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An Algebraic Approach for the NCE Principle with Massive Subpopulations

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Abstract— **We study large population stochastic dynamic games where each agent receives influences from multi-classes of agents according to intra- and inter-subpopulation cost coupling. The NCE principle developed in our previous works gave decentralized asymptotic Nash strategies; however, its solubility depends on a conservative fixed point analysis which does not lead to easy computation of the solution. In this paper we apply a different algebraic approach via a state space augmentation, and it is convenient for practical computation involving first a set of algebraic Riccati equations subject to consistency constraints and next a set of ordinary differential equations.**

I. Introduction

Noncooperative games with mean field coupling have broad backgrounds in economics, social science, biology and engineering, among others [18], [6], [7], [19] [2], [5], [10]. In stochastic dynamic models, the so-called Nash Certainty Equivalence (NCE) methodology [10], [15], [16], [11], [12] provides a conceptually simple approach for decentralized individual strategy design via decomposing a dynamic game of very high complexity into a family of much simpler localized optimal control problems, and this is achieved by identifying an individual-mass interaction consistency relationship. Closely related approaches are also developed by Lasry and Lions for stochastic differential games [20], [21] , and Weintraub, Benkard, and Van Roy for game models of many firm industry dynamics [24], [25].

In the work [13], [14], the NCE methodology has been generalized to models with interaction locality as motivated by social and economic interactions where the decay of the mutual influence of agents depends on how they are physically or socially distanced from each other. For instance, games considering the effect of local interactions arise in retailing service models [4] and social segregation phenomena [23]; a common feature of these works is their investigation of the relationship between microscopic local behavior of individual agents and the resulting macroscopic phenomena (also see, e.g., [8], [22], [3]).

In the setting of the NCE methodology with interaction locality, a model of particular interest considers a population consisting of several massive subpopulations so that the agents mutually interact according to intra- and inter-subpopulation coupling in their costs. For this class of models, the NCE approach leads to a set of coupled ordinary differential equations where some components (i.e., the mean trajectories of finite classes of agents) have given initial conditions while the initial conditions of some other components are only implicitly determined via the growth condition of the solution (see [13], [14]). A practical difficulty associated with the NCE equation system there is the computation of its solution, either analytically or numerically.

In this paper, we apply a new approach which will significantly facilitate the related computation. This approach by a state space augmentation was initially developed in mean field LQG games involving a major player [9] to handle the difficulty that the mean field of all the minor agents is inherently random and that there is a lack of aggregate quantities to capture its evolution. It turns out that this approach may also be used to circumvent the computational difficulty encountered by the NCE equation system obtained in [13], and the basic idea of this approach resembles the classic internal model principle developed by Wonham [26] in that by incorporating into an enlarged system the dynamics of the exogenous signals, which, in the current setup, amount to the mean field effect, one may apply standard tools in optimal control to determine the strategies of individual self-interest seeking agents. The only remaining issue is the determination of the dynamics of the mean field and this, in turn, is addressed by imposing appropriate interaction consistency conditions, following the spirit of the Nash certainty equivalence theory developed in our previous works.

The organization of the paper is as follows. The large-population LQG game with intra- and inter- subpopulation cost coupling is stated in Section II, and the existing NCE approach for decentralized strategy design is reviewed. For developing a computationally efficient scheme, the state space augmentation method is described in Section III, and the consistency conditions are specified in Section IV, leading to a new set of NCE equation system. Section V presents the asymptotic Nash equilibrium results. Section VI gives a computational example to show the effectiveness of the state space augmentation based approach. Section VII concludes the paper.

II. The Stochastic Dynamic Game Model

In a population of N agents, let the dynamics of the individual agents be given by

$$dz_i(t) = [Az_i(t) + Bu_i(t)]dt + DdW_i(t), \quad 1 \le i \le N, \quad t \ge 0, \tag{1}$$

where $z_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ is the control input, and $\{W_i, 1 \le i \le N\}$ denotes N independent n dimensional standard Wiener

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processes. The matrices A, B and D have compatible dimensions. The initial states $\{z_i(0), 1 \le i \le N\}$ are mutually independent and also independent of $\{W_i, 1 \le i \le N\}$. In addition, $E|z_i(0)|^2 < \infty$.

In specifying the cost coupling we assign the agents a "locality" (or "spatial") parameter α. Note that this locality parameter may have different interpretations and is not necessarily restricted to be a physical location, and may hold in a social interaction context [1]. The locality parameter for agent i will be denoted by p_i.

Let the cost for the ith agent be given by

$$J_i = E\int_0^\infty e^{-\rho t}\left\{[z_i-\Phi_i]^T Q[z_i-\Phi_i]+u_i^T R u_i\right\}dt, \quad (2)$$

where $\Phi_i = \gamma(\sum_{j=1}^N \omega_{p_i p_j}^{(N)} z_j + \eta)$, $\eta \in \mathbb{R}^n$, and $\rho > 0$, $\gamma > 0$. The two matrices $Q \ge 0$, $R > 0$. The set of weight coefficients $\omega_{p_i p_j}^{(N)}$ satisfies the condition

$$\omega_{p_i p_j}^{(N)} \ge 0, \quad \forall i,j, \qquad \sum_{j=1}^N \omega_{p_i p_j}^{(N)} = 1, \quad \forall i. \qquad (3)$$

For each fixed i, it is seen from (3) that the total weight of unit is allocated to all the N agents.

A. Interactions with Intra- and Inter-subpopulation Cost Coupling

In below we adapt the general cost structure (2)-(3) to the specific situation of modelling the interaction of agents from K groups or subpopulations within the overall population.

The locality parameter p_i indicates which subpopulation the ith agent belongs to, and the cost interaction for a pair of agents is determined by either the intra-subpopulation or the inter-subpopulation coupling parameter, whichever applicable. Denote $\Lambda \triangleq \{1,\dots,K\}$ for the subpopulation indices and suppose each p_i, $1 \le i < \infty$, takes a value from Λ. The coupling weight assignment will be constructed by using the $K \times K$ matrix

$$\omega_\Lambda = (\omega_{ij})_{K\times K} \qquad (4)$$

which satisfies $\omega_{ij} \ge 0$ and $\sum_{j=1}^K \omega_{ij} = 1$ for each i. Let $\sum_{i=1}^N 1_{(p_i=k)} = N_k$ for $k \in \Lambda$. If $p_i = k$ and $p_{i'} = k'$, define $\omega_{p_i p_{i'}}^{(N)} = \omega_{kk'}/N_{k'}$, which ensures the unit total weight condition $\sum_{j=1}^N \omega_{p_i p_j}^{(N)} = 1$.

B. Review of the NCE Equation System

Now we give a brief review of the approach in [13], [14]. Denote the algebraic Riccati equation

$$\rho\Pi = A^T\Pi + \Pi A - \Pi B R^{-1} B^T \Pi + Q. \qquad (5)$$

If (5) has a solution $\Pi > 0$, define

$$A_1 = A - BR^{-1}B^T\Pi, \qquad A_2 = A - BR^{-1}B^T\Pi - \rho I. \quad (6)$$

(**A1**) The pair $[A,B]$ is controllable and the pair $[Q^{1/2}, A]$ is observable. In addition, A_1 defined by (6) is Hurwitz (i.e., all its eigenvalues have negative real parts). □

(**A2**) The sequence $\{p_i, 1 \ge 1\}$ has the limiting empirical distribution $\lim_{N\to\infty}(1/N)\sum_{i=1}^N 1_{(p_i=k)} = \pi_k > 0$, $k \in \Lambda$. □

The K components in $(\pi_1,\dots,\pi_K)$ shows the relative frequency of the agents in each of the K subpopulations.

To simplify the analysis, we assume all agents have the same initial mean, which is further assumed to be zero. Now we write the NCE equation system in the form:

$$\rho s_\kappa = \frac{ds_\kappa}{dt} + A^T s_\kappa - \Pi B R^{-1} B^T s_\kappa - R_\kappa, \qquad (7)$$

$$\frac{d\bar z_\kappa}{dt} = (A - BR^{-1}B^T\Pi)\bar z_\kappa - BR^{-1}B^T s_\kappa, \qquad (8)$$

$$\bar r_\kappa(t) = \sum_{\kappa'\in\Lambda} \omega_{\kappa\kappa'}\bar z_{\kappa'}(t), \qquad (9)$$

$$R_\kappa = \gamma(\bar r_\kappa + \eta), \qquad (10)$$

where $\kappa \in \Lambda$, $\bar z_\kappa(0) = 0$ and $s_\kappa(t)$ is restricted to be a bounded function on $[0,\infty)$ without the necessity of separately specifying an initial condition $s_\kappa(0)$. The NCE equation system is derived based on the following idea: The agent with $p_i = \kappa$ optimally tracks R_κ (i.e., replace Φ_i by R_κ in (2)), and the K classes of agents' closed-loop produces the mean trajectories $\bar z_\kappa$, which are further used to determine R_κ. Due to the particular nature of the initial conditions, we will refer to (7)-(10) as the NCE equation system with mixed initial conditions. Let $C_b([0,\infty),\mathbb{R}^n)$ denote the set of n dimensional continuous and bounded functions on $[0,\infty)$. Denote

$$(\Gamma_l\zeta)(t) = \gamma\int_0^t e^{A_1(t-s)}BR^{-1}B^T\int_s^\infty e^{A_2^T(\tau-s)}\zeta(\tau)d\tau ds. \qquad (11)$$

Under (**A1**) it may be shown that Γ_l is a bounded linear operator from $C_b([0,\infty),\mathbb{R}^n)$ to $C_b([0,\infty),\mathbb{R}^n)$. Denote by $\|\Gamma_l\|$ the norm of the operator Γ_l. By adapting the result in [13], [14], we obtain the following theorem.

Theorem 1: Assume (**A1**) holds. If $\|\Gamma_l\| < 1$, the equation system (7)-(10) has a unique bounded solution $(s_\kappa(\cdot), \bar z_\kappa(\cdot), \bar r_\kappa(\cdot))$, $\kappa \in \Lambda$. □

Provided that (7)-(10) has a solution, we may construct the decentralized control laws for the N agents:

$$\hat u_i = -R^{-1}B^T(\Pi z_i + s_{p_i}), \qquad 1 \le i \le N,$$

where s_{p_i} is given by (7)-(10) via the substitution $\kappa = p_i$ in s_κ. Under mild conditions, we may further establish an ε-Nash equilibrium property for this set of control laws.

III. State Space Augmentation

For the approach described in Section II, the verification of the gain condition $\|\Gamma_l\| < 1$ and the computation of the solution for the NCE equation system are quite challenging when multiple classes of agents are involved. To overcome this difficulty, here we develop a different approach.

To simplify the analysis, we assume all agents have zero initial mean, i.e., $Ez_i(0) = 0$. Suppose agent i is within the kth subpopulation. Let $\bar y_\kappa$ be the mean process of an agent in the κth subpopulation. Denote $\bar y = [\bar y_1^T,\dots,\bar y_K^T]^T$. We consider the following control problem for agent i interacting with an infinite population consisting of K classes of agents:

$$dz_i(t) = [Az_i(t) + Bu_i(t)]dt + DdW_i(t), \qquad (12)$$

$$d\bar y(t) = \overline{A}\bar y(t)dt + \overline{m}(t)dt, \qquad (13)$$

with the cost

$$\overline{J}_i = E\int_0^\infty e^{-\rho t}[z_i - \overline{\Phi}_k]^T Q[z_i - \overline{\Phi}_k] + u_i^T R u_i]dt, \qquad (14)$$

where $\overline{\Phi}_k = \gamma(\sum_{\kappa=1}^K \omega_{k\kappa}\overline{y}_\kappa + \eta)$. The initial condition of $\overline{y}$ is taken as $\overline{y}(0) = 0$ due to the zero initial mean assumption for the individual agents. Since agent i is in the kth subpopulation, the coupling term $\overline{\Phi}_k$ is associated with an index k. Equation (13) is used to model the aggregate effect of the K classes of agents. The constant coefficient $\overline{A}$ and the function $\overline{m}$ on $[0,\infty)$ will be determined later on by imposing a consistency relationship. We may write $\overline{m}$ by its components

$$\overline{m}(t) = \begin{bmatrix} \overline{m}_1(t) \\ \vdots \\ \overline{m}_K(t) \end{bmatrix},$$

where each $\overline{m}_k$ is n dimensional. The motivation for proposing the structure (13) is as follows. By starting with a large but finite population Nash game model, one can formally obtain a linear full state based feedback for the individual agents where each control u_i possesses constant coefficients for the individual states $(z_1,\ldots,z_N)$ together with a time varying offset term due to the constant reference term η in the cost (see (2)). After a heuristic limiting argument one may obtain (13) for approximating the aggregate effects, respectively, generated by the K classes of agents. A more detailed illustration of such a procedure is given in [9] in a model with a major player and a large number of minor players. However, we stress that one may be saved from a difficult rigorous derivation of (13) through full state based feedback and, instead, the feasibility of introducing (13) may be addressed by the interaction consistency argument developed in the subsequent analysis.

Let I_n be the $n\times n$ identity matrix. Denote

$$\mathbb{A} = \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}, \quad \mathbb{B} = \begin{bmatrix} B \\ 0_{nK\times m} \end{bmatrix}, \quad \mathbb{M} = \begin{bmatrix} 0_{n\times 1} \\ \overline{m} \end{bmatrix}, \qquad (15)$$

$$H_k = \gamma[\omega_{k1}I_n, \ \ldots, \ \omega_{kK}I_n], \quad 1\le k\le K, \qquad (16)$$

$$Q_k = [I_n, -H_k]^T Q[I_n, -H_k], \qquad (17)$$

$$\overline{\eta}_k = [I_n, -H_k]^T Q\eta. \qquad (18)$$

In the following derivation, we assume the solvability of the LQG problem (12)-(14) by first presuming that $\overline{A}$ and $\overline{m}$ have been given and the related conditions for these calculations will be formalized later.

Define the function class: $C_{\rho/2}([0,\infty),\mathbb{R}^l) = \{f | f \in C([0,\infty),\mathbb{R}^l)$ and there exists $\rho' < \rho$ such that $\sup_{t\ge 0}|f(t)|e^{-(\rho'/2)t} < \infty\}$. The constant ρ' is allowed to change with f.

We introduce the algebraic Riccati equation

$$\rho P_k = P_k\mathbb{A} + \mathbb{A}^T P_k - P_k\mathbb{B}R^{-1}\mathbb{B}^T P_k + Q_k, \qquad (19)$$

and the ordinary differential equation

$$\rho s_k = \frac{ds_k}{dt} + (\mathbb{A}^T - P\mathbb{B}R^{-1}\mathbb{B}^T)s_k + P_k\mathbb{M} - \overline{\eta}_k, \qquad (20)$$

where the initial condition $s_k(0)$ is implicitly determined by the growth condition $s_k \in C_{\rho/2}([0,\infty),\mathbb{R}^{n(K+1)})$.

The optimal control law for the LQG problem (12)-(14) is given as

$$\hat{u}_i = -R^{-1}\mathbb{B}^T[P_k(z_i^T,\overline{y}^T)^T + s_k]. \qquad (21)$$

After substituting (21) into (12), the closed-loop dynamics for agent i takes the form

$$dz_i = \{Az_i - BR^{-1}\mathbb{B}^T[P_k(z_i^T,\overline{y}^T)^T + s_k]\}dt + DdW_i, \qquad (22)$$

where $\overline{y}$ is determined by (13).

IV. The New NCE Equation System

A. The Consistency Condition

We introduce the following matrix partition

$$P_k = \begin{bmatrix} P_{k,11} & P_{k,12} \\ P_{k,21} & P_{k,22} \end{bmatrix},$$

where $P_{k,11}$ and $P_{k,22}$ are, respectively, $n\times n$ and $nK\times nK$ submatrices.

Now from (22) we obtain

$$\begin{aligned} dz_i =& [(A - BR^{-1}B^T P_{k,11})z_i - BR^{-1}B^T P_{k,12}\overline{y}]dt \\ & - BR^{-1}\mathbb{B}^T s_k dt + DdW_i. \end{aligned} \qquad (23)$$

After averaging with (23) for all agents within the kth subpopulation, we obtain

$$\begin{aligned} d\overline{y}_k =& [(A - BR^{-1}B^T P_{k,11})\overline{y}_k - BR^{-1}B^T P_{k,12}\overline{y}]dt \\ & - BR^{-1}\mathbb{B}^T s_k dt, \quad 1\le k\le K. \end{aligned} \qquad (24)$$

Notice that in obtaining (24) the diffusion term in (23) has been averaged out. In fact, we may also interpret $\overline{y}_k$ as the mean process of z_i with $p_i = k$.

Now we impose the fundamental *consistency condition* that the equation system determined by (24) be the same as (13). Define

$$\mathbf{e}_k = [0_{n\times n},\ldots,I_n,\ldots,0_{n\times n}] \in \mathbb{R}^{n\times nK},$$

where only the kth block is nonzero, being equal to an identity matrix. We further write (24) in the form

$$\begin{aligned} d\overline{y}_k =& [(A - BR^{-1}B^T P_{k,11})\mathbf{e}_k - BR^{-1}B^T P_{k,12}]\overline{y}dt \\ & - BR^{-1}\mathbb{B}^T s_k dt. \end{aligned} \qquad (25)$$

Denote

$$\overline{A} = \begin{bmatrix} \overline{A}_1 \\ \vdots \\ \overline{A}_K \end{bmatrix}, \qquad (26)$$

where each $\overline{A}_k$ contains n rows. Now the consistency condition translates into

$$\overline{A}_k = (A - BR^{-1}B^T P_{k,11})\mathbf{e}_k - BR^{-1}B^T P_{k,12}, \qquad (27)$$

$$\overline{m}_k = -BR^{-1}\mathbb{B}^T s_k, \qquad (28)$$

for $1\le k\le K$.

By combining (19) with (27), we introduce the following algebraic equation system:

$$\begin{cases} \rho P_k = P_k\mathbb{A} + \mathbb{A}^T P_k - P_k\mathbb{B}R^{-1}\mathbb{B}^T P_k + Q_k, \\ \overline{A}_k = (A - BR^{-1}B^T P_{k,11})\mathbf{e}_k - BR^{-1}B^T P_{k,12}, \\ k = 1,\dots,K, \end{cases} \tag{29}$$

which will be called the consistency constrained algebraic Riccati equation system.

By combining (20) with (28), we introduce the following ordinary differential equation system:

$$\begin{cases} \rho s_k = \frac{ds_k}{dt} + (\mathbb{A}^T - P\mathbb{B}R^{-1}\mathbb{B}^T)s_k + P_k\mathbb{M} - \overline{\eta}_k, \\ \overline{m}_k(t) = -BR^{-1}\mathbb{B}^T s_k, \\ k = 1,\dots,K, \end{cases} \tag{30}$$

which will be called the consistency constrained ordinary differential equation system.

We will call (29) and (30) combined together the NCE equation system.

Denote

$$M_1 = \begin{bmatrix} A - BR^{-1}B^T P_{1,11} & & \\ & \ddots & \\ & & A - BR^{-1}B^T P_{K,11} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} BR^{-1}B^T P_{1,12} \\ \vdots \\ BR^{-1}B^T P_{K,12} \end{bmatrix}.$$

Then we may write the condition (27) in the compact form:

$$\overline{A} = M_1 - M_2. \tag{31}$$

When the condition (28) is imposed, (20) may be rewritten by expressing $\mathbb{M}$ in terms of $(s_1,\dots,s_K)$. This leads to a linear ordinary differential equation for $(s_1,\dots,s_K)$ as shown below. Let $\widehat{\mathbb{A}}_k = \mathbb{A} - \mathbb{B}R^{-1}\mathbb{B}^T P_k - \rho I$. Denote

$$\Gamma_1 = -\begin{bmatrix} \widehat{\mathbb{A}}_1 & & \\ & \ddots & \\ & & \widehat{\mathbb{A}}_K \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} BR^{-1}\mathbb{B}^T & & \\ & \ddots & \\ & & BR^{-1}\mathbb{B}^T \end{bmatrix}.$$

Denote $P_k = [P_{k,1}, P_{k,2}]$, where $P_{k,1}$ is the first n columns of P_k, and

$$\Gamma_2 = \begin{bmatrix} P_{1,2}\Lambda \\ \vdots \\ P_{K,2}\Lambda \end{bmatrix}.$$

Denote $\Gamma = \Gamma_1 + \Gamma_2$ and

$$s_* = \begin{bmatrix} s_1 \\ \vdots \\ s_K \end{bmatrix}, \quad \eta_* = \begin{bmatrix} \overline{\eta}_1 \\ \vdots \\ \overline{\eta}_K \end{bmatrix}.$$

After some elementary matrix calculation, we may show that (30) is equivalent to the equation

$$\frac{ds_*}{dt} = \Gamma s_* + \eta_*, \tag{32}$$

where the initial condition $s_*(0)$ is to be determined. So now the consistency condition (28) has been incorporated into (32).

B. Consistent Solutions

Definition 2: The set of constant matrices $(\overline{A}, P_\kappa, \kappa = 1,\dots,K)$ is said to be a consistent solution to (29) if

$$\begin{aligned} &P_\kappa \geq 0, \ \forall \kappa, \\ &\mathbb{A} - \mathbb{B}R^{-1}\mathbb{B}^T P_\kappa - (\rho/2)I \text{ is Hurwitz}, \ \forall \kappa, \end{aligned} \tag{33}$$

and (29) is satisfied. If, furthermore,

$$\mathbb{A} - \mathbb{B}R^{-1}\mathbb{B}^T P_\kappa \text{ is Hurwitz}, \ \forall \kappa, \tag{34}$$

we say $(\overline{A}, P_\kappa, \kappa = 1,\dots,K)$ is a stabilizing consistent solution to (29). □

Definition 3: Suppose $(\overline{A}, P_\kappa, \kappa = 1,\dots,K)$ is a consistent solution to (29), and the matrices $(P_\kappa, \kappa = 1,\dots,K)$ are further used to define the equation system (30). The set of $2K$ vector functions $(s_\kappa, \overline{m}_\kappa, \kappa = 1,\dots,K)$ is called a consistent solution to (30) if the two conditions hold:

(i) both s_κ and $\overline{m}_\kappa$ belong to the class $C_{\rho/2}([0,\infty), \mathbb{R}^{n(K+1)})$ for each κ;

(ii) (30) is satisfied. □

Definition 4: If $(\overline{A}, P_\kappa, \kappa = 1,\dots,K)$ and $(s_\kappa, \overline{m}_\kappa, \kappa = 1,\dots,K)$ are, respectively, a consistent solution to (29) and (30), we call $(\overline{A}, P_\kappa, s_\kappa, \overline{m}_\kappa, \kappa = 1,\dots,K)$ a solution to the NCE equation system (29)-(30). □

Theorem 5: Suppose (29) admits a consistent solution. Then we have:

(i) The equation system (30) always has a consistent solution.

(ii) If the real parts of all eigenvalues of Γ are at least $\rho/2$, (32) has a unique solution s_* in the class $C_{\rho/2}([0,\infty), \mathbb{R}^{nK(K+1)})$, which in fact is bounded.

Proof: (i) Since (30) is equivalent to (32), it suffices to find a solution in $C_{\rho/2}([0,\infty), \mathbb{R}^{nK(K+1)})$ for (32). If necessary, we may apply a nonsingular linear transformation to change Γ into two diagonal blocks Γ_a and Γ_b, where all eigenvalues of Γ_a have real parts less than $\rho/2$, and all eigenvalues of Γ_b have real parts at least equal to $\rho/2$. Without loss of generality, we may assume

$$\frac{ds_*}{dt} = \begin{bmatrix} \Gamma_a & \\ & \Gamma_b \end{bmatrix} s_* + \eta_*. \tag{35}$$

Let the components of s_* (resp., η_*) associated with Γ_a and Γ_b be denoted, respectively, by s_a and s_b (resp., η_a and η_b). Then we may take any initial condition $s_a(0)$ for s_a. Using the relation

$$s_b(t) = e^{\Gamma_b t}s_b(0) + \int_0^t e^{\Gamma_b(t-\tau)}\eta_b d\tau,$$

it may be further shown that

$$s_b(0) = -\int_0^\infty e^{-\Gamma_b \tau}\eta_b d\tau \tag{36}$$

is the only initial condition for s_b to generate a solution for s_* in $C_{\rho/2}([0,\infty),\mathbb{R}^{nK(K+1)})$; in fact the resulting solution

$$s_b(t) = \int_t^\infty e^{(t-\tau)\Gamma_b}\eta_b d\tau$$

is bounded, further implying the required growth rate of s_*. Let the dimension of s_b be d_b. For any other initial condition, we may show that the corresponding solution s_b will not be in the class $C_{\rho/2}([0,\infty),\mathbb{R}^{d_b})$. Finally, a consistent solution to (30) may be obtained from s_*.

(ii) This part follows from the proof of (i). □

By Theorem 5 we see the remarkable fact that the solubility of the state space augmentation based NCE equation system is completely reduced to the analysis of the algebraic Riccati equation system with constrains, and no solvability of coupled ODE systems with mixed initial conditions will be involved as in [13], [14].

From the point of view of numerical solutions, the state space augmentation based approach also exhibits its advantage. Once a numerical solution to (29) can be found, one may further perform eigenvalue check for the matrix Γ in (32), and find a solution s_* from the class $C_{\rho/2}([0,\infty),\mathbb{R}^{nK(K+1)})$. Thus, the calculation of the individual strategies may be performed solely by an algebraic approach which has low computational complexity.

V. The Equilibrium Analysis

Within the context of a population of N agents, for any $1 \le j \le N$, the jth agent's admissible control set $\mathscr{U}_j$ consists of all feedback controls u_j adapted to the σ-algebra $\sigma(z_i(\tau), \tau \le t, 1 \le i \le N)$ (i.e., $u_j(t)$ is a function of $(t, z_1(t), \ldots, z_N(t))$) such that a unique strong solution to the closed-loop system of the N agents exists on $[0,\infty)$. Note that $\mathscr{U}_j$ itself is not restricted to be decentralized. Denote $u_{-i} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N)$. To indicate the dependence of J_i on u_i and u_{-i}, we write it in the form $J_i(u_i, u_{-i})$.

Definition 6: A set of controls $u_j \in \mathscr{U}_j, 1 \le j \le N$, for N players is called an *ε-Nash equilibrium* with respect to the costs $J_j, 1 \le j \le N$, where $\varepsilon \ge 0$, if for any $1 \le i \le N$,

$$J_i(u_i, u_{-i}) \le J_i(u_i', u_{-i}) + \varepsilon,$$

when any alternative $u_i' \in \mathscr{U}_i$ is applied by the ith player. □

Theorem 7: Assume (i) (**A2**) holds, $Ez_i(0) = 0$ for all i, and $\sup_{i\ge 1} E|z_i(0)|^2 < \infty$; (ii) $(\overline{A}, P_k, s_k, \overline{m}_k, k = 1, \ldots, K)$ is a solution to the NCE equation system (29)-(30); (iii) each pair $[Q_k^{1/2}, \mathbb{A} - (\rho/2)I]$, $k \in \Lambda$, is detectable for $\mathbb{A}$ and Q_k defined by (15)-(17). Then for the N agents, the set of control laws

$$u_i = -R^{-1}\mathbb{B}^T[P_{p_i}(z_i^T, \overline{y}^T)^T + s_{p_i}], \qquad 1 \le i \le N \tag{37}$$

is an ε-Nash equilibrium, where $\varepsilon \to 0$ as $N \to \infty$. Here the matrix P_{p_i} and function s_{p_i} are obtained from (29)-(30) via the substitution of $k = p_i$ in P_k and s_k, and $\overline{y}$ is given by (13).

Proof: (Sketch) The proof uses a typical mean field approximation argument as in [11], [15], [9]. The basic steps are as follows. Consider agent i_0 with $p_{i_0} = k$. Suppose all other agents' strategies are given by (37) and u_{i_0} is replaced by u_{i_0}' attempting for improving J_{i_0}. For sufficiently large N, $\sum_{j=1}^N \omega_{p_{i_0}p_j}^{(N)} z_j$ may be tightly approximated by $\sum_{\kappa\in\Lambda}\omega_{k\kappa}\overline{y}_\kappa$. Next, the optimization of J_{i_0} may be approximated by an optimal tracking problem with respect to a deterministic reference trajectory $\overline{\Phi}_k$ as in $\overline{J}_i$ given by (14). Hence, the cost for agent i_0 cannot be considerably reduced and the ε-Nash equilibrium result may be obtained. □

In (37), since $\overline{y}$ may be calculated off-line as a deterministic function of t, u_i is in the admissible control set $\mathscr{U}_i$.

VI. Numerical Computation

We consider a model with two subpopulations. The coefficients in the dynamics (1) are given by

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad D = I_2. \tag{38}$$

The parameters in the cost (2) are given by

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \; \eta = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \; \rho = 2, \; R = 1, \; \gamma = 0.4, \tag{39}$$

and the cost coupling parameters are given by the matrix

$$\omega_\Lambda = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix},$$

where ω_Λ has been specified in (4). Since there are two subpopulations, the index set $\Lambda = \{1,2\}$.

We apply the following algorithm:

Step 1. Take $\overline{A} = 0$ for $\mathbb{A}$ in (19) to obtain two matrix solutions P_1 and P_2.

Step 2. Update $\overline{A}$ by (27). Go back to Step 1.

The above iterates are terminated when a prescribed precision has been reached in solving the consistency constrained Riccati equation system (29). In general, the initial choice for $\overline{A}$ should be carefully done so that the standard conditions for the solvability of the algebraic Riccati equation are ensured, and as long as such solvability conditions hold during each step, the iteration may be repeated. The convergence proof of the above procedure is an interesting issue, but is beyond the scope of the present work. Instead, we only examine the convergence behavior numerically.

For solving (29), we run the above algorithm for 20 iterates to obtain P_1, P_2 and $\overline{A}$ as displayed by (40)-(42). It can be checked that each equation in (29) holds with an error below 10^{-14}. The eigenvalues of Γ are given as

$$\begin{aligned}
&2.57230253242457 \pm 0.48070909418077i\\
&2.68103785278419 \pm 0.38081117340856i\\
&2.57100702022159 \pm 0.22850065102118i\\
&2.57717691998667 \pm 0.24343703885405i\\
&2.68367413806908 \pm 0.28640416034497i\\
&2.68305932883450 \pm 0.28643422379263i,
\end{aligned}$$

where i is the imaginary unit. Thus $-\Gamma + (\rho/2)I$ is Hurwitz and hence the condition in Theorem 5-(ii) is satisfied.

$$P_1 = \begin{bmatrix} 1.28768066344495 & 0.36396239307458 & -0.15515799172644 & 0.00250859772040 & -0.10850057039275 & -0.00067801115477 \\ 0.36396239307458 & 0.49502915405475 & 0.00250859772040 & -0.06571491538809 & -0.00067801115477 & -0.04491525210407 \\ -0.15515799172644 & 0.00250859772040 & 0.03236235594210 & -0.00286774478030 & 0.02230361429789 & -0.00158711992106 \\ 0.00250859772040 & -0.06571491538809 & -0.00286774478030 & 0.01470525882948 & -0.00158711992106 & 0.00994983522524 \\ -0.10850057039275 & -0.00067801115477 & 0.02230361429789 & -0.00158711992106 & 0.01541359759328 & -0.00081378950036 \\ -0.00067801115477 & -0.04491525210407 & -0.00158711992106 & 0.00994983522524 & -0.00081378950036 & 0.00674407673699 \end{bmatrix} \quad (40)$$

$$P_2 = \begin{bmatrix} 1.28768066344495 & 0.36396239307458 & -0.05425028519638 & -0.00033900557739 & -0.20940827692282 & 0.00216959214301 \\ 0.36396239307458 & 0.49502915405476 & -0.00033900557739 & -0.02245762605204 & 0.00216959214301 & -0.08817254144012 \\ -0.05425028519638 & -0.00033900557739 & 0.00385339939832 & -0.00020344737509 & 0.01500520654726 & -0.00099700733562 \\ -0.00033900557739 & -0.02245762605204 & -0.00020344737509 & 0.00168601918425 & -0.00099700733562 & 0.00666093679687 \\ -0.20940827692282 & 0.00216959214301 & 0.01500520654726 & -0.00099700733562 & 0.05851936963831 & -0.00465831207645 \\ 0.00216959214301 & -0.08817254144012 & -0.00099700733562 & 0.00666093679687 & -0.00465831207645 & 0.02634111323897 \end{bmatrix} \quad (41)$$

$$\overline{A} = \begin{bmatrix} -0.49899366251348 & 0.20421477053836 & 0.10917858154753 & 0.04559326325885 \\ -0.49899366251348 & -0.79578522946164 & 0.10917858154753 & 0.04559326325885 \\ 0.05458929077376 & 0.02279663162942 & -0.44440437173972 & 0.22701140216778 \\ 0.05458929077376 & 0.02279663162942 & -0.44440437173972 & -0.77298859783222 \end{bmatrix}. \quad (42)$$

VII. Conclusion

In this paper we consider a class of LQG games with massive subpopulations where the agents interact with each other according to intra- and inter-subpopulation cost coupling. To obtain computationally efficient decentralized control design, a state space augmentation approach is applied and our previous NCE methodology is extended to this augmented model. The advantage of this approach is that the computation of the solution reduces to solving some algebraic equations, and also, the existence of a solution to the associated NCE equation system reduces to the solvability of purely algebraic equations. A numerical example is used to illustrate the effectiveness of this approach. For future work, it would be of interest to establish an equivalence relationship of the solution determined by the previous approach [13], [14] and the solution based on the state space augmentation approach.

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References

[1] G. A. Akerlof. Social distance and social decisions. *Econometrica*, vol. 65, no. 5, pp. 1005-1027, Sept. 1997.

[2] C. T. Bauch and D. J. D. Earn. Vaccination and the theory of games. *Proc. Natl. Acad. Sci., U.S.A.*, vol. 101, pp. 13391-13394, Sept. 2004.

[3] A. Bisin, U. Horst, and O. Özgür. Rational expectations equilibria of economies with local interactions. *J. Econ. Theory*, vol. 127, pp. 74-116, 2006.

[4] L. E. Blume. The statistical mechanics of strategic interaction. *Games and Economic Behavior*, vol. 5, pp. 387-424, 1993.

[5] R. Breban, R. Vardavas, and S. Blower. Mean-field analysis of an inductive reasoning game: application to influenza vaccination. *Physical Review E*, vol. 76, 2007. DOI: 10.1103/PhysRevE.76.031127.

[6] G. M. Erickson. Differential game models of advertsing competition. *Europ. J. Oper. Res.*, vol. 83, pp. 431-438, 1995.

[7] E. J. Green. Continuum and finite-player noncooperative models of competition. *Econometrica*, vol. 52, no. 4, pp. 975-993, 1984.

[8] U. Horsta and J. A. Scheinkman. Equilibria in systems of social interactions. *J. Economic Theory*, vol. 130, no. 1, pp. 44-77, 2006.

[9] M. Huang. Mean field stochastic differential games involving a major player. Presented at *the Canadian Mathematical Society Winter Meeting*, Ottawa, ON, Abstract Volume pp. 105, Dec. 2008.

[10] M. Huang, P. E. Caines, and R. P. Malhamé. Individual and mass behaviour in large population stochastic wireless power control problems: centralized and Nash equilibrium solutions. *Proc. 42nd IEEE Conf. Decision Contr.*, Maui, HI, pp. 98-103, December 2003.

[11] M. Huang, P. E. Caines, and R. P. Malhamé. Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ε-Nash equilibria. *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1560-1571, 2007.

[12] M. Huang, P. E. Caines, and R. P. Malhamé. The Nash certainty equivalence principle and McKean-Vlasov systems: an invariance principle and entry adaptation. *Proc. 46th IEEE Conference on Decision and Control*, New Orleans, pp. 121-126, December 2007.

[13] M. Huang, P. E. Caines and, R. P. Malhamé. A locality generalization of the NCE (mean field) principle: agent specific cost interactions. *Proc. 47th IEEE CDC*, Cancun, Mexico, pp. 5539-5544, Dec. 2008.

[14] M. Huang, P. E. Caines, and R. P. Malhamé. The NCE (mean field) principle with locality dependent cost interactions. Submitted to *IEEE Trans. Auotmatic Control*, Jan. 2009.

[15] M. Huang, R. P. Malhamé, and P. E. Caines. Nash equilibria for large-population linear stochastic systems of weakly coupled agents. In *Analysis, Contr. Optim. Complex Dyn. Syst.*, E. K. Boukas and R. P. Malhamé eds., pp. 215-252, Springer, New York, 2005.

[16] M. Huang, R. P. Malhamé, and P. E. Caines. Nash certainty equivalence in large population stochastic dynamic games: connections with the physics of interacting particle systems. *Proc. 45th IEEE CDC Conf.*, San Diego, CA, pp. 4921-4926, Dec. 2006

[17] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Communications in Information and Systems*, vol. 6, no. 3, pp. 221-252, 2006.

[18] M. A. Khan and Y. Sun, Non-cooperative games with many players. In *Handbook of Game Theory with Economic Applications*, vol. 3, R. J. Aumann and S. Hart eds., North-Holland, 2002.

[19] V. E. Lambson. Self-enforcing collusion in large dynamic markets. *J. Econ. Theory*, vol. 34, pp. 282-291, 1984.

[20] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I – Le cas stationnaire, *C. R. Acad. Sci. Paris, Ser. I*, vol. 343, pp. 619-625, 2006.

[21] J.-M. Lasry and P.-L. Lions. Mean field games. *Japan. J. Math.*, no. 2, pp. 229-260, 2007.

[22] M. A. Nowak and K. Sigmund. Games on grids. In *The Geometry of Ecological Interactions: Simplifying Spatial Complexity*, U. Dieckmann, R. Law, and J. A. J. Metz eds., pp. 135-150, Cambridge University Press, 2000.

[23] T. C. Schelling. Dynamic models of segregation. *J. Math. Sociology*, vol. 1, pp. 143-186, 1971.

[24] G. Y. Weintraub, C. L. Benkard, and B. Van Roy. Oblivious equilibrium: a mean field approximation for large-scale dynamic games. In *Advances in Neural Information Processing Systems*, MIT Press, 2005.

[25] G. Y. Weintraub, C. L. Benkard, and B. Van Roy. Markov perfect industry dynamics with many firms. *Econometrica*, vol. 76, no. 6, pp. 1375-1411, Nov. 2008.

[26] W. M. Wonham. *Linear Multivariable Control: A Geometric Approach*, Springer-Verlag, New York, 1979.