 80 90
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