Graphs and Algebra in Modern Communication.

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Joint work with
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Kimberly (Bowdoin College) and Nathan (Haverford College)

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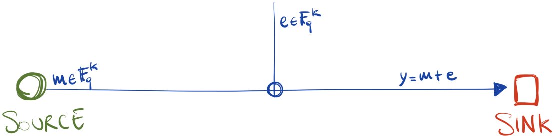
Noisy-Channel Coding Theorem (1948)



Theorem (Noisy-Channel Coding Theorem - Shannon - 1948)

"Any channel, however affected by noise, possesses a specific channel capacity - a rate of conveying information that can never be exceeded without error, but that can always be attained with an arbitrarily small probability of error."

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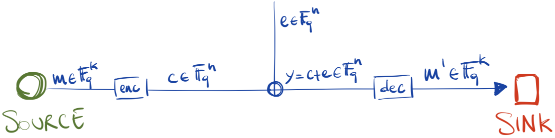

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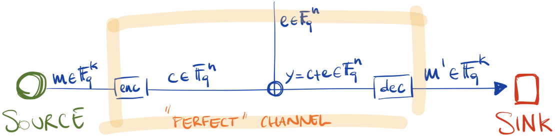

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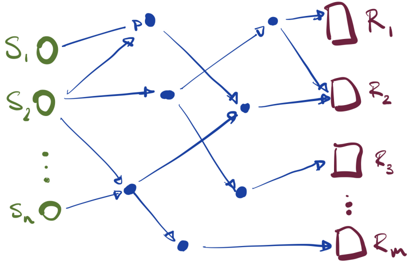
Networks - a graph perspective

A network is a *directed acyclic graph* $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \mathcal{S}, \mathcal{R}, \mathbb{F}_q)$.



- Sources: nodes with no incoming edges, $\mathcal{S} \subsetneq \mathcal{V}$.
- Sinks: nodes with no outgoing edges, $\mathcal{R} \subsetneq \mathcal{V}$.
- Edges represent perfect unit capacity channels.
- each sink $R \in \mathcal{R}$ demands messages from $D_R \subseteq \mathcal{S}$.

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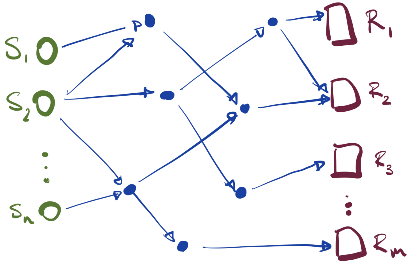
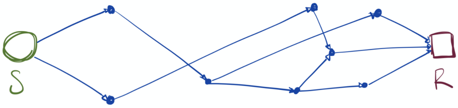


- *Unicast Problem*: $|\mathcal{S}| = |\mathcal{R}| = 1$ and $D_R = \mathcal{S}$.
- *Multicast Problem*: $|\mathcal{R}| \geq 1$ and $D_R = \mathcal{S}$ for all $R \in \mathcal{R}$.
- *Multiple Unicast problem*: $\mathcal{S} = \{S_1, \ldots, S_n\}$, $\mathcal{R} = \{R_1, \ldots, R_n\}$, and $D_{R_i} = \{S_i\}$.

Unicast Network



- Communication rate: $\rho \leq \mathrm{mincut}(S, R)$.

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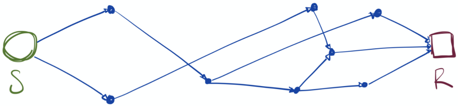

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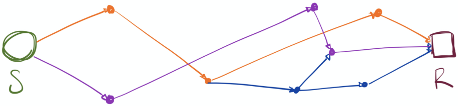
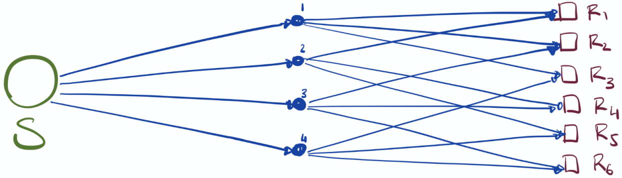


- Communication rate: $\rho \leq \mathrm{mincut}(S, R)$.
- Menger's Theorem: $\mathrm{mincut}(S, R) =$ maximum number of pairwise edge–disjoint paths.
- Routing maximizes R.

Multicast network [Li et al., 2003, Koetter and Medard, 2003]



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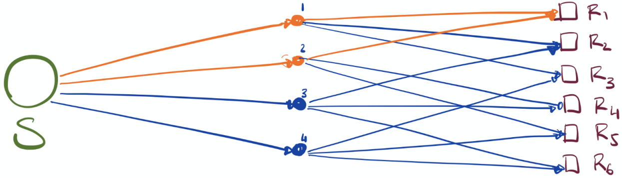

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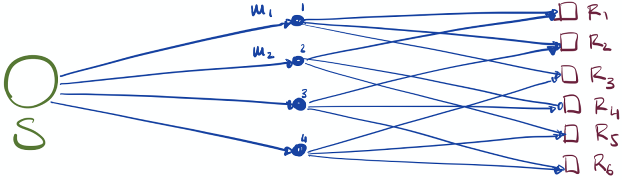

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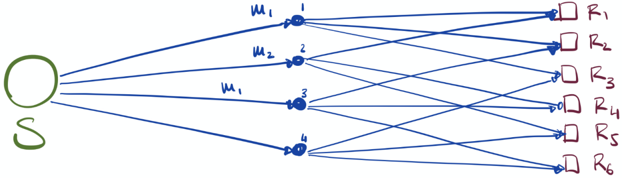

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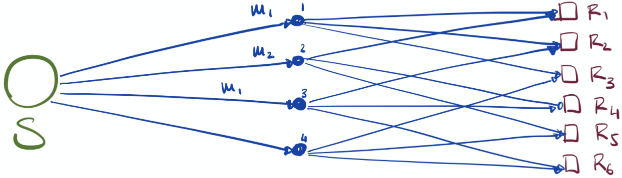

Theorem (Linear Network Multicasting Theorem)

Let $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \mathcal{S}, \mathcal{R}, \mathbb{F}_q)$. *A multicast rate of*

$$\min_{R \in \mathcal{R}} \mathrm{mincut}(\mathcal{S}, R)$$

is achievable, for sufficiently large q, with linear network coding.

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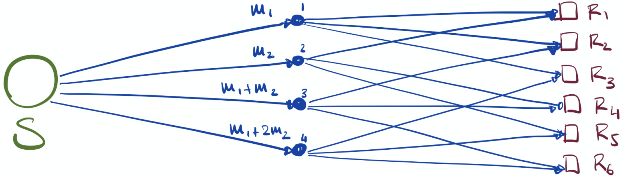

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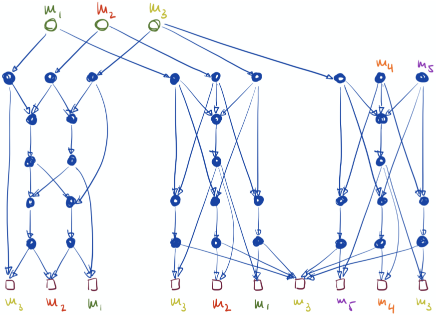
Insufficiency of LNC [Dougherty et al., 2005]



Theorem

There exists an solvable network that has no linear solution over any finite field and any vector dimension.

1. Introduction

2. Interference Networks
   - Definition
   - Multiple Unicast network
   - Linear Achievability and complexity

3. Achievability bounds for Multiple Unicast Networks
   - Lower Bound
   - Upper Bounds

4. Conclusions

Networks and Interference

"Interference is a major impairment to the reliable communication in multi-user wireless networks, due to the broadcast and superposition nature of wireless medium." [Zhao et al., 2016]

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Let $\mathcal{N}$ be a network with $\mathcal{S}$ sources set and $\mathcal{R}$ receivers set. A network has interference if

- $D_R \neq \mathcal{S}$ for some $R \in \mathcal{R}$
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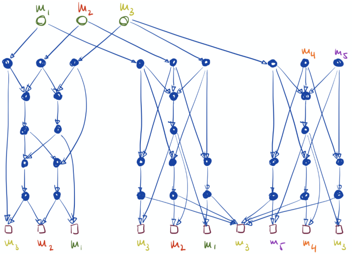

Multiple Unicast network

Definition

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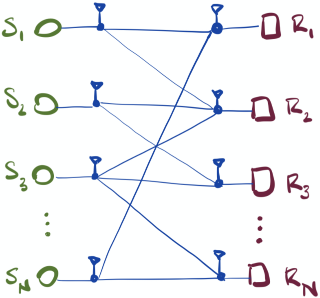

Communication strategy:

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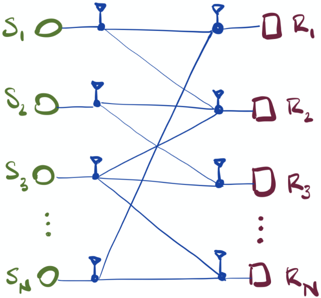

Communication strategy:

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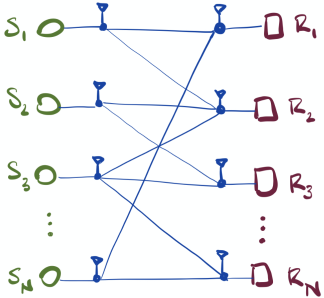

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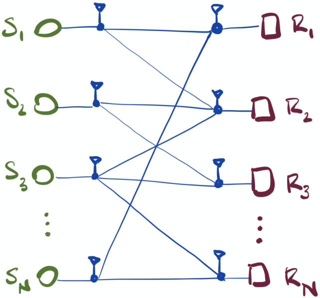

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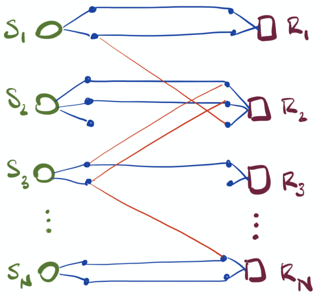
- multiple rounds $\rightarrow$ time sharing.

Multiple Unicast network (cont'd)



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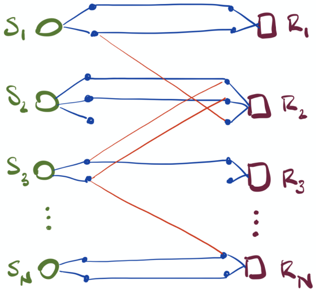

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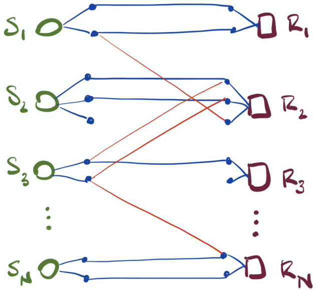

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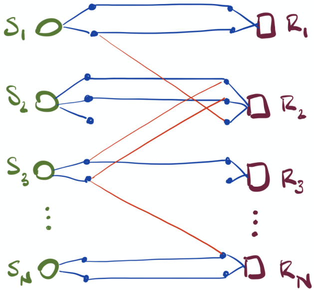
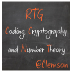


Communication strategy:

- multiple rounds $\to$ time sharing.
- 1 round $\to$ *interference alignment*, meaning use of subspaces to communicate without interference
- original paper [Cadambe and Jafar, 2008].



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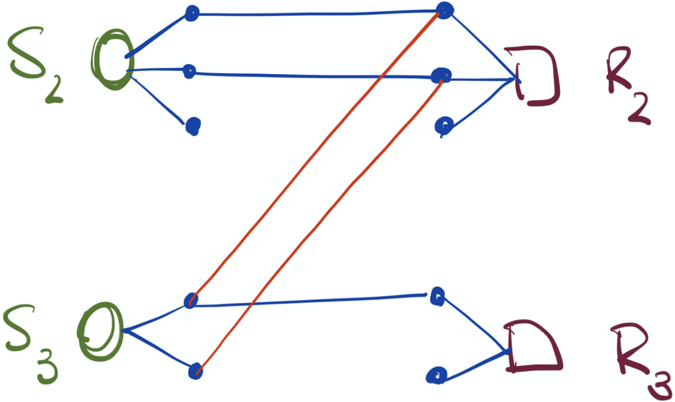

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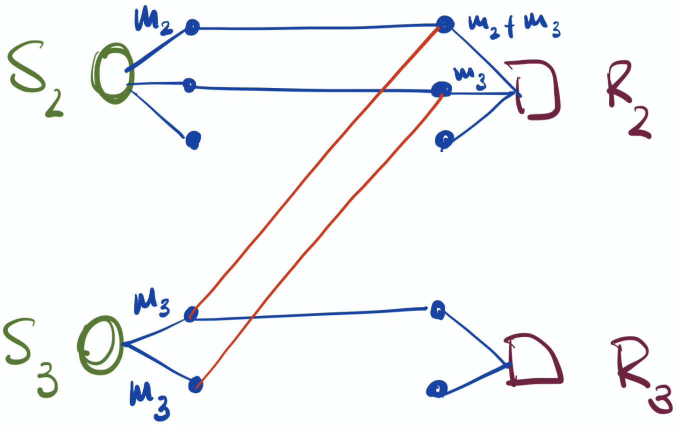

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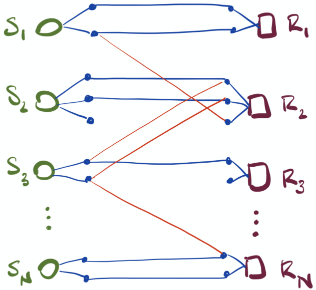

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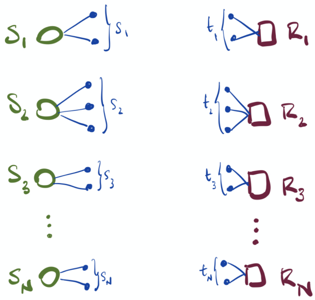
Multiple Unicast network - Notation



- s_i number of antennas available to source S_i and $s = \sum_{i=1}^{N} s_i$.
- t_i number of antennas available to source R_i and $t = \sum_{i=1}^{N} t_i$.

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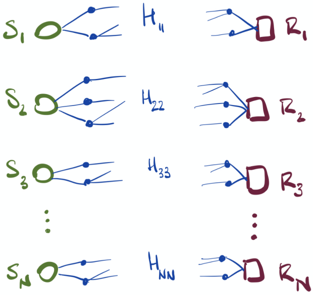

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$$H = \begin{pmatrix} H_{11} & & \\ & H_{22} & \\ & & H_{NN} \end{pmatrix}$$

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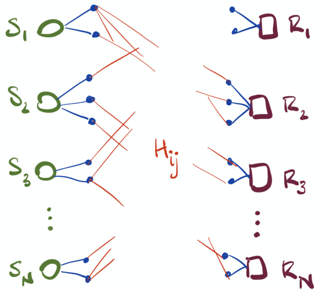

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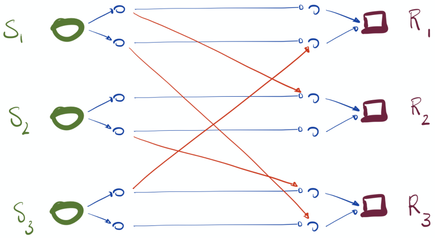
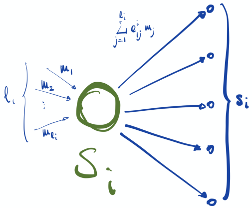
Example



$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Linear Encoders and Decoders



$$E = \begin{pmatrix} E_1 & 0 & \dots & 0 \\ 0 & E_2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & E_N \end{pmatrix}$$

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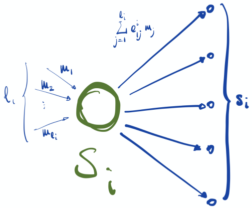

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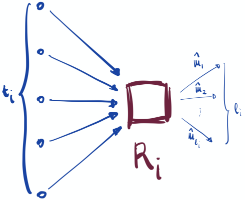

Linear Network Communication

$$\begin{pmatrix} m_{i,1} \\ \vdots \\ m_{i,\ell_i} \end{pmatrix} \qquad \overset{j \neq i}{} \qquad \begin{pmatrix} m_{j,1} \\ \vdots \\ m_{j,\ell_j} \end{pmatrix}$$

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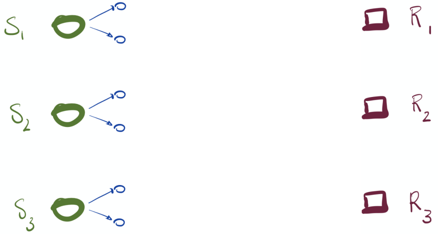

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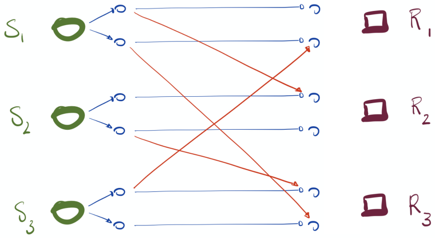
$$H_{ii}E_i \begin{pmatrix} m_{i,1} \\ \vdots \\ m_{i,\ell_i} \end{pmatrix} + \sum_{j \neq i} H_{ij}E_j \begin{pmatrix} m_{j,1} \\ \vdots \\ m_{j,\ell_j} \end{pmatrix}$$



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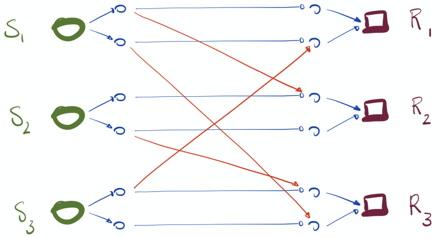
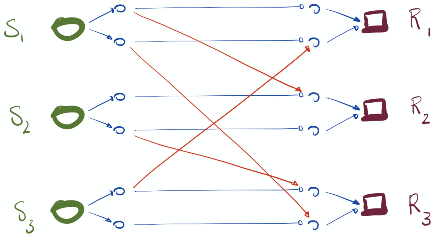
$$D_i H_{ii} E_i \begin{pmatrix} m_{i,1} \\ \vdots \\ m_{i,\ell_i} \end{pmatrix} + \sum_{j \neq i} D_i H_{ij} E_j \begin{pmatrix} m_{j,1} \\ \vdots \\ m_{j,\ell_j} \end{pmatrix}$$



Linear Network Communication

$$\begin{pmatrix} \hat{m}_{i,1} \\ \vdots \\ \hat{m}_{i,\ell_i} \end{pmatrix} = D_i H_{ii} E_i \begin{pmatrix} m_{i,1} \\ \vdots \\ m_{i,\ell_i} \end{pmatrix} + \sum_{j \neq i} D_i H_{ij} E_j \begin{pmatrix} m_{j,1} \\ \vdots \\ m_{j,\ell_j} \end{pmatrix}$$



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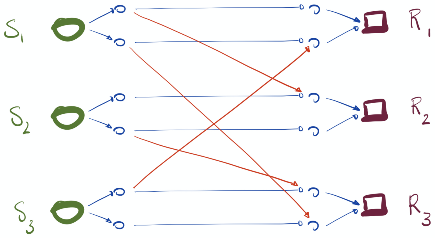


Linear Achievability of Multiple Unicast networks

Definition

A network $\mathcal{N}$ is linearly achievable for $\rho = (\rho_1, \dots, \rho_N) \in \mathbb{Z}^N$, or simply ρ-linearly achievable, if there exist two matrices D, E (with entries in $\mathbb{F}_q$) such that

$$D_i H_{ij} E_j = 0 \text{ and } \operatorname{rank}(D_i H_{ii} E_i) = \rho_i.$$

- $D_i H_{ii} E_i$ is a $\ell_i \times \ell_i$ matrix.
- wlog D_i can be chosen to be in RCEF and messages are sent at pivot positions.

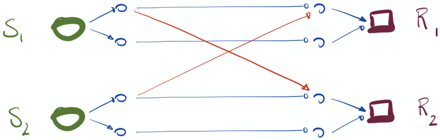
$$D_i H_{ii} E_i \begin{pmatrix} m_1 \\ 0 \\ m_3 \\ \vdots \\ m_{\rho_i} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ \star & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} m_1 \\ 0 \\ m_3 \\ \vdots \\ m_{\rho_i} \end{pmatrix} = \begin{pmatrix} m_1 \\ \star \\ m_3 \\ \vdots \\ m_{\rho_i} \end{pmatrix}$$

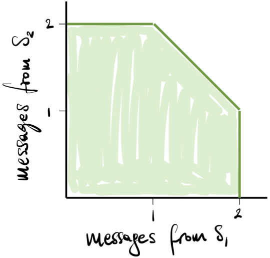
Linear Achievability Region

Definition

The *linear achievability region* of network $\mathcal{N}$, denoted $\mathrm{Lin}(\mathcal{N})$, is the subset of $\mathbb{R}^N$ for which $\mathcal{N}$ is ρ-linearly achievable.

$$H = \begin{pmatrix} 1 & 0 & \color{red}{1} & 0 \\ 0 & 1 & 0 & 0 \\ \color{red}{1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$





Algebraic model

We represent a network $\mathcal{N}$ by matrices with

$$DHE = \begin{pmatrix} D_1 H_{11} E_1 & \dots & D_1 H_{1N} E_N \\ D_2 H_{21} E_1 & \dots & D_2 H_{2N} E_N \\ \vdots & \ddots & \vdots \\ D_N H_{N1} E_1 & \dots & D_N H_{NN} E_N \end{pmatrix}$$

where $E \in \mathbb{F}_q[\underline{e}, \underline{d}]^{\{s \times \ell\}}$ and $D \in \mathbb{F}_q[\underline{e}, \underline{d}]^{\{\ell \times t\}}$.

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Optimization problem. Find matrices E, D such that

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Optimization problem. Find matrices E, D such that

- $D_i H_{ij} E_j = 0$

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- $D_i H_{ij} E_j = 0 \quad \longrightarrow \quad$ homogeneous bilinear system.

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- $D_i H_{ij} E_j = 0 \quad \longrightarrow \quad$ homogeneous bilinear system.
- rank $D_i H_{ii} E_i$ is maximal.

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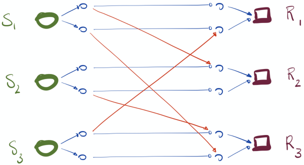
Some preliminary results

Lemma

If $\mathcal{N}$ is ρ-linearly achievable then
$\rho_i \leq \text{rank } H_{ii} = r_i$.



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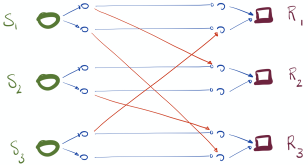

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Lemma

Let $\mathcal{N}$ be such that $t_i = s_i = r_i = \ell_i$ for all $1 \leq i \leq N$. Then $\mathcal{N}$ is ℓ-linearly achievable for $\ell = (\ell_1, \ldots, \ell_N)$ only if D, $H$and E are fullrank.

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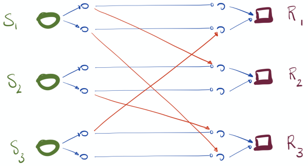

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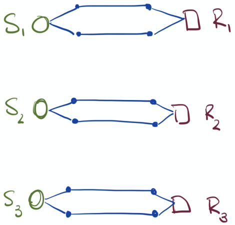
Theorem

Let $\mathcal{N}$ be such that $t_i = s_i = r_i = \ell_i$ for all $1 \leq i \leq N$. If $\mathcal{N}$ has interference then $\mathcal{N}$ is not linearly achievable for ℓ (for all finite fields).

Lower achievability bound



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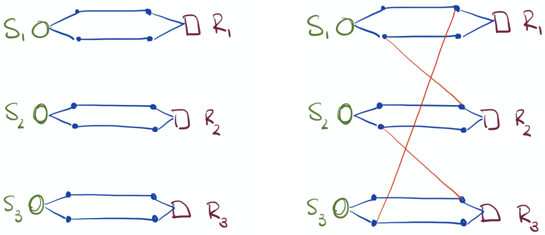
Lower achievability bound



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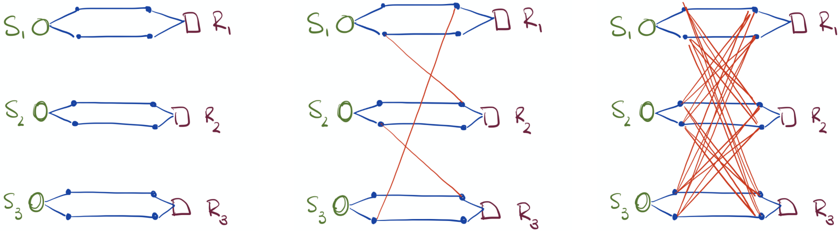

Sufficient condition for solvability

$$H = \begin{pmatrix} H_{11} & H_{12} & \dots & H_{1N} \\ H_{21} & H_{22} & \dots & H_{2N} \\ \vdots & & & \vdots \\ H_{N1} & H_{N2} & \dots & H_{NN} \end{pmatrix}$$

Definition

We call rank of interference the value $o_i := rank(H_{ij} \mid j \neq i)$.

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Definition

We call rank of interference the value $o_i := rank(H_{ij} \mid j \neq i)$.

$$o_1 = 1, \; o_2 = 1, \; o_3 = 2$$

Theorem

A network $\mathcal{N}$ is $(r_1 - o_1, \dots, r_N - o_N)$-linearly achievable and matrices E and D can be computed using Gaussian elimination 2N times.

$\mathcal{N}$ is $(1, 1, 0)$-linearly achievable.

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Proof of the lower bound

Denote by $\ell\ker(H_{ij} \mid j \neq i)$ be the left kernel of $(H_{ij} \mid j \neq i)$.

- $D_i H_{ij} = 0$ if and only if the rows of D_i are contained in $\ell\ker\left(H_{ij} \mid j \neq i\right)$.

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- $\dim \ell\ker\left(H_{ij} \mid j \neq i\right)$ is $t_i - o_i$.

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- Without loss of generality H_{ii} is a partial identity of rank r_i.
- $\operatorname{rank} D_i H_{ii} \geq (t_i - o_i) - (t_i - r_i) = r_i - o_i$.
- Let E_i be any invertible matrix, then $\operatorname{rank} D_i H_{ii} E_i \geq r_i - o_i$.

Encoding-Dependent Linear Achievability

Theorem

A network $\mathcal{N}$ is $\rho = (\rho_1, \ldots, \rho_n)$-linearly achievable if and only if there exist $E_1, \ldots, E_n$ such that for all $i \in [N]$ if holds that

$$\rho_i \leq \dim \,{}^{\ell}\mathrm{ker}(H_{ij}E_j \mid j \neq i) / {}^{\ell}\mathrm{ker}(H_{ij}E_j \mid \forall\, j).$$

Moreover the bound is always tight.

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- Let $v_{i,1}, \ldots, v_{i,\ell_i}$ be the rows of D_i, then

$$\begin{aligned}\operatorname{rank} D_i H_{ii} E_i &= \dim\langle v_{i,1} H_{ii} E_i, \ldots, v_{i,\ell_i} H_{ii} E_i \rangle \\ &\leq \dim {}^{\ell\ker(H_{ij}E_j \mid j \neq i)}/_{\ell\ker(H_{ij}E_j \mid \forall j)}\end{aligned}$$

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- $\operatorname{rank} D_i H_{ii} E_i = \dim V_i$ iff $\{[v_{i,1}], \ldots, [v_{i,\ell_i}]\}$ spans V_i.

Example

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

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$$H_{31}E_1 = 0 \qquad H_{32}E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad H_{33}E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\ell\ker H_{31}E_1 = \mathbb{F}_q^2 \qquad \ell\ker H_{32}E_2 = \langle (0,1) \rangle \qquad \ell\ker H_{33}E_2 = \langle (1,0) \rangle$$

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$$D_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Conclusions

Future projects

- Find Lin($\mathcal{N}$) for all multiple unicast networks.
- Find good algorithmic methods to solve the optimization problem.
- Prove that linearity is actually optimal for the explained multiple unicast networks.
- Generalize these results to other interference networks

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Thank you.

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