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Linear-quadratic mean field games with a major player: Nash certainty equivalence versus master equations*

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In honor of Professor Tyrone Duncan on the occasion of his 80th birthday

Mean field games with a major player were introduced in [26] within a linear-quadratic (LQ) modeling framework. Due to the rich structure of major-minor player models, the past ten years have seen significant research efforts for different solution notions and analytical techniques. For LQ models, we address the relation between three solution frameworks: the Nash certainty equivalence (NCE) approach in [26], master equations, and asymptotic solvability, which have been developed starting with different ideas. We establish their equivalence relationships.

1. Introduction

Mean field game theory was first developed for a population of comparably small but possibly heterogeneous players; see an overview in [9]. A generalization of this theory is to consider a major player interacting with a large number of minor players as initially introduced in [26]. Historically, games with major and minor players have been studied in the literature, but usually for static cooperative decision models (see e.g. [24, 37]).

When a major player interacts with a large number of minor players, a crucial feature is that the mean field generated by the minor players is a random process even if the population size of the minor players tends to infinity. This necessitates the characterization of the dynamic property of the mean field when the players choose their strategies based on mean field approximations. This analytical complication is similar to what happens in common noise models [7, 11, 14].

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Following the major-minor player mean field game model introduced in [26], the same LQ framework has been greatly extended by different authors. The reader is referred to [39] for non-uniform minor players, [10, 18] for partial information, [31] for an application to optimal execution in finance, [25] for system dynamics via backward stochastic differential equations, [33] for random entrance of agents, [36] for multi-scale analysis and the notion of asymptotic solvability, and [19] for a convex analysis approach. Meanwhile, the study of major-minor players in nonlinear models can be found in [6, 7, 8, 15, 16, 40]. Nourian and Caines [40] treat the mean field as a random measure flow driven by the major player's Brownian motion and use stochastic Hamilton-Jacobi-Bellman (HJB) equations to solve the best responses. Bensoussan et al. [6] use stochastic adjoint equations to handle a pair of optimal control problems with a dominating player. Carmona and Zhu [16] apply a stochastic maximum principle under a conditional law of the minor player. Sen and Caines [41] consider partial information and control with nonlinear filtering. Lasry and Lions [35] introduce master equations for a nonlinear major-minor player model. Cardaliaguet et al. [12] analyze a convergence problem for a major-minor player model with general nonlinear cost integrands, and it is shown that the pair of master equations can be obtained as the limit of the HJB equations for the $N+1$ players as $N \to \infty$.

The analysis of leadership or Stackelberg equilibria can be found in [44] for a discrete time model, [4, 38] for LQ Stackelberg games, [5] for control delay, [29] for state feedback and time-consistency, [17] for mean field principal-agent problems, [21] for an application to optimal portfolio liquidation, and [32] for evolutionary inspection games under a major player's pressure.

For LQ mean field games, the current literature has provided several routes: The Nash certainty equivalence (NCE) approach, master equations, and asymptotic solvability. These solution methods start with different ideas although they all look for a certain Nash equilibrium with state feedback information.

The NCE approach was initially developed for mean field games with all comparably small players. One starts by considering an infinite population and fixing the mean field [27, 28]. Next a special stochastic optimal control problem for a representative agent is solved by finding its best response strategy with respect to the given mean field and subsequently letting the overall population implementing such strategies regenerate the same mean field, leading to a fixed point problem. The set of strategies obtained is an ε-Nash equilibrium for the finite population.

When extending the NCE approach in [28, 27] to the major player model, the method in [26] is to augment the state space by an extra state describing the mean field generated by different types of minor players. This Markovianizes the decision problems to be solved by the major player and a representative minor player. A key step is to assume a linear structure of the mean field dynamics and further find appropriate constraints on its parameters by imposing consistency conditions.

The master equation approach assigns the player in question with its own value function and next determines the optimization rule of all players. The mean field evolution is naturally determined by system dynamics under the chosen strategies.

The notion of asymptotic solvability attempts to understand the large population decision problem from a different point of view. It shares similarity to convergence problems in [13, 20, 34]. One can formally start to write the dynamic programming equations for a sequence of Nash games with population sizes tending to infinity. If for all sufficiently large populations, the game has a solution via the coupled Riccati equations and moreover, the Riccati equation solutions maintain certain boundedness properties, we say the sequence of games has asymptotic solvability. A basic question is how to characterize asymptotic solvability in terms of some low dimensional structure that captures all essential information of the model. This question has been answered for LQ mean field games in [30] without a major player and in [36] with a major player.

When taking the approaches of Nash certainty equivalence or master equations or asymptotic solvability, one typically obtains seemingly different solution structures. We are interested in studying the relationship between these solution equations, and their uniqueness properties. This will shed light into the intrinsic nature of these solutions and will be especially valuable when many solution formulations via different information or interaction patterns exist for major-minor player models and when non-uniqueness results of mean field games have been frequently seen in the literature [1, 3, 22, 23, 30, 43]. The main contribution of the paper is to show that the NCE equation system is equivalent to the master equations when the latter are restricted to quadratic solutions, and that for homogeneous minor players, the NCE equation system has a solution if and only if asymptotic solvability holds. In contrast, for LQ mean field games without a major player, the NCE equation system via consistent mean field approximations may have a solution but asymptotic solvability fails [30].

The organization of the paper is as follows. Section 2 presents a summary of the NCE approach adopted in [26]. Section 3 introduces the master equations for the LQ mean field games with a major player and K types of minor players and compares with the NCE equation system. Section 4 analyzes the special case of homogeneous minor players; it first reviews the asymptotic solvability problem in [36] and next shows an equivalence relationship between the solution of the NCE approach and asymptotic solvability. Section 5 concludes the paper.

Notation We use $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ to denote an underlying filtered probability space. Let S^n be the set of $n \times n$ real and symmetric matrices, S^n_+ its subset of positive semi-definite matrices, and I_k the $k \times k$ identity matrix. Let $\mathcal{P}_2(\mathbb{R}^n)$ be the set of probability measures μ on $\mathbb{R}^n$ that have finite second moment. Denote $\langle y\rangle_\mu = \int_{\mathbb{R}^n} y\mu(dy)$ for the probability measure μ. Given a symmetric matrix $M \geq 0$, the quadratic form $z^T Mz$ may be denoted as $|z|^2_M$. For $Z = (z_{jk}) \in \mathbb{R}^{l\times m}$, denote the l_1-norm $\|Z\|_{l_1} = \sum_{j,k} |z_{jk}|$. For matrices $A = (a_{ij}) \in \mathbb{R}^{l_1\times l_2}$, $\hat{A} \in \mathbb{R}^{l_3\times l_4}$, the Kronecker product $A \otimes \hat{A} = (a_{ij}\hat{A})_{1\leq i\leq l_1, 1\leq j\leq l_2} \in \mathbb{R}^{(l_1 l_3)\times(l_2 l_4)}$.

2. The LQ mean field game with a major player

This section summarizes the methodology developed in [26] which considered an infinite horizon. The model below takes a finite horizon for convenience of later comparison with master equations and asymptotic solvability by analyzing ordinary differential equations (ODEs). The approach in [26] is applied in a straightforward manner.

We consider the LQ game with a major player $\mathcal{A}_0$ and N minor players $\mathcal{A}_i$, $1 \leq i \leq N$. The states of $\mathcal{A}_0$ and $\mathcal{A}_i$ are, respectively, denoted by $X_0(t)$ and $X_i(t)$, $1 \leq i \leq N$, which satisfy the linear stochastic differential equations (SDEs):

$$(1) \quad dX_0(t) = \big(A_0X_0(t) + B_0u_0(t) + F_0X^{(N)}(t)\big)dt + D_0dW_0(t),$$

$$(2) \quad dX_i(t) = \big(A_{\theta_i}X_i(t) + Bu_i(t) + FX^{(N)}(t) + GX_0(t)\big)dt + DdW_i(t), \quad 1 \leq i \leq N, \quad t \geq 0,$$

where we have state $X_j \in \mathbb{R}^n$, control $u_j \in \mathbb{R}^{n_1}$, and $X^{(N)} = \frac{1}{N}\sum_{k=1}^N X_k(t)$. The initial states $\{X_j(0), 0 \leq j \leq N\}$ are independent with $EX_j(0) = x_j(0)$ and finite second moment. The $N+1$ standard n_2-dimensional Brownian motions $\{W_j, 0 \leq j \leq N\}$ are independent and also independent of the initial states. The subscript θ_i is a dynamic parameter to model a heterogenous population of minor players. We assume θ_i takes values from the finite set $\Theta = \{1, \ldots, K\}$ modeling K types of minor players. If $\theta_i = k$, $\mathcal{A}_i$ is called a k-type minor player. The constant matrices A_0, B_0, F_0, D_0, A_k, B, F, G, D have compatible dimensions. Denote $u = (u_0, \cdots, u_N)$. The costs of the players are given by

$$J_0(u) = E\int_0^T e^{-\rho t}\Big[|X_0(t) - \mathit{\Gamma}_0 X^{(N)}(t) - \eta_0|^2_{Q_0} + |u_0(t)|^2_{R_0}\Big]\,dt$$
$$+\,e^{-\rho T}E|X_0(T) - \mathit{\Gamma}_{0f}X^{(N)}(T) - \eta_{0f}|^2_{Q_{0f}}, \tag{3}$$
$$J_i(u) = E\int_0^T e^{-\rho t}\Big[|X_i(t) - \mathit{\Gamma}_1 X_0(t) - \mathit{\Gamma}_2 X^{(N)}(t) - \eta|^2_{Q} + |u_i(t)|^2_{R}\Big]\,dt$$
$$+\,e^{-\rho T}E|X_i(T) - \mathit{\Gamma}_{1f}X_0(T) - \mathit{\Gamma}_{2f}X^{(N)}(T) - \eta_f|^2_{Q_f}, \quad 1 \le i \le N, \tag{4}$$

where $\rho \ge 0$ is a discount factor. The constant matrices (or vectors) Q_0, $\mathit{\Gamma}_0$, η_0, R_0, Q_{0f}, $\mathit{\Gamma}_{0f}$, η_{0f}, Q, $\mathit{\Gamma}_1$, $\mathit{\Gamma}_2$, η, R, Q_f, $\mathit{\Gamma}_{1f}$, $\mathit{\Gamma}_{2f}$, η_f above have compatible dimensions, and $Q_0 \ge 0$, $Q_{0f} \ge 0$, $Q \ge 0$, $Q_f \ge 0$, $R_0 > 0$, $R > 0$. Our analysis can be easily extended to the case of time-dependent parameters. We only take $A_{(\cdot)}$ in (2) to be dependent on θ_i for the purpose of notational simplicity. When other parameters for $\mathcal{A}_i$ also depend on θ_i, the analysis is similar.

For given N, define $\mathcal{I}_k = \{i|\theta_i = k, 1 \le i \le N\}$, $N_k = |\mathcal{I}_k|$, where $|\mathcal{I}_k|$ is the cardinality of $\mathcal{I}_k$, $1 \le k \le K$. Let $\pi_k^{(N)} = N_k/N$. Then $\pi^{(N)} = (\pi_1^{(N)}, \cdots, \pi_K^{(N)})$ is the empirical distribution of $\theta_1, \cdots, \theta_N$. We make the following assumptions.

(**A1**) There exists a probability vector π such that $\lim_{N\to\infty}\pi^{(N)} = \pi$, where $\pi = (\pi_1, \cdots, \pi_K)$ and $\min_{1\le k\le K}\pi_k > 0$.

(**A2**) The initial states $X_j(0)$, $0 \le j \le N$, are independent, $EX_i(0) = \alpha_0$ for all $i \ge 1$, and there exists a fixed constant c_0 such that $\sup_{j\ge 0} E|X_j(0)|^2 \le c_0$.

2.1. The limiting two-player model

Below we overview the steps in [26]. The key idea is to introduce a new process $\bar{Z}(t) = [\bar{Z}_1^T(t), \cdots, \bar{Z}_K^T(t)]^T$ as a state component in an augmented state space, where $\bar{Z}_k$ specifies the mean field generated by all k-type minor players. Given the major player's state $\bar{X}_0$, the process $\bar{Z}$ is described by the following equation

$$d\bar{Z}(t) = [\overline{A}(t)\bar{Z}(t) + \overline{G}(t)\bar{X}_0(t) + \overline{m}(t)]dt, \tag{5}$$

where $\bar{Z}_k(0) = \alpha_0$, and $\overline{A}(t) \in \mathbb{R}^{nK\times nK}$, $\overline{G}(t) \in \mathbb{R}^{nK\times n}$, and $\overline{m}(t)$ are continuous matrix or vector valued functions on $[0, T]$. The initial condition for (5) is due to the initial mean for the minor players as specified in (**A2**). The triple $(\overline{A}, \overline{G}, \overline{m})$ is not known in advance and needs to be determined as part of the solution.

After replacing $X^{(N)}$ in (1)–(4) by $\sum_{k=1}^{K} \pi_k \bar{Z}_k$, we introduce two limiting optimal control problems for the major player and a representative minor player. Let $\bar{\mathcal{A}}_0$ and $\bar{\mathcal{A}}_i$ stand for the two players.

Problem (P1): The major player $\bar{\mathcal{A}}_0$ with state $\bar{X}_0$ has dynamics and cost

$$
\begin{aligned}
d\bar{X}_0 &= \Big[A_0\bar{X}_0 + B_0u_0 + F_0\sum_{k=1}^{K}\pi_k\bar{Z}_k\Big]dt + D_0dW_0,\\
d\bar{Z} &= (\overline{A}\bar{Z} + \overline{G}\bar{X}_0 + \overline{m})dt, \qquad\qquad t\geq 0,\\
\bar{J}_0(u_0(\cdot)) &= E\int_0^T e^{-\rho t}\Big\{\Big|\bar{X}_0 - \Gamma_0\sum_{k=1}^{K}\pi_k\bar{Z}_k - \eta_0\Big|^2_{Q_0} + u_0^TR_0u_0\Big\}dt,\\
&\quad + e^{-\rho T}E\Big|\bar{X}_0(T) - \Gamma_{0f}\sum_{k=1}^{K}\pi_k\bar{Z}_k(T) - \eta_{0f}\Big|^2_{Q_{0f}},
\end{aligned}
$$

where $\bar{X}_0(0) = X_0(0)$ and $\bar{Z}_k(0) = \alpha_0$.

Problem (P2): The minor player $\bar{\mathcal{A}}_i$ with state $\bar{X}_i$ has dynamics and cost

$$
\begin{aligned}
d\bar{X}_i &= \Big[A_{\theta_i}\bar{X}_i + Bu_i + F\sum_{k=1}^{K}\pi_k\bar{Z}_k + G\bar{X}_0\Big]dt + DdW_i,\\
d\bar{X}_0 &= \Big[A_0\bar{X}_0 + B_0\hat{u}_0 + F_0\sum_{k=1}^{K}\pi_k\bar{Z}_k\Big]dt + D_0dW_0,\\
d\bar{Z} &= (\overline{A}\bar{Z} + \overline{G}\bar{X}_0 + \overline{m})dt, \qquad\qquad t\geq 0,\\
\bar{J}_i(u_i(\cdot),\hat{u}_0(\cdot)) &= E\int_0^T e^{-\rho t}\Big\{\Big|\bar{X}_i - \Gamma_1\bar{X}_0 - \Gamma_2\sum_{k=1}^{K}\pi_k\bar{Z}_k - \eta\Big|^2_{Q} + u_i^TRu_i\Big\}dt,\\
&\quad + e^{-\rho T}E\Big|\bar{X}_i(T) - \Gamma_{1f}\bar{X}_0(T) - \Gamma_{2f}\sum_{k=1}^{K}\pi_k\bar{Z}_k(T) - \eta_f\Big|^2_{Q_f},
\end{aligned}
\tag{6}
$$

where $\bar{X}_i(0) = X_i(0)$, $\bar{X}_0(0) = X_0(0)$, $\bar{Z}_k(0) = \alpha_0$, and $\hat{u}_0$ is the optimal control law solved from (P1).

To distinguish from the original model with $N+1$ players, we use the new state variables $\bar{X}_0$ and $\bar{X}_i$. But we still reuse the same set of variables

u_0, u_i, W_0 and W_i in this infinite population limit model without the risk of confusion.

2.2. Nash certainty equivalence solution

Step 1: Solution of Problem (P1)

We start by assuming that $(\overline{A}, \overline{G}, \overline{m})$ has been known. Define

$$
\begin{aligned}
&F_0^\pi = \pi \otimes F_0, \quad \Gamma_0^\pi = \pi \otimes \Gamma_0, \quad \Gamma_{0f}^\pi = \pi \otimes \Gamma_{0f}, \\
&\mathbb{A}_0(t) = \begin{bmatrix} A_0 & F_0^\pi \\ \overline{G} & \overline{A} \end{bmatrix}, \quad \mathbb{B}_0 = \begin{bmatrix} B_0 \\ 0_{nK\times n_1} \end{bmatrix}, \quad \mathbb{M}_0(t) = \begin{bmatrix} 0_{n\times 1} \\ \overline{m} \end{bmatrix}, \\
&\mathbb{Q}_0^\pi = [I, -\Gamma_0^\pi]^T Q_0 [I, -\Gamma_0^\pi], \qquad \bar{\eta}_0^\pi = [I, -\Gamma_0^\pi]^T Q_0 \eta_0, \\
&\mathbb{Q}_{0f}^\pi = [I, -\Gamma_{0f}^\pi]^T Q_{0f} [I, -\Gamma_{0f}^\pi], \qquad \bar{\eta}_{0f}^\pi = [I, -\Gamma_{0f}^\pi]^T Q_{0f} \eta_{0f}.
\end{aligned}
$$

We introduce the ODE system

$$
\rho P_0 = \frac{dP_0}{dt} + P_0 \mathbb{A}_0 + \mathbb{A}_0^T P_0 - P_0 \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T P_0 + \mathbb{Q}_0^\pi, \tag{7}
$$

$$
\rho s_0 = \frac{ds_0}{dt} + (\mathbb{A}_0^T - P_0 \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T) s_0 + P_0 \mathbb{M}_0 - \bar{\eta}_0^\pi, \tag{8}
$$

where $P_0(T) = \mathbb{Q}_{0f}^\pi$ and $s_0(T) = -\bar{\eta}_{0f}^\pi$. The ODE system has a unique solution. The optimal control law for $\bar{\mathcal{A}}_0$ is

$$
\hat{u}_0 = -R_0^{-1} \mathbb{B}_0^T \left[P_0 (\bar{X}_0^T, \bar{Z}^T)^T + s_0 \right].
$$

Step 2: Solution of Problem (P2)

Suppose $\bar{\mathcal{A}}_i$ has its dynamic parameter $\theta_i = \kappa$ so that $A_{\theta_i} = A_\kappa$. Denote $F^\pi = \pi \otimes F$. After taking the feedback control law $\hat{u}_0$ for $\bar{\mathcal{A}}_0$, we have

$$
\begin{aligned}
d \begin{bmatrix} \bar{X}_i \\ \bar{X}_0 \\ \bar{Z} \end{bmatrix} = &\begin{bmatrix} A_\kappa & [G \quad F^\pi] \\ 0_{n(K+1)\times n} & \mathbb{A}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T P_0 \end{bmatrix} \begin{bmatrix} \bar{X}_i \\ \bar{X}_0 \\ \bar{Z} \end{bmatrix} dt \\
&+ \begin{bmatrix} B \\ 0_{n(K+1)\times n_1} \end{bmatrix} u_i dt + \begin{bmatrix} 0_{n\times 1} \\ \mathbb{M}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T s_0 \end{bmatrix} dt + \begin{bmatrix} D dW_i \\ D_0 dW_0 \\ 0_{nK\times 1} \end{bmatrix},
\end{aligned}
$$

where $\bar{X}_i(0) = X_i(0)$, $\bar{X}_0(0) = X_0(0)$, $\bar{Z}(0) = \alpha_0$ and P_0 is solved from (7). Define

$$
\mathbb{A}_\kappa(t) = \begin{bmatrix} A_\kappa & [G \quad F^\pi] \\ 0_{n(K+1)\times n} & \mathbb{A}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T P_0 \end{bmatrix}, \quad \mathbb{B} = \begin{bmatrix} B \\ 0_{n(K+1)\times n_1} \end{bmatrix},
$$

$$\mathbb{M}(t)=\begin{bmatrix}0_{n\times 1}\\ \mathbb{M}_0-\mathbb{B}_0R_0^{-1}\mathbb{B}_0^Ts_0\end{bmatrix},\quad \Gamma_2^\pi=\pi\otimes\Gamma_2,\quad \Gamma_{2f}^\pi=\pi\otimes\Gamma_{2f},$$
$$\mathbb{Q}^\pi=[I,-\Gamma_1,-\Gamma_2^\pi]^TQ[I,-\Gamma_1,-\Gamma_2^\pi],\quad \bar\eta^\pi=[I,-\Gamma_1,-\Gamma_2^\pi]^TQ\eta,$$
$$\mathbb{Q}_f^\pi=[I,-\Gamma_{1f},-\Gamma_{2f}^\pi]^TQ_f[I,-\Gamma_{1f},-\Gamma_{2f}^\pi],\quad \bar\eta_f^\pi=[I,-\Gamma_{1f},-\Gamma_{2f}^\pi]^TQ_f\eta_f.$$

We introduce

$$\rho P_\kappa=\frac{dP_\kappa}{dt}+P_\kappa\mathbb{A}_\kappa+\mathbb{A}_\kappa^TP_\kappa-P_\kappa\mathbb{B}R^{-1}\mathbb{B}^TP_\kappa+\mathbb{Q}^\pi, \tag{9}$$

$$\rho s_\kappa=\frac{ds_\kappa}{dt}+\left(\mathbb{A}_\kappa^T-P_\kappa\mathbb{B}R^{-1}\mathbb{B}^T\right)s_\kappa+P_\kappa\mathbb{M}-\bar\eta^\pi, \tag{10}$$

where $P_\kappa(T)=\mathbb{Q}_f^\pi$ and $s_\kappa(T)=-\bar\eta_f^\pi$. The optimal control law for $\bar{\mathcal{A}}_i$ is given by

$$\hat u_i=-R^{-1}\mathbb{B}^T\left[P_\kappa(\bar X_i^T,\bar X_0^T,\bar Z^T)^T+s_\kappa\right]. \tag{11}$$

Finally, substituting (11) into (6) gives

$$\begin{aligned}d\bar X_i=&(A_\kappa\bar X_i+G\bar X_0+F^\pi\bar Z)dt-BR^{-1}\mathbb{B}^TP_\kappa(\bar X_i^T,\bar X_0^T,\bar Z^T)^Tdt\\&-BR^{-1}\mathbb{B}^Ts_\kappa dt+DdW_i,\end{aligned} \tag{12}$$

where $\bar X_i(0)=X_i(0)$.

Step 3: The consistency condition

For the matrices P_κ, $\kappa=1,\cdots,K$, we introduce the partition

$$P_\kappa=(P_{\kappa,jl})_{1\le j,l\le 3}, \tag{13}$$

where $P_{\kappa,11}$, $P_{\kappa,22}\in\mathbb{R}^{n\times n}$, and $P_{\kappa,33}\in\mathbb{R}^{nK\times nK}$. The matrix functions $\overline{A}(t)$, $\overline{G}(t)$ and vector function $\overline{m}(t)$ are represented in the form

$$\overline{A}(t)=\begin{bmatrix}\overline{A}_1\\ \vdots\\ \overline{A}_K\end{bmatrix},\qquad \overline{G}(t)=\begin{bmatrix}\overline{G}_1\\ \vdots\\ \overline{G}_K\end{bmatrix},\qquad \overline{m}(t)=\begin{bmatrix}\overline{m}_1\\ \vdots\\ \overline{m}_K\end{bmatrix}, \tag{14}$$

where $\overline{A}_k(t)\in\mathbb{R}^{n\times nK}$, $\overline{G}_k(t)\in\mathbb{R}^{n\times n}$ and $\overline{m}_k(t)\in\mathbb{R}^n$ for $1\le k\le K$. Denote

$$\mathbf{e}_k=[0_{n\times n},\cdots,0_{n\times n},I_n,0_{n\times n},\cdots,0_{n\times n}]\in\mathbb{R}^{n\times nK}, \tag{15}$$

where the identity matrix I_n is at the kth block, $1 \leq k \leq K$. Now we consider the average state $(1/N_\kappa)\sum_{i\in\mathcal{I}_\kappa} X_i$ of N_κ κ-type minor players with closed-loop dynamics of the form (12). When $N \to \infty$ so that $N_\kappa \to \infty$, the limit of the state average is required to regenerate $\bar{Z}_\kappa$ (which has been intended for the approximation of $(1/N_\kappa)\sum_{i\in\mathcal{I}_\kappa} X_i$) such that

$$
\begin{aligned}
d\bar{Z}_\kappa = & \left\{\left[A_\kappa - BR^{-1}B^T P_{\kappa,11}\right]\mathbf{e}_\kappa + F^\pi - BR^{-1}B^T P_{\kappa,13}\right\}\bar{Z}dt \\
& + (G - BR^{-1}B^T P_{\kappa,12})\bar{X}_0 dt - BR^{-1}\mathbb{B}^T s_\kappa dt,
\end{aligned} \tag{16}
$$

where $\bar{Z} = [\bar{Z}_1^T, \cdots, \bar{Z}_K^T]^T$ and $\bar{Z}_\kappa(0) = \alpha_0$. Now, under the NCE methodology, the resulting equation system (16) is required to coincide with (5) which had been presumed in the first place. We call this requirement the *consistency condition.*

2.3. The Nash certainty equivalence equation system

Based on Steps 1, 2 and 3, we introduce the first differential-algebraic system of equations (DAEs):

$$
\begin{cases}
\rho P_0 = \dot{P}_0 + P_0\mathbb{A}_0 + \mathbb{A}_0^T P_0 - P_0\mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T P_0 + \mathbb{Q}_0^\pi, \\
\rho P_\kappa = \dot{P}_\kappa + P_\kappa\mathbb{A}_\kappa + \mathbb{A}_\kappa^T P_\kappa - P_\kappa\mathbb{B}R^{-1}\mathbb{B}^T P_\kappa + \mathbb{Q}^\pi, \quad 1 \leq \kappa \leq K, \\
\overline{A}_\kappa = (A_\kappa - BR^{-1}B^T P_{\kappa,11})\mathbf{e}_\kappa + F^\pi - BR^{-1}B^T P_{\kappa,13}, \qquad \forall\kappa, \\
\overline{G}_\kappa = G - BR^{-1}B^T P_{\kappa,12}, \qquad \forall\kappa,
\end{cases} \tag{17}
$$

where $P_0(T) = \mathbb{Q}_{0f}^\pi$ and $P_\kappa(T) = \mathbb{Q}_f^\pi$, and the second differential-algebraic system of equations (DAEs):

$$
\begin{cases}
\rho s_0 = \dot{s}_0 + (\mathbb{A}_0^T - P_0\mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T)s_0 + P_0\mathbb{M}_0 - \bar{\eta}_0^\pi, \\
\rho s_\kappa = \dot{s}_\kappa + (\mathbb{A}_\kappa^T - P_\kappa\mathbb{B}R^{-1}\mathbb{B}^T)s_\kappa + P_\kappa\mathbb{M} - \bar{\eta}^\pi, \quad 1 \leq \kappa \leq K, \\
\overline{m}_\kappa = -BR^{-1}\mathbb{B}^T s_\kappa, \qquad \forall\kappa,
\end{cases} \tag{18}
$$

where $s_0(T) = -\bar{\eta}_{0f}^\pi$ and $s_\kappa(T) = -\bar{\eta}_f^\pi$. Note that $\overline{m}$ has been used in defining $\mathbb{M}_0$ and $\mathbb{M}$. The equality constraints for $\overline{A}_\kappa$, $\overline{G}_\kappa$ in (17) and $\overline{m}_\kappa$ in (18) result from the consistency condition specified in Step 3. The combined differential-algebraic system of equations (17)–(18) will be called the NCE equation system. The matrices $\mathbb{A}_0$ and $\mathbb{A}_\kappa$ now depend on the solution of the equation system. The solution of (17), if it exists, can be solved without involving (18).

Denote

$$
\mathcal{X}_P = C([0,T];\mathbb{R}^{n(K+1)\times n(K+1)}) \times (C([0,T];\mathbb{R}^{n(K+2)\times n(K+2)}))^K
$$

$$\times C([0,T];\mathbb{R}^{nK\times nK}) \times C([0,T];\mathbb{R}^{nK\times n}),$$
$$\mathcal{X}_s = C([0,T];\mathbb{R}^{n(K+1)}) \times (C([0,T];\mathbb{R}^{n(K+2)}))^K \times C([0,T];\mathbb{R}^{nK}).$$

Definition 1. A set of functions

$$(P_0, P_1, \cdots, P_K, \overline{A}, \overline{G}) \in \mathcal{X}_P, \quad (s_0, s_1, \cdots, s_K, \overline{m}) \in \mathcal{X}_s$$

satisfying (17)–(18) on $[0,T]$ is called a solution of the NCE equation system.

Lemma 2. *We have the following assertions:*

i) *If $(P_0, P_1, \cdots, P_K, \overline{A}, \overline{G}) \in \mathcal{X}_P$ is a solution of (17), then it is unique and $P_0 \in C([0,T]; S_+^{n(K+1)})$, $P_\kappa \in C([0,T]; S_+^{n(K+1)})$ for each $1 \le \kappa \le K$.*

ii) *The NCE equation system has a solution if and only if (17) has a solution.*

iii) *If the NCE equation system has a solution in $\mathcal{X}_P \times \mathcal{X}_s$, it is unique.*

Proof. i) By eliminating $(\overline{A}, \overline{G})$, we write $\mathbb{A}_0$ as a function of $(P_1, \cdots, P_K)$, and $\mathbb{A}_\kappa$ as a function of $(P_0, P_1, \cdots, P_K)$, and accordingly denote $\mathbb{A}_0(P_1, \cdots, P_K)$, $\mathbb{A}_\kappa(P_0, P_1, \cdots, P_K)$. Next we write the ODEs of $(P_0, P_1, \cdots, P_K)$ where the vector field is a continuous function of $(P_0, P_1, \cdots, P_K)$ with local Lipschitz continuity. Hence the solution $(P_0, P_1, \cdots, P_K)$ is unique, which further uniquely determines $(\overline{A}, \overline{G})$. After solving $P_0(t)$ and $P_1(t), \cdots, P_K(t)$, we denote

$$\hat{\mathbb{A}}_0(t) = \mathbb{A}_0(P_1(t), \cdots, P_K(t)), \quad \hat{\mathbb{A}}_\kappa(t) = \mathbb{A}_\kappa(P_0(t), P_1(t), \cdots, P_K(t)).$$

Consider the standard Riccati ODE

$$\rho \hat{P}_0 = \dot{\hat{P}}_0 + \hat{P}_0 \hat{\mathbb{A}}_0 + \hat{\mathbb{A}}_0^T \hat{P}_0 - \hat{P}_0 \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T \hat{P}_0 + \mathbb{Q}_0^\pi, \qquad \hat{P}_0(T) = \mathbb{Q}_{0f}^\pi. \tag{19}$$

We obtain a unique solution $\hat{P}_0 \in C([0,T]; S_+^{n(K+1)})$ [42]. Since P_0 also satisfies (19), we necessarily have $\hat{P}_0 = P_0$. It follows that $P_0 \in C([0,T]; S_+^{n(K+1)})$. Similarly we obtain $P_\kappa \in C([0,T]; S_+^{n(K+2)})$ for $1 \le \kappa \le K$.

ii) Necessity is trivial. We show sufficiency. After solving $(P_0,P_1,\cdots,P_K)$, we write $\mathbb{M}_0(s_1, \cdots, s_K)$ as a linear function of $s_1, \cdots, s_K$, and $\mathbb{M}(s_0, s_1, \cdots, s_K)$ as a linear function of $(s_0, s_1, \cdots, s_K)$. Then $(s_0, s_1, \cdots, s_K)$ is determined by a linear ODE system with time dependent coefficients and can be uniquely solved. We further solve $\overline{m}$.

iii) Suppose $(P_0, P_1, \cdots, P_K \overline{A}, \overline{G}, s_0, s_1, \cdots, s_K, \overline{m})$ is a solution of the NCE equation system. Let $(P_0', P_1', \cdots, P_K', \overline{A}', \overline{G}', s_0', s_1', \cdots, s_K', \overline{m}')$ be another solution. By part i), we have $(P_0, P_1, \cdots, P_K, \overline{A}, \overline{G}) = (P_0', P_1', \cdots,$

$P'_K, \overline{A}', \overline{G}'$). Consequently, $(s_0, s_1, \ldots, s_K, \overline{m}) = (s'_0, s'_1, \cdots, s'_K, \overline{m}')$. Uniqueness follows. $\square$

Remark 3. When T is replaced by ∞, the system specified by (1)–(4) reduces to the model in [26], and (17) is modified by replacing the two ODEs by two algebraic equations. For the infinite horizon case, (18) can be retained as a differential-algebraic system of equations with no terminal condition, for which one can impose appropriate growth conditions on the solutions (s_0, s_κ). The consideration of time-varying s_0, s_κ means searching in a larger space than constant solutions.

Remark 4. A class of LQ stochastic optimal control problems is solved in [19] based on convex analysis and the Gâteaux derivative of the cost; this approach is applied to solve the limiting optimal control problems of the LQ mean field game with a major player for both the finite and infinite horizon cases. This recovers both the NCE equation system obtained in [26] and (17)–(18).

3. The master equations and their relation to the NCE approach

Let $X_0^\dagger(t)$ and $Z_\kappa^\dagger(t)$, $1 \le \kappa \le K$, be the states of the major player and a minor player of type κ, respectively. Let $\{\mu_\kappa^\dagger(t), 0 \le t \le T\}$ denote the (random) measure flow as the mean field generated by all minor players of type κ. Due to the correlation of the κ-type minor players' states, the limit $\mu_\kappa^\dagger(t)$ of their empirical distribution is still random except for trivial cases.

For $\mu = (\mu_1, \cdots, \mu_K)$, where each $\mu_\kappa \in \mathcal{P}_2(\mathbb{R}^n)$, define the mean function

$$\bar{z}_\kappa^\mu = \bar{z}_\kappa = \langle y \rangle_{\mu_\kappa} \in \mathbb{R}^n, \quad \bar{z}^\mu = [\bar{z}^T, \cdots, \bar{z}_K^T]^T. \tag{20}$$

More generally, for the measure flow $\{\mu^\dagger(t) = (\mu_1^\dagger(t), \cdots, \mu_K^\dagger(t)), 0 \le t \le T\}$, we denote $\bar{z}_\kappa^{\mu^\dagger(t)} = \bar{z}_\kappa(t) = \langle y \rangle_{\mu_\kappa^\dagger(t)}$ and $\bar{z}^{\mu^\dagger}(t) = \bar{z}^{\mu^\dagger(t)} = [\bar{z}_1^T(t), \cdots, \bar{z}_K^T(t)]^T$. For this section, μ and $\mu^\dagger$ always contain K components, and this should be clear from the context.

We introduce the following model with $K+1$ players with dynamics

$$dX_0^\dagger = (A_0 X_0^\dagger + B_0 u_0 + F_0^\pi \bar{z}^{\mu^\dagger})dt + D_0 dW_0, \qquad 0 \le t \le T,$$
$$dZ_\kappa^\dagger = (A_\kappa Z_\kappa^\dagger + B u_\kappa + G X_0^\dagger + F^\pi \bar{z}^{\mu^\dagger})dt + D dW_\kappa, \quad 1 \le \kappa \le K,$$

and costs

$$J_0(u_0,\cdots,u_K)=E\int_0^T e^{-\rho t}\Big\{\Big|X_0^\dagger-\mathit{\Gamma}_0^\pi\bar z^{\mu^\dagger}-\eta_0\Big|_{Q_0}^2+u_0^TR_0u_0\Big\}dt,$$
$$+e^{-\rho T}E\Big|X_0^\dagger(T)-\mathit{\Gamma}_{0f}^\pi\bar z^{\mu^\dagger}(T)-\eta_{0f}\Big|_{Q_{0f}}^2,$$
$$J_\kappa(u_0,\cdots,u_K)=E\int_0^T e^{-\rho t}\Big\{\Big|Z_\kappa^\dagger-\mathit{\Gamma}_1X_0^\dagger-\mathit{\Gamma}_2^\pi\bar z^{\mu^\dagger}-\eta\Big|_Q^2+u_\kappa^TRu_\kappa\Big\}dt,$$
$$+e^{-\rho T}E\Big|Z_\kappa^\dagger(T)-\mathit{\Gamma}_{1f}X_0^\dagger(T)-\mathit{\Gamma}_{2f}^\pi\bar z^{\mu^\dagger}(T)-\eta_f\Big|_{Q_f}^2,\quad 1\le\kappa\le K.$$

The parameters in the dynamics and costs are the same as in (1)–(4). The major player takes $(X_0^\dagger,\mu^\dagger)$ as the state variable in its control problem, and the κ-type minor player takes $(Z_\kappa^\dagger,X_0^\dagger,\mu^\dagger)$ as the state variable.

Take the initial time t and initial state (x_0,μ) (resp., (z_κ,x_0,μ)) for the major player (resp., the κ-type minor player). Denote the value functions $V_0(t,x_0,\mu)$, $V_\kappa(t,z_\kappa,x_0,\mu)$, where $t\in[0,T]$, $x_0,z_\kappa\in\mathbb{R}^n$ and $\mu=(\mu_1,\cdots,\mu_K)$, $\mu_\kappa\in\mathcal{P}_2(\mathbb{R}^n)$.

We have the Hamilton-Jacobi-Bellman (master) equations

$$\begin{aligned}(21)\qquad &-\partial_tV_0+\rho V_0\\ &=\partial_{x_0}^TV_0(A_0x_0+B_0\hat u_0+F_0^\pi\bar z^\mu)\\ &\quad+|x_0-\mathit{\Gamma}_0^\pi\bar z^\mu-\eta_0|_{Q_0}^2+\hat u_0^TR_0\hat u_0+(1/2)\mathrm{Tr}(\partial_{x_0x_0}V_0D_0D_0^T)\\ &\quad+\sum_{l=1}^K\int\partial_{y_l}^T(\partial_{\mu_l}V_0)(t,x_0,\mu;y_l)\big[A_ly_l+B\hat u_l(t,y_l,x_0,\mu)\\ &\qquad\qquad+Gx_0+F^\pi\bar z^\mu\big]\mu_l(dy_l)\\ &\quad+\sum_{l=1}^K\int\frac{1}{2}\mathrm{Tr}[\partial_{y_ly_l}(\partial_{\mu_l}V_0)(t,x_0,\mu;y_l)DD^T]\mu_l(dy_l),\end{aligned}$$

where $V_0(T,x_0,\mu)=|x_0-\mathit{\Gamma}_{0f}^\pi\bar z^\mu-\eta_{0f}|_{Q_{0f}}^2$, and

$$\begin{aligned}(22)\qquad &-\partial_tV_\kappa+\rho V_\kappa\\ &=\partial_{x_0}^TV_\kappa(A_0x_0+B_0\hat u_0+F_0^\pi\bar z^\mu)+(1/2)\mathrm{Tr}(\partial_{x_0x_0}V_\kappa D_0D_0^T)\\ &\quad+\partial_{z_\kappa}^TV_\kappa(A_\kappa z_k+B\hat u_\kappa+Gx_0+F^\pi\bar z^\mu)\\ &\quad+|z_\kappa-\mathit{\Gamma}_1x_0-\mathit{\Gamma}_2^\pi\bar z^\mu-\eta|_Q^2+\hat u_\kappa^TR\hat u_\kappa+(1/2)\mathrm{Tr}(\partial_{z_\kappa z_\kappa}V_\kappa DD^T)\end{aligned}$$

$$+\sum_{l=1}^{K}\int \partial_{y_l}^T(\partial_{\mu_l}V_\kappa)(t,z_k,x_0,\mu;y_l)\big[A_l y_l + B\hat{u}_l(t,y_l,x_0,\mu) \\ + Gx_0 + F^\pi \bar{z}^\mu\big]\mu_l(dy_l) \\ +\sum_{l=1}^{K}\int \frac{1}{2}\mathrm{Tr}[\partial_{y_l y_l}(\partial_{\mu_l}V_\kappa)(t,z_k,x_0,\mu;y_l)DD^T]\mu_l(dy_l),$$

where $V_\kappa(T,z_\kappa,x_0,\mu)=|z_\kappa-\Gamma_{1f}x_0-\Gamma_{2f}^\pi\bar{z}^\mu-\eta_f|_{Q_f}^2$.

We may regard (21)–(22) as $K+1$ dynamic programming equations which are formally derived by a local expansion of $V_0(t+\epsilon, X_0^\dagger(t+\epsilon), \mu^\dagger(t+\epsilon))$ and $V_\kappa(t+\epsilon, Z_\kappa^\dagger(t+\epsilon), X_0^\dagger(t+\epsilon), \mu^\dagger(t+\epsilon))$ given $X_0^\dagger(t)=x_0$, $Z_\kappa^\dagger(t)=z_\kappa$, $\mu^\dagger(t)=\mu$. Each of the equilibrium strategies in $(\hat{u}_0,\hat{u}_1,\cdots,\hat{u}_K)$ is selected as a best response to maximize its own Hamiltonian. This amounts to finding

$$\hat{u}_0 = \arg\max_{u_0}(-\partial_{x_0}^T V_0 B_0 u_0 - u_0^T R_0 u_0),$$
$$\hat{u}_\kappa = \arg\max_{u_\kappa}(-\partial_{z_\kappa}^T V_\kappa B u_\kappa - u_\kappa^T R u_\kappa). \tag{23}$$

The integral terms in (21)–(22) account for the variation of the value function that is caused by the small perturbation of the mean field term. The choice of $\hat{u}_\kappa$ in (23) does not directly use the integral terms since the control of the minor player in question has little impact on the mean field. The reader may consult [12, 35] for master equations of nonlinear major player models, and [13] for differentiation with respect to probability measures (if $K=1$, our notation $\partial_\mu(V_0(\cdot,\mu))$ is equivalent to $\frac{\delta(V_0(\cdot,\mu))}{\delta\mu}$ in [13]). Note that after differentiation of the value functions, a new variable y_l arises so that we write

$$(\partial_{\mu_l}V_0)(t,x_0,\mu;y_l), \qquad (\partial_{\mu_l}V_\kappa)(t,z_k,x_0,\mu;y_l). \tag{24}$$

We determine $B_0^T\partial_{x_0}V_0 + 2R_0\hat{u}_0 = 0$, $B^T\partial_{z_\kappa}^T V_\kappa + 2R\hat{u}_\kappa = 0$, which gives

$$\hat{u}_0(t,x_0,\mu) = -(1/2)R_0^{-1}B_0^T\partial_{x_0}V_0(t,x_0,\mu), \tag{25}$$
$$\hat{u}_\kappa(t,z_\kappa,x_0,\mu) = -(1/2)R^{-1}B^T\partial_{z_\kappa}V_\kappa(t,z_\kappa,x_0,\mu). \tag{26}$$

Next, substituting (25)–(26) into (21)–(22) we obtain

$$-\partial_t V_0 + \rho V_0 \\ = \partial_{x_0}^T V_0(A_0 x_0 + F_0^\pi \bar{z}^\mu) - (1/4)\partial_{x_0}^T V_0 B_0 R_0^{-1} B_0^T \partial_{x_0} V_0 \tag{27}$$

$$
\begin{aligned}
&+ |x_0 - \Gamma_0^\pi \bar{z}^\mu - \eta_0|^2_{Q_0} + (1/2)\mathrm{Tr}(\partial_{x_0x_0} V_0 D_0 D_0^T) \\
&+ \sum_{l=1}^K \int \partial_{y_l}^T (\partial_{\mu_l} V_0)(t, x_0, \mu; y_l)(A_l y_l + G x_0 + F^\pi \bar{z}^\mu)\mu_l(dy_l) \\
&- \sum_{l=1}^K \int \frac{1}{2}\partial_{y_l}^T (\partial_{\mu_l} V_0)(t, x_0, \mu; y_l) B R^{-1} B^T \partial_{y_l} V_l(t, y_l, x_0, \mu)\mu_l(dy_l) \\
&+ \sum_{l=1}^K \int \frac{1}{2}\mathrm{Tr}[\partial_{y_l y_l}(\partial_{\mu_l} V_0)(t, x_0, \mu; y_l) D D^T]\mu_l(dy_l),
\end{aligned}
$$

where $V_0(T, x_0, \mu) = |x_0 - \Gamma_{0f}^\pi \bar{z}^\mu - \eta_{0f}|^2_{Q_{0f}}$, and

$$
\begin{aligned}
(28)\quad & -\partial_t V_\kappa + \rho V_\kappa \\
&= \partial_{x_0}^T V_\kappa (A_0 x_0 + F_0^\pi \bar{z}^\mu) - (1/2)\partial_{x_0}^T V_\kappa B_0 R_0^{-1} B_0^T \partial_{x_0} V_0 \\
&\quad + (1/2)\mathrm{Tr}(\partial_{x_0 x_0} V_\kappa D_0 D_0^T) \\
&\quad + \partial_{z_\kappa}^T V_\kappa (A_\kappa z_\kappa + G x_0 + F^\pi \bar{z}^\mu) - (1/4)\partial_{z_\kappa}^T V_\kappa B R^{-1} B^T \partial_{z_\kappa} V_\kappa \\
&\quad + |z_\kappa - \Gamma_1 x_0 - \Gamma_2^\pi \bar{z}^\mu - \eta|^2_Q + (1/2)\mathrm{Tr}(\partial_{z_\kappa z_\kappa} V_\kappa D D^T) \\
&\quad + \sum_{l=1}^K \int \partial_{y_l}^T (\partial_{\mu_l} V_\kappa)(t, z_\kappa, x_0, \mu; y_l)(A_l y_l + G x_0 + F^\pi \bar{z}^\mu)\mu_l(dy_l) \\
&\quad - \sum_{l=1}^K \int \frac{1}{2}\partial_{y_l}^T (\partial_{\mu_l} V_\kappa)(t, z_\kappa, x_0, \mu; y_l) B R^{-1} B^T \partial_{y_l} V_l(t, x_0, y_l, \mu)\mu_l(dy_l) \\
&\quad + \sum_{l=1}^K \int \frac{1}{2}\mathrm{Tr}[\partial_{y_l y_l}(\partial_{\mu_l} V_\kappa)(t, z_\kappa, x_0, \mu; y_l) D D^T]\mu_l(dy_l),
\end{aligned}
$$

where $V_\kappa(T, z_\kappa, x_0, \mu) = |z_\kappa - \Gamma_{1f} x_0 - \Gamma_{2f}^\pi \bar{z}^\mu - \eta_f|^2_{Q_f}$. We call (27) and (28) the master equations.

For the right hand side of (27), we denote it as

$$\chi_0 \coloneqq \chi_{0,1} - \chi_{0,2} + \chi_{0,3} + \chi_{0,4} + \chi_{0,5} - \chi_{0,6} + \chi_{0,7},$$

where each constituent term stands for the term in the master equation taking the same place. Similarly, the right hand side of (28) is written as

$$\chi \coloneqq \chi_1 - \chi_2 + \chi_3 + \chi_4 - \chi_5 + \chi_6 + \chi_7 + \chi_8 - \chi_9 + \chi_{10}.$$

3.1. Quadratic solutions

Recalling (20), for $\mu = (\mu_1, \cdots, \mu_K)$, we now simply denote $\bar{z}_\kappa = \langle y \rangle_{\mu_\kappa} \in \mathbb{R}^n$ and $\bar{z} = [\bar{z}_1^T, \cdots, \bar{z}_K^T]^T$. Denote $\xi_0 = [x_0^T, \bar{z}^T]^T$ and $\xi_\kappa = [z_\kappa^T, x_0^T, \bar{z}^T]^T$. We are interested in solutions of the following form:

$$V_0(t, x_0, \mu) = \xi_0^T \mathbf{P}_0^\dagger(t)\xi_0 + 2\mathbf{s}_0^{\dagger T}(t)\xi_0 + \mathbf{r}_0^\dagger(t), \tag{29}$$

$$V_\kappa(t, z_\kappa, x_0, \mu) = \xi_\kappa^T \mathbf{P}_\kappa^\dagger(t)\xi_\kappa + 2\mathbf{s}_\kappa^{\dagger T}(t)\xi_\kappa + \mathbf{r}_\kappa^\dagger(t), \quad 1 \le \kappa \le K, \tag{30}$$

where $\mathbf{P}_0^\dagger(t)$ and $\mathbf{P}_\kappa^\dagger(t)$ are symmetric matrix functions of $t \in [0, T]$, and the coefficient functions are differentiable on $[0, T]$. Such a solution $(V_0, V_1, \cdots, V_K)$ is called a quadratic solution to the master equations.

Denote the partition $\mathbf{P}_0^\dagger(t) = (\mathbf{P}_{0,jl}^\dagger)_{1\le j,l\le 2}$ and $\mathbf{P}_\kappa^\dagger(t) = (\mathbf{P}_{\kappa,jl}^\dagger)_{1\le j,l\le 3}$, where $\mathbf{P}_{0,11}^\dagger \in \mathbb{R}^{n\times n}$, $\mathbf{P}_{0,22}^\dagger \in \mathbb{R}^{nK\times nK}$, $\mathbf{P}_{\kappa,11}^\dagger$, $\mathbf{P}_{\kappa,22}^\dagger \in \mathbb{R}^{n\times n}$, $\mathbf{P}_{\kappa,33}^\dagger \in \mathbb{R}^{nK\times nK}$. Denote

$$\mathbf{s}_0^\dagger(t) = \begin{bmatrix} \mathbf{s}_{0,1}^\dagger \\ \mathbf{s}_{0,2}^\dagger \end{bmatrix}, \qquad \mathbf{s}_\kappa^\dagger(t) = \begin{bmatrix} \mathbf{s}_{\kappa,1}^\dagger \\ \mathbf{s}_{\kappa,2}^\dagger \\ \mathbf{s}_{\kappa,3}^\dagger \end{bmatrix},$$

where $\mathbf{s}_{0,1}^\dagger \in \mathbb{R}^n$, $\mathbf{s}_{0,2}^\dagger \in \mathbb{R}^{nK}$, $\mathbf{s}_{\kappa,1}^\dagger \in \mathbb{R}^n$, $\mathbf{s}_{\kappa,2}^\dagger \in \mathbb{R}^n$, and $\mathbf{s}_{\kappa,3}^\dagger \in \mathbb{R}^{nK}$.

In order to calculate the integral terms in the master equations to analyze quadratic solutions, we introduce some notation. For $1 \le l \le K$, denote

$$\begin{aligned} \overline{\mathbf{A}}_l^\dagger(t) &= (A_l - BR^{-1}B^T\mathbf{P}_{l,11}^\dagger)\mathbf{e}_K + F^\pi - BR^{-1}B^T\mathbf{P}_{l,13}^\dagger, \\ \overline{\mathbf{G}}_l^\dagger(t) &= G - BR^{-1}B^T\mathbf{P}_{l,12}^\dagger, \\ \overline{\mathbf{m}}_l^\dagger(t) &= -BR^{-1}B^T\mathbf{s}_{l,1}^\dagger, \end{aligned} \tag{31}$$

and

$$\overline{\mathbf{A}}^\dagger(t) = \begin{bmatrix} \overline{\mathbf{A}}_1^\dagger \\ \vdots \\ \overline{\mathbf{A}}_K^\dagger \end{bmatrix}, \quad \overline{\mathbf{G}}^\dagger(t) = \begin{bmatrix} \overline{\mathbf{G}}_1^\dagger \\ \vdots \\ \overline{\mathbf{G}}_K^\dagger \end{bmatrix}, \quad \overline{\mathbf{m}}^\dagger(t) = \begin{bmatrix} \overline{\mathbf{m}}_1^\dagger \\ \vdots \\ \overline{\mathbf{m}}_K^\dagger \end{bmatrix}, \tag{32}$$

$$\mathbb{A}_\kappa^\dagger(t) = \begin{bmatrix} A_\kappa & G & F^\pi \\ 0_{n\times n} & A_0 - B_0R_0^{-1}B_0^T\mathbf{P}_{0,11}^\dagger & F_0^\pi - B_0R_0^{-1}B_0^T\mathbf{P}_{0,12}^\dagger \\ 0_{nK\times n} & \overline{\mathbf{G}}^\dagger & \overline{\mathbf{A}}^\dagger \end{bmatrix}.$$

We introduce the system of ODEs:

$$\rho \mathbf{P}_0^\dagger - \dot{\mathbf{P}}_0^\dagger = \mathbf{P}_0^\dagger \begin{bmatrix} A_0 & F_0^\pi \\ \overline{\mathbf{G}}^\dagger & \overline{\mathbf{A}}^\dagger \end{bmatrix} + \begin{bmatrix} A_0 & F_0^\pi \\ \overline{\mathbf{G}}^\dagger & \overline{\mathbf{A}}^\dagger \end{bmatrix}^T \mathbf{P}_0^\dagger - \mathbf{P}_0^\dagger \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T \mathbf{P}_0^\dagger + \mathbb{Q}_0^\pi \tag{33}$$

$$\rho \mathbf{P}_\kappa^\dagger - \dot{\mathbf{P}}_\kappa^\dagger = \mathbf{P}_\kappa^\dagger \mathbb{A}_\kappa^\dagger + \mathbb{A}_\kappa^{\dagger T} \mathbf{P}_\kappa^\dagger - \mathbf{P}_\kappa^\dagger \mathbb{B} R^{-1} \mathbb{B}^T \mathbf{P}_\kappa^\dagger + \mathbb{Q}^\pi, \quad 1 \le \kappa \le K, \tag{34}$$

where $\mathbf{P}_0^\dagger(T) = \mathbb{Q}_{0f}^\pi$ and $\mathbf{P}_\kappa^\dagger(T) = \mathbb{Q}_f^\pi$. The $K+1$ equations are all coupled together through the dependence of $(\overline{\mathbf{G}}^\dagger, \overline{\mathbf{A}}^\dagger, \mathbb{A}_\kappa^\dagger)$ on $(\mathbf{P}_0^\dagger, \mathbf{P}_1^\dagger, \cdots, \mathbf{P}_K^\dagger)$.

We further introduce the ODE system:

(35)

$$\rho \mathbf{s}_0^\dagger - \dot{\mathbf{s}}_0^\dagger = \begin{bmatrix} A_0 & F_0^\pi \\ \overline{\mathbf{G}}^\dagger & \overline{\mathbf{A}}^\dagger \end{bmatrix}^T \mathbf{s}_0^\dagger - \begin{bmatrix} \mathbf{P}_{0,11}^\dagger \\ \mathbf{P}_{0,21}^\dagger \end{bmatrix} B_0 R_0^{-1} B_0^T \mathbf{s}_{0,1}^\dagger + \begin{bmatrix} \mathbf{P}_{0,12}^\dagger \\ \mathbf{P}_{0,22}^\dagger \end{bmatrix} \overline{\mathbf{m}}^\dagger - \bar{\eta}_0^\pi,$$

(36)

$$\rho \mathbf{s}_\kappa^\dagger - \dot{\mathbf{s}}_\kappa^\dagger = (\mathbb{A}_\kappa^{\dagger T} - \mathbf{P}_\kappa^\dagger \mathbb{B} R^{-1} \mathbb{B}^T) \mathbf{s}_\kappa^\dagger - \begin{bmatrix} \mathbf{P}_{\kappa,12}^\dagger \\ \mathbf{P}_{\kappa,22}^\dagger \\ \mathbf{P}_{\kappa,32}^\dagger \end{bmatrix} B_0 R_0^{-1} B_0^T \mathbf{s}_{0,1}^\dagger + \begin{bmatrix} \mathbf{P}_{\kappa,13}^\dagger \\ \mathbf{P}_{\kappa,23}^\dagger \\ \mathbf{P}_{\kappa,33}^\dagger \end{bmatrix} \overline{\mathbf{m}}^\dagger - \bar{\eta}^\pi,$$

where $\mathbf{s}_0^\dagger(T) = -\bar{\eta}_{0f}^\pi$ and $\mathbf{s}_\kappa^\dagger(T) = -\bar{\eta}_f^\pi$, $1 \le \kappa \le K$. Note that by (32), $\overline{\mathbf{m}}^\dagger$ is expressed in terms of $(\mathbf{s}_1^\dagger, \cdots, \mathbf{s}_K^\dagger)$.

Theorem 5. *The system of master equations (27)–(28) has a quadratic solution (29)–(30) if and only if $(\mathbf{P}_0^\dagger, \cdots, \mathbf{P}_K^\dagger)$ satisfies (33)–(34) on $[0,T]$, which further determines $(\mathbf{s}_0^\dagger, \mathbf{s}_1^\dagger, \cdots, \mathbf{s}_K^\dagger)$ as a unique solution of (35)–(36) on $[0,T]$.*

Proof. See Appendix A. □

Theorem 6. *The NCE equation system (17)–(18) has a solution as specified in Definition 1 if and only if the master equation system (27)–(28) has a quadratic solution of the form (29)–(30). If their solutions exist, they are unique and moreover*

$$P_k = \mathbf{P}_k^\dagger, \quad s_k = \mathbf{s}_k^\dagger, \quad \textit{for} \quad k = 0, 1, \cdots, K. \tag{37}$$

Proof. First of all, the NCE equation system has a solution if and only if (17) has a solution, which is necessarily unique. By Theorem 5, the master equation system has a quadratic solution if and only if (33)–(34) has a solution.

We rewrite the ODE system of $(P_0, \cdots, P_K)$ by expressing $(\overline{A}, \overline{G})$ in (17) in terms of $(P_1, \cdots, P_K)$. The gives a new ODE system where the vector field only has the unknowns $(P_0, P_1, \cdots, P_K)$. Subsequently we see that the above new vector field is the same as the one for $(\mathbf{P}_0^\dagger, \cdots, \mathbf{P}_K^\dagger)$ in (33)–(34). This proves the first part of the theorem together with $P_k = \mathbf{P}_k^\dagger$ for all $0 \le k \le K$. By showing the equivalence between the two ODE systems (18) and (35)–(36), we further obtain $s_k = \mathbf{s}_k^\dagger$ for all $0 \le k \le K$. □

3.2. Comparison of feedback control laws

Suppose (27)–(28) has a quadratic solution. By (25)–(26) and (29)–(30), we have

$$u_0(t) = -R_0^{-1} B_0^T (\mathbf{P}_{0,11}^\dagger X_0^\dagger(t) + \mathbf{P}_{0,12}^\dagger \bar{z}(t) + \mathbf{s}_{0,1}^\dagger(t)), \tag{38}$$

$$u_\kappa(t) = -R^{-1} B^T (\mathbf{P}_{\kappa,11}^\dagger Z_\kappa^\dagger(t) + \mathbf{P}_{\kappa,12}^\dagger X_0^\dagger(t) + \mathbf{P}_{\kappa,13}^\dagger \bar{z}(t) + \mathbf{s}_{\kappa,1}^\dagger(t)), \tag{39}$$

where the right hand sides use the value of the processes at time t. We need to determine the equation of $\bar{z}$. Under (38)–(39) we have the closed-loop equation

$$\begin{aligned} dZ_\kappa^\dagger &= (A_\kappa Z_\kappa^\dagger + G X_0^\dagger + F^\pi \bar{z})dt + D dW_\kappa \\ &\quad - BR^{-1}B^T(\mathbf{P}_{\kappa,11}^\dagger Z_\kappa^\dagger + \mathbf{P}_{\kappa,12}^\dagger X_0^\dagger + \mathbf{P}_{\kappa,13}^\dagger \bar{z} + \mathbf{s}_{\kappa,1}^\dagger)dt. \end{aligned} \tag{40}$$

Consider N_κ κ-type players with independent Brownian motions and initial states of mean α_0. We take their empirical mean by averaging (40) and let $N_\kappa \to \infty$. The limit of the empirical mean should regenerate $\bar{z}$, and this derives

$$\begin{aligned} d\bar{z}_\kappa &= ((A_\kappa - BR^{-1}B^T \mathbf{P}_{\kappa,11}^\dagger)\mathbf{e}_\kappa + F^\pi - BR^{-1}B^T\mathbf{P}_{\kappa,13}^\dagger)\bar{z}dt \\ &\quad + (G - BR^{-1}B^T\mathbf{P}_{\kappa,12}^\dagger)X_0^\dagger dt - BR^{-1}B^T \mathbf{s}_{\kappa,1}^\dagger dt, \end{aligned} \tag{41}$$

where $\bar{z}_\kappa(0) = \alpha_0$.

By (37) and (16), if we set $\theta_i = \kappa$, $X_\kappa^\dagger(0) = \bar{X}_i(0)$, $X_0^\dagger(0) = \bar{X}_0(0)$, and $W_\kappa = W_i$, the process $(X_\kappa^\dagger, X_0^\dagger, \bar{z})$ under the strategies (38)–(39) is equal to the process $(\bar{X}_i, \bar{X}_0, \bar{Z})$ under the NCE based strategies. Accordingly, the two sets of control laws are equivalent.

4. Homogeneous minor players

For this section, all minor players form a single type so that $A_{\theta_i} \equiv A$. The state processes of the $N+1$ players $\mathcal{A}_k$, $0 \le k \le N$, satisfy the SDEs:

$$dX_0(t) = \big(A_0 X_0(t) + B_0 u_0(t) + F_0 X^{(N)}(t)\big)dt + D_0 dW_0(t), \tag{42}$$

$$\begin{aligned} dX_i(t) &= \big(AX_i(t) + Bu_i(t) + FX^{(N)}(t) + GX_0(t)\big)dt + DdW_i(t), \\ &\quad 1 \le i \le N, \quad t \ge 0, \end{aligned} \tag{43}$$

The costs are given by (3)–(4).

4.1. The Nash certainty equivalence equation system

We follow the notation in Section 2.2 to denote

$$\begin{aligned} &\mathbb{A}_0(t) = \begin{bmatrix} A_0 & F_0 \\ \overline{G}(t) & \overline{A}(t) \end{bmatrix}, \quad \mathbb{B}_0 = \begin{bmatrix} B_0 \\ 0_{n\times n_1} \end{bmatrix}, \quad \mathbb{M}_0(t) = \begin{bmatrix} 0_{n\times 1} \\ \overline{m}(t) \end{bmatrix} \\ &\mathbb{Q}_0 = [I, -\varGamma_0]^T Q_0 [I, -\varGamma_0], \quad \bar\eta_0 = [I, -\varGamma_0]^T Q_0 \eta_0 \\ &\mathbb{Q}_{0f} = [I, -\varGamma_{0f}]^T Q_{0f} [I, -\varGamma_{0f}], \quad \bar\eta_{0f} = [I, -\varGamma_{0f}]^T Q_{0f} \eta_{0f}, \end{aligned}$$

where we have $\overline{A}(t), \overline{G}(t) \in \mathbb{R}^{n\times n}$, $\overline{m}(t) \in \mathbb{R}^n$. Denote

$$\begin{aligned} &\mathbb{A}(t) = \begin{bmatrix} A & [G \quad F] \\ 0_{2n\times n} & \mathbb{A}_0(t) - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T P_0(t) \end{bmatrix}, \quad \mathbb{B} = \begin{bmatrix} B \\ 0_{2n\times n_1} \end{bmatrix}, \\ &\mathbb{M}(t) = \begin{bmatrix} 0_{n\times 1} \\ \mathbb{M}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T s_0 \end{bmatrix}, \\ &\mathbb{Q} = [I, -\varGamma_1, -\varGamma_2]^T Q [I, -\varGamma_1, -\varGamma_2], \qquad \bar\eta = [I, -\varGamma_1, -\varGamma_2]^T Q\eta, \\ &\mathbb{Q}_f = [I, -\varGamma_{1f}, -\varGamma_{2f}]^T Q_f [I, -\varGamma_{1f}, -\varGamma_{2f}], \qquad \bar\eta_f = [I, -\varGamma_{1f}, -\varGamma_{2f}]^T Q_f \eta_f. \end{aligned}$$

The NCE equation system (17)–(18) now reduces to i)

$$\begin{cases} \rho P_0 = \dot P_0 + P_0\mathbb{A}_0 + \mathbb{A}_0^T P_0 - P_0 \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T P_0 + \mathbb{Q}_0, \\ \rho P_1 = \dot P_1 + P_1\mathbb{A} + \mathbb{A}^T P_1 - P_1\mathbb{B}R^{-1}\mathbb{B}^T P_1 + \mathbb{Q}, \\ \overline{A}(t) = A + F - BR^{-1}B^T P_{1,11} - BR^{-1}B^T P_{1,13}, \\ \overline{G}(t) = G - BR^{-1}B^T P_{1,12}, \end{cases} \tag{44}$$

where $P_0(T) = \mathbb{Q}_{0f}$, $P_1(T) = \mathbb{Q}_f$,

$$P_1 = (P_{1,jk})_{1\le j,k\le 3} \quad P_{1,jk}(t) \in \mathbb{R}^{n\times n},$$

and ii)

$$
\begin{cases}
\rho s_0 = \dot{s}_0 + (\mathbb{A}_0^T - P_0\mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T)s_0 + P_0\mathbb{M}_0 - \bar{\eta}_0, \\
\rho s_1 = \dot{s}_1 + (\mathbb{A}^T - P_1\mathbb{B}R^{-1}\mathbb{B}^T)s_1 + P_1\mathbb{M} - \bar{\eta}, \\
\overline{m}(t) = -BR^{-1}\mathbb{B}^T s_1(t),
\end{cases}
\tag{45}
$$

where $s_0(T) = -\bar{\eta}_{0f}$, $s_1(T) = -\bar{\eta}_f$.

4.2. The asymptotic solvability problem

To begin with, we provide some background on asymptotic solvability based on [36]. Denote by $\mathbf{1}_{k\times l}$ a $k \times l$ matrix with all entries equal to 1, and by the column vectors $\{e_1^k, \dots, e_k^k\}$ the canonical basis of $\mathbb{R}^k$. For instance, $e_1^k = [1, 0, \cdots, 0]^T \in \mathbb{R}^k$. Define

$$
\begin{aligned}
&\boldsymbol{x} = (\boldsymbol{x}_0^T, \boldsymbol{x}_1^T, \cdots, \boldsymbol{x}_N^T)^T \in \mathbb{R}^{(N+1)n}, \\
&X(t) = \begin{bmatrix} X_0(t) \\ \vdots \\ X_N(t) \end{bmatrix} \in \mathbb{R}^{(N+1)n}, \quad W(t) = \begin{bmatrix} W_0(t) \\ \vdots \\ W_N(t) \end{bmatrix} \in \mathbb{R}^{(N+1)n_2}, \\
&\widehat{\boldsymbol{A}} = \mathrm{diag}[A_0, A, \cdots, A] + \begin{bmatrix} 0_{n\times n}, & \mathbf{1}_{1\times N} \otimes \frac{F_0}{N} \\ \mathbf{1}_{N\times 1} \otimes G, & \mathbf{1}_{N\times N} \otimes \frac{F}{N} \end{bmatrix}, \\
&\widehat{\boldsymbol{D}} = \mathrm{diag}[D_0, D, \cdots, D] \in \mathbb{R}^{(N+1)n\times(N+1)n_2}, \\
&\boldsymbol{B}_0 = e_1^{N+1} \otimes B_0 \in \mathbb{R}^{(N+1)n\times n_1}, \\
&\boldsymbol{B}_k = e_{k+1}^{N+1} \otimes B \in \mathbb{R}^{(N+1)n\times n_1}, \qquad 1 \le k \le N.
\end{aligned}
$$

Now we write (42) and (43) in the form

$$
dX(t) = \Big(\widehat{\boldsymbol{A}}X(t) + \sum_{k=0}^{N} \boldsymbol{B}_k u_k(t)\Big)dt + \widehat{\boldsymbol{D}}dW(t), \quad t \ge 0. \tag{46}
$$

We consider closed-loop perfect state (CLPS) information [2] so that $X(t)$ is observed by each player, and look for Nash strategies. Let u_{-k} denote the strategies of all players other than $\mathcal{A}_k$. A set of strategies $(\hat{u}_0, \cdots, \hat{u}_N)$ is a Nash equilibrium if for all $0 \le k \le N$, we have $J_k(\hat{u}_k, \hat{u}_{-k}) \le J_k(u_k, \hat{u}_{-k})$, for any state feedback based strategy u_k which together with $\hat{u}_{-k}$ ensures a unique solution of $X(t)$ on $[0, T]$. Denote

$$
\boldsymbol{K}_0 = [I_n, 0, \cdots, 0] - \tfrac{1}{N}[0, \Gamma_0, \cdots, \Gamma_0],
$$

$$
\begin{aligned}
&\boldsymbol{K}_{0f} = [I_n, 0, \cdots, 0] - \tfrac{1}{N}[0, \Gamma_{0f}, \cdots, \Gamma_{0f}],\\
&\boldsymbol{Q}_0 = \boldsymbol{K}_0^T Q_0 \boldsymbol{K}_0, \qquad \boldsymbol{Q}_{0f} = \boldsymbol{K}_{0f}^T Q_{0f} \boldsymbol{K}_{0f},\\
&\boldsymbol{K}_i = [0, 0, \cdots, I_n, 0, \cdots, 0] - [\Gamma_1, 0, \cdots, 0] - \tfrac{1}{N}[0, \Gamma_2, \cdots, \Gamma_2],\\
&\boldsymbol{K}_{if} = [0, 0, \cdots, I_n, 0, \cdots, 0] - [\Gamma_{1f}, 0, \cdots, 0] - \tfrac{1}{N}[0, \Gamma_{2f}, \cdots, \Gamma_{2f}],\\
&\boldsymbol{Q}_i = \boldsymbol{K}_i^T Q \boldsymbol{K}_i, \qquad \boldsymbol{Q}_{if} = \boldsymbol{K}_{if}^T Q_f \boldsymbol{K}_{if},\\
&\widehat{\boldsymbol{A}}_{\rho/2} = \widehat{\boldsymbol{A}} - \rho I/2, \qquad \widehat{\boldsymbol{A}}_{\rho} = \widehat{\boldsymbol{A}} - \rho I,
\end{aligned} \tag{47}
$$

where I_n is the $(i+1)$th submatrix in (47). We have $\boldsymbol{K}_0, \boldsymbol{K}_i \in \mathbb{R}^{n\times(N+1)n}$ and $\boldsymbol{Q}_0, \boldsymbol{Q}_i \in \mathbb{R}^{(N+1)n\times(N+1)n}$.

Based on [36], we introduce the equation system:

(48)
$$
\begin{cases}
\dot{\boldsymbol{P}}_0(t) = -\big(\boldsymbol{P}_0\widehat{\boldsymbol{A}}_{\rho/2} + \widehat{\boldsymbol{A}}_{\rho/2}^T\boldsymbol{P}_0\big) + \boldsymbol{P}_0\boldsymbol{B}_0R_0^{-1}\boldsymbol{B}_0^T\boldsymbol{P}_0\\
\qquad\qquad + \boldsymbol{P}_0\sum_{k=1}^N \boldsymbol{B}_kR^{-1}\boldsymbol{B}_k^T\boldsymbol{P}_k + \sum_{k=1}^N \boldsymbol{P}_k\boldsymbol{B}_kR^{-1}\boldsymbol{B}_k^T\boldsymbol{P}_0 - \boldsymbol{Q}_0,\\
\boldsymbol{P}_0(T) = \boldsymbol{Q}_{0f},
\end{cases}
$$

$$
\begin{cases}
\dot{\boldsymbol{S}}_0(t) = -\widehat{\boldsymbol{A}}_\rho^T\boldsymbol{S}_0 + \boldsymbol{P}_0\boldsymbol{B}_0R_0^{-1}\boldsymbol{B}_0^T\boldsymbol{S}_0 + \sum_{k=1}^N \boldsymbol{P}_k\boldsymbol{B}_kR^{-1}\boldsymbol{B}_k^T\boldsymbol{S}_0\\
\qquad\qquad + \boldsymbol{P}_0\sum_{k=1}^N \boldsymbol{B}_kR^{-1}\boldsymbol{B}_k^T\boldsymbol{S}_k + \boldsymbol{K}_0^TQ_0\eta_0,\\
\boldsymbol{S}_0(T) = -\boldsymbol{K}_{0f}^TQ_{0f}\eta_{0f},
\end{cases} \tag{49}
$$

(50)
$$
\begin{cases}
\dot{\boldsymbol{P}}_i(t) = -\big(\boldsymbol{P}_i\widehat{\boldsymbol{A}}_{\rho/2} + \widehat{\boldsymbol{A}}_{\rho/2}^T\boldsymbol{P}_i\big) - \boldsymbol{P}_i\boldsymbol{B}_iR^{-1}\boldsymbol{B}_i^T\boldsymbol{P}_i\\
\qquad\qquad + \big(\boldsymbol{P}_i\boldsymbol{B}_0R_0^{-1}\boldsymbol{B}_0^T\boldsymbol{P}_0 + \boldsymbol{P}_0\boldsymbol{B}_0R_0^{-1}\boldsymbol{B}_0^T\boldsymbol{P}_i\big)\\
\qquad\qquad + \Big(\boldsymbol{P}_i\sum_{k=1}^N \boldsymbol{B}_kR^{-1}\boldsymbol{B}_k^T\boldsymbol{P}_k + \sum_{k=1}^N \boldsymbol{P}_k\boldsymbol{B}_kR^{-1}\boldsymbol{B}_k^T\boldsymbol{P}_i\Big) - \boldsymbol{Q}_i,\\
\boldsymbol{P}_i(T) = \boldsymbol{Q}_{if}, \qquad 1 \le i \le N,
\end{cases}
$$

(51)
$$
\begin{cases}
\dot{\boldsymbol{S}}_i(t) = -\widehat{\boldsymbol{A}}_\rho^T\boldsymbol{S}_i + \boldsymbol{P}_0\boldsymbol{B}_0R_0^{-1}{\boldsymbol{B}_0}^T\boldsymbol{S}_i\\
\qquad\qquad + \boldsymbol{P}_i\boldsymbol{B}_0R_0^{-1}\boldsymbol{B}_0^T\boldsymbol{S}_0 - \boldsymbol{P}_i\boldsymbol{B}_iR^{-1}\boldsymbol{B}_i^T\boldsymbol{S}_i\\
\qquad\qquad + \sum_{k=1}^N \boldsymbol{P}_k\boldsymbol{B}_kR^{-1}\boldsymbol{B}_k^T\boldsymbol{S}_i + \boldsymbol{P}_i\sum_{k=1}^N \boldsymbol{B}_kR^{-1}\boldsymbol{B}_k^T\boldsymbol{S}_k + \boldsymbol{K}_i^TQ\eta,\\
\boldsymbol{S}_i(T) = -\boldsymbol{K}_{if}^TQ_f\eta_f, \qquad 1 \le i \le N.
\end{cases}
$$

The analysis in [36] is for costs without discount. The notion of asymptotic solvability and main results in [36] can be translated to the discounted case verbatim once we let $\widehat{\boldsymbol{A}}_{\rho/2}$ take the role of $\widehat{\boldsymbol{A}}$ used in [36] for Riccati ODEs.

Suppose that (48) and (50) have a unique solution $(\boldsymbol{P}_0, \cdots, \boldsymbol{P}_N)$ on $[0,T]$. Then we can uniquely solve (49), (51), and the Nash game of $N+1$ players has a set of feedback Nash strategies given by $\hat{u}_0(t) = -R_0^{-1}\boldsymbol{B}_0^T(\boldsymbol{P}_0 X(t) + \boldsymbol{S}_0)$, $\hat{u}_i(t) = -R^{-1}\boldsymbol{B}_i^T(\boldsymbol{P}_i X(t)+\boldsymbol{S}_i)$, $1 \le i \le N$. The solution of the feedback Nash strategies completely reduces to the study of (48) and (50).

Definition 7 ([36])**.** The sequence of Nash games specified by (42)–(43) and (3)–(4) has asymptotic solvability if there exists N_0 such that for all $N \ge N_0$, the Riccati ODE system consisting of (48) and (50) has a solution $(\boldsymbol{P}_0, \cdots, \boldsymbol{P}_N)$ on $[0,T]$ and $\sup_{N\ge N_0, 0\le t\le T}\left(\|\boldsymbol{P}_0(t)\|_{l_1} + \|\boldsymbol{P}_1(t)\|_{l_1}\right) < \infty$.

Denote $M_0 = B_0 R_0^{-1} B_0^T$, $M = BR^{-1}B^T$. We introduce the ODE system:

$$
\begin{cases}
\dot{\Lambda}_1^0 = \rho\Lambda_1^0 + \Lambda_1^0 M_0 \Lambda_1^0 - (\Lambda_1^0 A_0 + A_0^T \Lambda_1^0) \\
\qquad +\Lambda_2^0(M\Lambda_a^T - G) + (\Lambda_a M - G^T)\Lambda_2^{0^T} - Q_0, \\
\dot{\Lambda}_2^0 = \rho\Lambda_2^0 + (\Lambda_1^0 M_0 - A_0^T)\Lambda_2^0 + \Lambda_2^0(M(\Lambda_1+\Lambda_2) - A - F) \\
\qquad -\Lambda_1^0 F_0 + (\Lambda_a M - G^T)\Lambda_3^0 + Q_0\Gamma_0, \\
\dot{\Lambda}_3^0 = \rho\Lambda_3^0 + \Lambda_2^{0^T} M_0 \Lambda_2^0 - \Lambda_2^{0^T} F_0 - F_0^T \Lambda_2^0 \\
\qquad +\Lambda_3^0(M(\Lambda_1+\Lambda_2) - A - F) \\
\qquad +((\Lambda_1 + \Lambda_2^T)M - A^T - F^T)\Lambda_3^0 - \Gamma_0^T Q_0 \Gamma_0, \\
\dot{\Lambda}_0 = \rho\Lambda_0 + \Lambda_a M \Lambda_a^T - \Lambda_b G - G^T \Lambda_b^T \\
\qquad +\Lambda_0(M_0\Lambda_1^0 - A_0) + (\Lambda_1^0 M_0 - A_0^T)\Lambda_0 \\
\qquad -\Lambda_a(G - M\Lambda_b^T) - (G^T - \Lambda_b M)\Lambda_a^T - \Gamma_1^T Q \Gamma_1, \\
\dot{\Lambda}_1 = \rho\Lambda_1 + \Lambda_1 M \Lambda_1 - \Lambda_1 A - A^T\Lambda_1 - Q, \\
\dot{\Lambda}_2 = \rho\Lambda_2 + \Lambda_a^T(M_0\Lambda_2^0 - F_0) - \Lambda_1 F + (\Lambda_1 M - A^T)\Lambda_2 \\
\qquad +\Lambda_2(M(\Lambda_1+\Lambda_2) - A - F) + Q\Gamma_2, \\
\dot{\Lambda}_3 = \rho\Lambda_3 + \Lambda_b^T M_0 \Lambda_2^0 + \Lambda_2^{0^T} M_0 \Lambda_b + \Lambda_2^T M \Lambda_2 \\
\qquad -\Lambda_b^T F_0 - F_0^T\Lambda_b - \Lambda_2^T F - F^T\Lambda_2 \\
\qquad +\Lambda_3(M(\Lambda_1+\Lambda_2) - A - F) \\
\qquad +((\Lambda_1 + \Lambda_2^T)M - A^T - F^T)\Lambda_3 - \Gamma_2^T Q \Gamma_2, \\
\dot{\Lambda}_a = \rho\Lambda_a + (\Lambda_1^0 M_0 - A_0^T)\Lambda_a + \Lambda_a(M\Lambda_1 - A) \\
\qquad -G^T\Lambda_1 + (\Lambda_a M - G^T)\Lambda_2^T + \Gamma_1^T Q, \\
\dot{\Lambda}_b = \rho\Lambda_b + \Lambda_0 M_0 \Lambda_2^0 + (\Lambda_a M - G^T)(\Lambda_2 + \Lambda_3) - \Lambda_0 F_0 \\
\qquad -\Lambda_a F + \Lambda_b(M(\Lambda_1+\Lambda_2) - A - F) \\
\qquad +(\Lambda_1^0 M_0 - A_0^T)\Lambda_b - \Gamma_1^T Q \Gamma_2,
\end{cases}
\tag{52}
$$

where the terminal conditions are

$$\begin{cases} \Lambda_1^0(T) = Q_{0f}, \quad \Lambda_2^0(T) = -Q_{0f}\Gamma_{0f}, \quad \Lambda_3^0(T) = \Gamma_{0f}^T Q_{0f}\Gamma_{0f}, \\ \Lambda_0(T) = \Gamma_{1f}^T Q_f \Gamma_{1f}, \quad \Lambda_1(T) = Q_f, \\ \Lambda_2(T) = -Q_f\Gamma_{2f}, \quad \Lambda_3(T) = \Gamma_{2f}^T Q_f \Gamma_{2f}, \\ \Lambda_a(T) = -\Gamma_{1f}^T Q_f, \quad \Lambda_b(T) = \Gamma_{1f}^T Q_f \Gamma_{2f}. \end{cases}$$

To give the reader some insights, we explain from where the equations in (52) arise. Note that asymptotic solvability is stated in terms of $(\boldsymbol{P}_0, \boldsymbol{P}_1, \cdots, \boldsymbol{P}_K)$, the dimension of which increases with the population size. The discount factor having been absorbed in $\widehat{\boldsymbol{A}}_{\rho/2}$, we follow the procedure in [36] to isolate a low dimensional structure from $(\boldsymbol{P}_0, \boldsymbol{P}_1, \cdots, \boldsymbol{P}_K)$. Specifically, by exploiting symmetry of the ODEs (48) and (50), it can be shown that the large matrix $\boldsymbol{P}_0$ is formed from 3 distinct $n \times n$ submatrices by arranging them into N^2 places. Similarly, $\boldsymbol{P}_1$ is formed from 6 distinct $n \times n$ submatrices. Any other matrix $\boldsymbol{P}_k$, $k \geq 2$, is determined from $\boldsymbol{P}_1$ by appropriate simultaneous row and column permutations. By using the above 9 submatrices and applying appropriate re-scaling to individual submatrices, we derive the ODE system (52) as $N \to \infty$; see [36] for more details.

Theorem 8 ([36])**.** *The sequence of games with dynamics (42)–(43) and costs (3)–(4) has asymptotic solvability if and only if (52) has a solution on $[0, T]$.*

Theorem 9. *The NCE equation system (44)–(45) has a solution on $[0, T]$ if and only if asymptotic solvability holds.*

Proof. By Lemma 2, (44)–(45) has a solution if and only if (44) has a solution on $[0, T]$; in addition, P_0 and P_1 in such a solution are symmetric. We denote the two matrix functions P_0 and P_1 in (44) in the form

$$P_0(t) = \begin{bmatrix} \Phi_1^0 & \Phi_2^0 \\ \Phi_2^{0T} & \Phi_3^0 \end{bmatrix}, \quad P_1(t) = \begin{bmatrix} \Phi_1 & \Phi_a^T & \Phi_2 \\ \Phi_a & \Phi_0 & \Phi_b \\ \Phi_2^T & \Phi_b^T & \Phi_3 \end{bmatrix},$$

where each submatrix is $n \times n$. For $\mathbb{A}_0$ and $\mathbb{A}$ in (44), we rewrite

$$\mathbb{A}_0(t) = \begin{bmatrix} A_0 & F_0 \\ G - M\Phi_a^T & A + F - M(\Phi_1 + \Phi_2) \end{bmatrix},$$

$$\mathbb{A}(t) = \begin{bmatrix} A & G & F \\ 0_{n\times n} & A_0 - M_0\Phi_1^0 & F_0 - M_0\Phi_2^0 \\ 0_{n\times n} & G - M\Phi_a^T & A + F - M(\Phi_1 + \Phi_2) \end{bmatrix}.$$

From (44) we obtain the following ODE system

$$
(53)\quad
\begin{cases}
\dot{\Phi}_1^0 = \rho\Phi_1^0 - \Phi_1^0 A_0 - A_0^T\Phi_1^0 - \Phi_2^0(G - M\Phi_a^T) \\
\qquad -(G^T - \Phi_a M)\Phi_2^{0T} + \Phi_1^0 M_0 \Phi_1^0 - Q_0, \\
\dot{\Phi}_2^0 = \rho\Phi_2^0 - \Phi_1^0 F_0 - \Phi_2^0(A + F - M(\Phi_1 + \Phi_2)) \\
\qquad -A_0^T\Phi_2^0 + (\Phi_a M - G^T)\Phi_3^0 + \Phi_1^0 M_0 \Phi_2^0 + Q_0\Gamma_0, \\
\dot{\Phi}_3^0 = \rho\Phi_3^0 - \Phi_2^{0T} F_0 - F_0^T\Phi_2^0 - \Phi_3^0(A + F - M(\Phi_1 + \Phi_2)) \\
\qquad -(A + F - M(\Phi_1 + \Phi_2))^T\Phi_3^0 + \Phi_2^{0T} M_0 \Phi_2^0 - \Gamma_0^T Q_0 \Gamma_0, \\
\dot{\Phi}_0 = \rho\Phi_0 - \Phi_a G - \Phi_0(A_0 - M_0\Phi_1^0) - \Phi_b(G - M\Phi_a^T) - G^T\Phi_a^T \\
\qquad -(A_0^T - \Phi_1^0 M_0)\Phi_0 - (G^T - \Phi_a M)\Phi_b^T + \Phi_a M \Phi_a^T - \Gamma_1^T Q \Gamma_1, \\
\dot{\Phi}_1 = \rho\Phi_1 - \Phi_1 A - A^T\Phi_1 + \Phi_1 M \Phi_1 - Q, \\
\dot{\Phi}_2 = \rho\Phi_2 - \Phi_1 F - \Phi_a^T(F_0 - M_0\Phi_2^0) - \Phi_2(A + F - M(\Phi_1 + \Phi_2)) \\
\qquad -A^T\Phi_2 + \Phi_1 M \Phi_2 + Q\Gamma_2, \\
\dot{\Phi}_3 = \rho\Phi_3 - \Phi_2^T F - F^T\Phi_2 - \Phi_b^T(F_0 - M_0\Phi_2^0) - (F_0^T - \Phi_2^{0T} M_0)\Phi_b \\
\qquad -\Phi_3(A + F - M(\Phi_1 + \Phi_2)) - (A + F - M(\Phi_1 + \Phi_2))^T\Phi_3 \\
\qquad +\Phi_2^T M \Phi_2 - \Gamma_2^T Q \Gamma_2, \\
\dot{\Phi}_a = \rho\Phi_a - \Phi_a A - G^T\Phi_1 - (A_0^T - \Phi_1^0 M_0)\Phi_a \\
\qquad -(G^T - \Phi_a M)\Phi_2^T + \Phi_a M \Phi_1 + \Gamma_1^T Q, \\
\dot{\Phi}_b = \rho\Phi_b - \Phi_a F - \Phi_0(F_0 - M_0\Phi_2^0) - \Phi_b(A + F - M(\Phi_1 + \Phi_2)) \\
\qquad -G^T\Phi_2 - (A_0^T - \Phi_1^0 M_0)\Phi_b - (G^T - \Phi_a M)\Phi_3 \\
\qquad +\Phi_a M \Phi_2 - \Gamma_1^T Q \Gamma_2,
\end{cases}
$$

where the terminal conditions are given by

$$
\begin{cases}
\Phi_1^0(T) = Q_{0f}, \quad \Phi_2^0(T) = -Q_{0f}\Gamma_{0f}, \quad \Phi_3^0(T) = \Gamma_{0f}^T Q_{0f}\Gamma_{0f}, \\
\Phi_0(T) = \Gamma_{1f}^T Q_f \Gamma_{1f}, \quad \Phi_1(T) = Q_f, \\
\Phi_2(T) = -Q_f\Gamma_{2f}, \quad \Phi_3(T) = \Gamma_{2f}^T Q_f \Gamma_{2f}, \\
\Phi_a(T) = -\Gamma_{1f}^T Q_f, \quad \Phi_b(T) = \Gamma_{1f}^T Q_f \Gamma_{2f}.
\end{cases}
$$

By comparing the individual equations at the corresponding place of (52) and (53), we see the two equation systems are determined by the same vector field with the same terminal conditions, and therefore they have the same solution. In view of Theorem 8 and Lemma 2, the theorem follows. □

Remark 10. The equation system (52) originates from the ODE system of $\boldsymbol{P}_0, \boldsymbol{P}_1, \cdots, \boldsymbol{P}_K$. But (53), which is equivalent to (52), arises from solving two optimal control problems in a low dimensional space as in Section 2.2.

5. Conclusion

This paper considers LQ mean field games with a major player and investigates the relationship between several solution frameworks. For a model of minor players of several subpopulations, an equivalence relationship is established between the Nash certainty equivalence approach and master equations. For a model with homogeneous minor players, it is shown that the Nash certainty equivalence based solution exists if and only if asymptotic solvability holds.

Appendix A. Proof of Theorem 5

Note that if $(\mathbf{P}_0^\dagger, \cdots, \mathbf{P}_K^\dagger)$ is a solution of (33)–(34) on $[0, T]$, it is the unique solution by the local Lipschitz continuity of the vector fields in the $K+1$ matrix ODEs. By (31), $\overline{\mathbf{m}}^\dagger$ depends linearly on $(\mathbf{s}_1^\dagger, \cdots, \mathbf{s}_K^\dagger)$. If $(\mathbf{P}_0^\dagger, \cdots, \mathbf{P}_K^\dagger)$ exists on $[0, T]$, we may uniquely solve $(\mathbf{s}_0^\dagger, \mathbf{s}_1^\dagger, \cdots, \mathbf{s}_K^\dagger)$ from a system of linear ODEs with bounded coefficients.

We now take (29)–(30) defined for $t \in [0, T]$ as a candidate solution of the master equations. Our plan is to substitute $(V_0, \cdots, V_K)$ into the right hand side of each of the $K+1$ equations in (27)–(28) and simplify the expression into a quadratic form of ξ_0 or ξ_κ. We directly compute the derivatives:

$$\begin{aligned}
&\partial_{x_0} V_0 = 2[\mathbf{P}_{0,11}^\dagger, \mathbf{P}_{0,12}^\dagger]\xi_0 + 2\mathbf{s}_{0,1}^\dagger, \qquad \partial_{x_0 x_0} V_0 = 2\mathbf{P}_{0,11}^\dagger, \\
&\partial_{x_0} V_\kappa = 2[\mathbf{P}_{\kappa,21}^\dagger, \mathbf{P}_{\kappa,22}^\dagger, \mathbf{P}_{\kappa,23}^\dagger]\xi_\kappa + 2\mathbf{s}_{\kappa,2}^\dagger, \qquad \partial_{x_0 x_0} V_\kappa = 2\mathbf{P}_{\kappa,22}^\dagger, \\
&\partial_{z_\kappa} V_\kappa = 2[\mathbf{P}_{\kappa,11}^\dagger, \mathbf{P}_{\kappa,12}^\dagger, \mathbf{P}_{\kappa,13}^\dagger]\xi_\kappa + 2\mathbf{s}_{\kappa,1}^\dagger, \qquad \partial_{z_\kappa z_\kappa} V_\kappa = 2\mathbf{P}_{\kappa,11}^\dagger.
\end{aligned}$$

To facilitate the subsequent computation, we state two lemmas involving derivatives with respect to probability measures. Recall the notation in (24).

Lemma A.1. *Suppose V_0 and V_κ are given by (29)–(30). Then we have*

$$\begin{aligned}
&[\partial_{y_1}^T \partial_{\mu_1} V_0, \cdots, \partial_{y_K}^T \partial_{\mu_K} V_0] = 2\xi_0^T \begin{bmatrix} \mathbf{P}_{0,12}^\dagger \\ \mathbf{P}_{0,22}^\dagger \end{bmatrix} + 2\mathbf{s}_{0,2}^{\dagger T}, \\
&[\partial_{y_1}^T \partial_{\mu_1} V_\kappa, \cdots, \partial_{y_K}^T \partial_{\mu_K} V_\kappa] = 2(z_\kappa^T \mathbf{P}_{\kappa,13}^\dagger + x_0^T \mathbf{P}_{\kappa,23}^\dagger + \bar{z}^T \mathbf{P}_{\kappa,33}^\dagger) + 2\mathbf{s}_{\kappa,3}^{\dagger T}.
\end{aligned}$$

Proof. Let $\mathbf{P}_{0,22}^{\dagger\mathrm{col1}}$ denote the sub-matrix consisting of the first n columns of $\mathbf{P}_{0,22}^\dagger$. Let $\mathbf{P}_{0,21}^{\dagger\mathrm{row1}}$ denote the submatrix consisting of the first n rows of $\mathbf{P}_{0,21}^\dagger$, and $\mathbf{s}_{0,2}^{\dagger\mathrm{row1}}$ be the subvector of the first n entries of $\mathbf{s}_{0,2}^\dagger$. We obtain

$$\partial_{\mu_1} V_0 = 2\bar{z}^T \mathbf{P}_{0,22}^{\dagger\mathrm{col1}} y_1 + 2y_1^T \mathbf{P}_{0,21}^{\dagger\mathrm{row1}} x_0 + 2y_1^T \mathbf{s}_{0,2}^{\dagger\mathrm{row1}}.$$

Therefore

$$\partial_{y_1}^T \partial_{\mu_1} V_0 = 2\bar{z}^T \mathbf{P}_{0,22}^{\dagger\text{col1}} + 2x_0^T \mathbf{P}_{0,21}^{\dagger\text{row1}T} + 2\mathbf{s}_{0,2}^{\dagger\text{row1}T}.$$

We similarly calculate $\partial_{y_l}^T \partial_{\mu_l} V_0$ for $l \geq 2$, and obtain the first equality in the lemma. Next we have

$$\partial_{\mu_l} V_\kappa = \partial_{\mu_l}(\bar{z}^T \mathbf{P}_{\kappa,33}^\dagger \bar{z} + 2\bar{z}^T \mathbf{P}_{\kappa,31}^\dagger z_\kappa + 2\bar{z}^T \mathbf{P}_{\kappa,32}^\dagger x_0 + 2\bar{z}^T \mathbf{s}_{\kappa,3}^\dagger).$$

Let $\mathbf{P}_{\kappa,33}^{\dagger\text{col1}}$ be the submatrix consisting of the first n columns of $\mathbf{P}_{\kappa,33}^\dagger$. Then $\partial_{\mu_1}(\bar{z}^T \mathbf{P}_{\kappa,33}^\dagger \bar{z}) = 2\bar{z}^T \mathbf{P}_{\kappa,33}^{\dagger\text{col1}} y_1$, which gives $\partial_{y_1}^T \partial_{\mu_1}(\bar{z}^T \mathbf{P}_{\kappa,33}^\dagger \bar{z}) = 2\bar{z}^T \mathbf{P}_{\kappa,33}^{\dagger\text{col1}}$. We further obtain the second equality. The lemma follows. □

Lemma A.2. *We have* $\partial_{y_l y_l} \partial_{\mu_l} V_0 = 0$ *and* $\partial_{y_l y_l} \partial_{\mu_l} V_\kappa = 0$ *for all* $1 \leq l \leq K$.

Proof. This follows from Lemma A.1. □

We proceed to evaluate the right hand side of (27) with the candidate solution $(V_0, \cdots, V_K)$. We have

$$\begin{aligned}
\chi_{0,1} &= \partial_{x_0}^T V_0(A_0 x_0 + F_0^\pi \bar{z}) = 2(\xi_0^T [\mathbf{P}_{0,11}^\dagger, \mathbf{P}_{0,12}^\dagger]^T + \mathbf{s}_{0,1}^{\dagger T})(A_0 x_0 + F_0^\pi \bar{z}), \\
\chi_{0,2} &= |[\mathbf{P}_{0,11}^\dagger, \mathbf{P}_{0,12}^\dagger]\xi_0 + \mathbf{s}_{0,1}^\dagger|^2_{B_0 R_0^{-1} B_0^T}, \\
\chi_{0,3} &= |[I, -\Gamma_0^\pi]\xi_0 - \eta_0|^2_{Q_0}, \\
\chi_{0,4} &= \mathrm{Tr}(\mathbf{P}_{0,11}^\dagger D_0 D_0).
\end{aligned}$$

Since $\partial_{y_l} V_l(t, y_l, x_0, \mu) = 2[\mathbf{P}_{l,11}^\dagger, \mathbf{P}_{l,12}^\dagger, \mathbf{P}_{l,13}^\dagger]\xi_l|_{z_l = y_l} + 2\mathbf{s}_{l,1}^\dagger$, we further write

$$\begin{aligned}
&A_l y_l + G x_0 + F^\pi \bar{z} - (1/2) B R^{-1} B^T \partial_{y_l} V_l(t, y_l, x_0, \mu) \\
&= (G - BR^{-1}B^T \mathbf{P}_{l,12}^\dagger) x_0 + (A_l - BR^{-1}B^T \mathbf{P}_{l,11}^\dagger) y_l \\
&\quad + (F^\pi - BR^{-1}B^T \mathbf{P}_{l,13}^\dagger)\bar{z} - BR^{-1}B^T \mathbf{s}_{l,1}^\dagger. \qquad \text{(A.1)}
\end{aligned}$$

By (32), (A.1) and Lemma A.1,

$$\begin{aligned}
\chi_{0,5} - \chi_{0,6} &= [\partial_{y_1}^T \partial_{\mu_1} V_0, \cdots, \partial_{y_K}^T \partial_{\mu_K} V_0]([\overline{\mathbf{G}}^\dagger, \overline{\mathbf{A}}^\dagger]\xi_0 + \overline{\mathbf{m}}^\dagger) \\
&= 2(\xi_0^T \begin{bmatrix} \mathbf{P}_{0,12}^\dagger \\ \mathbf{P}_{0,22}^\dagger \end{bmatrix} + \mathbf{s}_{0,2}^{\dagger T})([\overline{\mathbf{G}}^\dagger, \overline{\mathbf{A}}^\dagger]\xi_0 + \overline{\mathbf{m}}^\dagger). \qquad \text{(A.2)}
\end{aligned}$$

By Lemma A.2, $\chi_{0,7} = 0$.

We further evaluate the right hand side of (28). We have

$$
\begin{aligned}
\chi_1-\chi_2 &= \partial_{x_0}^T V_\kappa (A_0x_0+F_0^\pi \bar z) - (1/2)\partial_{x_0}^T V_\kappa B_0R_0^{-1}B_0^T\partial_{x_0}V_0 \\
&= 2([\mathbf{P}_{\kappa,21}^\dagger, \mathbf{P}_{\kappa,22}^\dagger, \mathbf{P}_{\kappa,23}^\dagger]\xi_k + \mathbf{s}_{\kappa,2}^\dagger)^T(A_0x_0+F_0^\pi \bar z \\
&\qquad - B_0R_0^{-1}B_0^T([\mathbf{P}_{0,11}^\dagger, \mathbf{P}_{0,12}^\dagger]\xi_0+\mathbf{s}_{0,1}^\dagger)) \\
&= 2([\mathbf{P}_{\kappa,21}^\dagger, \mathbf{P}_{\kappa,22}^\dagger, \mathbf{P}_{\kappa,23}^\dagger]\xi_k + \mathbf{s}_{\kappa,2}^\dagger)^T\cdot[(A_0-B_0R_0^{-1}B_0^T\mathbf{P}_{0,11}^\dagger)x_0 \\
&\qquad + (F_0^\pi - B_0R_0^{-1}B_0^T\mathbf{P}_{0,12}^\dagger)\bar z - B_0R_0^{-1}B_0^T\mathbf{s}_{0,1}^\dagger], \\
\chi_3+\chi_7 &= \mathrm{Tr}(\mathbf{P}_{\kappa,22}^\dagger D_0D_0 + \mathbf{P}_{\kappa,11}^\dagger DD),
\end{aligned}
$$

and

$$
\begin{aligned}
\chi_4 &= \partial_{z_\kappa}^T V_\kappa (A_\kappa z_\kappa + Gx_0 + F^\pi \bar z) \\
&= 2([\mathbf{P}_{\kappa,11}^\dagger, \mathbf{P}_{\kappa,12}^\dagger, \mathbf{P}_{\kappa,13}^\dagger]\xi_\kappa + \mathbf{s}_{\kappa,1}^\dagger)^T(A_\kappa z_\kappa + Gx_0 + F^\pi \bar z), \\
\chi_5 &= |[\mathbf{P}_{\kappa,11}^\dagger, \mathbf{P}_{\kappa,12}^\dagger, \mathbf{P}_{\kappa,13}^\dagger]\xi_\kappa + \mathbf{s}_{\kappa,1}^\dagger|_{BR^{-1}B}^2, \\
\chi_6 &= \xi_\kappa^T[I,-\Gamma_1,-\Gamma_2^\pi]^TQ[I,-\Gamma_1,-\Gamma_2^\pi]\xi_\kappa \\
&\quad - 2\eta^TQ[I,-\Gamma_1,-\Gamma_2^\pi]\xi_\kappa + \eta^TQ\eta.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\chi_8-\chi_9 &= [\partial_{y_1}^T\partial_{\mu_1}V_\kappa,\cdots,\partial_{y_K}^T\partial_{\mu_K}V_\kappa]([\overline{\mathbf{G}}^\dagger,\overline{\mathbf{A}}^\dagger]\xi_0+\overline{\mathbf{m}}^\dagger) \\
&= 2(z_\kappa^T\mathbf{P}_{\kappa,13}^\dagger + x_0^T\mathbf{P}_{\kappa,23}^\dagger + \bar z^T\mathbf{P}_{\kappa,33}^\dagger + \mathbf{s}_{\kappa,3}^{\dagger T})([\overline{\mathbf{G}}^\dagger,\overline{\mathbf{A}}^\dagger]\xi_0+\overline{\mathbf{m}}^\dagger),
\end{aligned}
$$

and by Lemma A.2, $\chi_{10}=0$.

By the above calculations, the right hand sides of (27)–(28) may be written as

$$\chi_0 = \xi_0^T\Theta_0(t)\xi_0 + 2\xi_0^T\theta_{0,1}(t) + \theta_{0,2}(t), \tag{A.3}$$

$$\chi = \xi_\kappa^T\Theta_\kappa(t)\xi_\kappa + 2\xi_\kappa^T\theta_{\kappa,1}(t) + \theta_{\kappa,2}(t), \tag{A.4}$$

where

$$
\begin{aligned}
\Theta_0 &= \mathbf{P}_0^\dagger\begin{bmatrix} A_0 & F_0^\pi \\ \overline{\mathbf{G}}^\dagger & \overline{\mathbf{A}}^\dagger\end{bmatrix} + \begin{bmatrix} A_0 & F_0^\pi \\ \overline{\mathbf{G}}^\dagger & \overline{\mathbf{A}}^\dagger\end{bmatrix}^T\mathbf{P}_0^\dagger - \mathbf{P}_0^\dagger\mathbb{B}_0R_0^{-1}\mathbb{B}_0^T\mathbf{P}_0^\dagger + \mathbb{Q}_0^\pi, \\
\theta_{0,1} &= \begin{bmatrix} A_0 & F_0^\pi \\ \overline{\mathbf{G}}^\dagger & \overline{\mathbf{A}}^\dagger\end{bmatrix}^T\mathbf{s}_0^\dagger - \begin{bmatrix}\mathbf{P}_{0,11}^\dagger \\ \mathbf{P}_{0,21}^\dagger\end{bmatrix}B_0R_0^{-1}B_0^T\mathbf{s}_{0,1}^\dagger + \begin{bmatrix}\mathbf{P}_{0,12}^\dagger \\ \mathbf{P}_{0,22}^\dagger\end{bmatrix}\overline{\mathbf{m}}^\dagger - \bar\eta_0^\pi,
\end{aligned}
$$

and

$$\Theta_\kappa = \mathbf{P}_\kappa^\dagger \mathbb{A}_\kappa^\dagger + \mathbb{A}_\kappa^{\dagger T} \mathbf{P}_\kappa^\dagger - \mathbf{P}_\kappa^\dagger \mathbb{B} R^{-1} \mathbb{B}^T \mathbf{P}_\kappa^\dagger + \mathbb{Q}^\pi,$$

$$\theta_{\kappa,1} = (\mathbb{A}_\kappa^{\dagger T} - \mathbf{P}_\kappa^\dagger \mathbb{B} R^{-1} \mathbb{B}^T) \mathbf{s}_\kappa^\dagger - \begin{bmatrix} \mathbf{P}_{\kappa,12}^\dagger \\ \mathbf{P}_{\kappa,22}^\dagger \\ \mathbf{P}_{\kappa,32}^\dagger \end{bmatrix} B_0 R_0^{-1} B_0^T \mathbf{s}_{0,1}^\dagger + \begin{bmatrix} \mathbf{P}_{\kappa,13}^\dagger \\ \mathbf{P}_{\kappa,23}^\dagger \\ \mathbf{P}_{\kappa,33}^\dagger \end{bmatrix} \overline{\mathbf{m}}^\dagger - \bar{\eta}^\pi.$$

The two terms $\theta_{0,2}(t)$ and $\theta_{\kappa,2}(t)$ can be determined but we omit the detail here.

Now if $(V_0, \cdots, V_K)$ in (29)–(30) is indeed a solution of the master equations, we necessarily have

$$\rho \mathbf{P}_0^\dagger - \dot{\mathbf{P}}_0^\dagger = \Theta_0, \qquad \rho \mathbf{P}_\kappa^\dagger - \dot{\mathbf{P}}_\kappa^\dagger = \Theta_\kappa, \quad 1 \le \kappa \le K.$$

Hence, (33) and (34) hold with the corresponding terminal conditions. In addition, $(\mathbf{s}_0^\dagger, \mathbf{s}_1^\dagger, \cdots, \mathbf{s}_K^\dagger)$ satisfies (35)–(36) as its unique solution.

Conversely, if (33)–(34) has a solution $(\mathbf{P}_0^\dagger, \cdots, \mathbf{P}_K^\dagger)$, we further uniquely solve $(\mathbf{s}_0^\dagger, \mathbf{s}_1^\dagger, \cdots, \mathbf{s}_K^\dagger)$ from (35)–(36), and next solve

$$\rho \mathbf{r}_0^\dagger - \dot{\mathbf{r}}_0^\dagger = \theta_{0,2}, \qquad \rho \mathbf{r}_\kappa^\dagger - \dot{\mathbf{r}}_\kappa^\dagger = \theta_{\kappa,2}, \quad 1 \le \kappa \le K,$$

where $\mathbf{r}_0^\dagger(T) = \eta_{0f}^T Q_{0f} \eta_{0f}$ and $\mathbf{r}_\kappa^\dagger(T) = \eta_f^T Q_f \eta_f$. By the relation (A.3)–(A.4), then $(V_0, \cdots, V_K)$ given by (29)–(30) is a solution to the master equations. This completes the proof of the theorem.

References

[1] M. Bardi and M. Fischer (2017), *On non-uniqueness and uniqueness of solutions in finite-horizon mean field games*, arXiv:1707.00628. MR4009550

[2] T. Basar and G. J. Olsder (1999), *Dynamic Noncooperative Game Theory*, 2nd ed., SIAM, Philadelphia. MR1657965

[3] E. Bayraktar and X. Zhang (2020), *On non-uniqueness in mean field games*, Proc. Amer. Math. Soc., **148**:4091–4106. MR4127851

[4] A. Bensoussan, M. Chau, Y. Lai and P. Yam (2017), *Linear-quadratic mean field Stackelberg games with state and control delays*, SIAM Journal on Control and Optimization, **55**(4):2748–2781. MR3691722

[5] A. Bensoussan, M. H. M. Chau and S. C. P. Yam (2015), *Mean field Stackelberg games: Aggregation of delayed instructions*, SIAM Journal on Control and Optimization, **53**(4):2237–2266. MR3376121

[6] A. Bensoussan, M. H. M. Chau and S. C. P. Yam (2016), *Mean field games with a dominating player*, Appl. Math. Optim., **74**(1):91–128. MR3518868

[7] A. Bensoussan, J. Frehse and P. Yam (2013), *Mean Field Games and Mean Field Type Control Theory*, New York: Springer. MR3134900

[8] R. Buckdahn, J. Li and S. Peng (2014), *Nonlinear stochastic differential games involving a major player and a large number of collectively acting minor agents*, SIAM Journal on Control and Optimization, **52**(1):451–492. MR3163233

[9] P. E. Caines, M. Huang and R. P. Malhamé (2017), *Mean Field Games*, In Handbook of Dynamic Game Theory, T. Basar and G. Zaccour Eds., 345–372, Berlin: Springer. MR4139285

[10] P. E. Caines and A. C. Kizilkale (2017), *ϵ-Nash equilibria for partially observed LQG mean field games with a major player*, IEEE Trans. Autom. Control, **62**(7):3225–3234. MR3669444

[11] P. Cardaliaguet (2012), *Notes on mean field games*, Univ. Paris, Dauphine.

[12] P. Cardaliaguet, M. Cirant and A. Porretta (2018), *Remarks on Nash equilibria in mean field game models with a major player*, arXiv:1811.02811. MR4135293

[13] P. Cardaliaguet, F. Delarue, J.-M. Lasry and P.-L. Lions (2015), *The master equation and the convergence problem in mean field games*, arXiv:1509.02505. MR3616319

[14] R. Carmona and F. Delarue (2018), *Probabilistic Theory of Mean Field Games with Applications*, vols. I and II, Cham: Springer. MR3753660

[15] R. Carmona and P. Wang (2017), *An alternative approach to mean field game with major and minor players, and applications to herders impacts*, Applied Mathematics and Optimization, **76**(1):5–27. MR3679337

[16] R. Carmona and X. Zhu (2016), *A probabilistic approach to mean field games with major and minor players*, Ann. Appl. Probab., **26**(3):1535–1580. MR3513598

[17] R. Elie, T. Mastrolia and D. Possamai (2019), *A tale of a principal and many, many agents*, Mathematics of Operations Research, **44**(2):440–467. MR3959080

[18] D. Firoozi and P.E. Caines (2015), *Nash equilibria for partially observed LQG mean field games with major agent: Partial observations by all agents*, In Proc. 54th IEEE Conference on Decision and Control, Osaka, Japan, pp. 4430–4437. MR3579694

[19] D. Firoozi, S. Jaimungal and P. E. Caines (2018), *Convex analysis for LQG systems with applications to major minor LQG mean field game systems*, arXiv:1810.07551. MR4120002

[20] M. Fischer (2017), *On the connection between symmetric N-player games and mean field games*, Ann. Appl. Probab., **27**(2):757–810. MR3655853

[21] G. Fu and U. Horst (2018), *Mean-field leader-follower games with terminal state constraint*, arXiv:1809.04401. MR4127094

[22] D. A. Gomes, L. Nurbekyan, and M. Prazeres (2018), *One-dimensional stationary mean-field games with local coupling*, Dynamic Games Appl., **8**(2):315–351. MR3784965

[23] B. Hajek and M. Livesay (2019), *On non-unique solutions in mean field games*, arXiv:1903.05788.

[24] S. Hart (1973), *Values of mixed games*, Internat. J. Game Theory, **2**:69–86. MR0329654

[25] J. Huang, S. Wang and Z. Wu (2016), *Backward-forward linear-quadratic mean-field games with major and minor agents*, Probability, Uncertainty and Quantitative Risk, 1:8 DOI 10.1186/s41546-016-0009-9. MR3583181

[26] M. Huang (2010), *Large-population LQG games involving a major player: the Nash certainty equivalence principle*, SIAM J. Control Optim., **48**(5):3318–3353. MR2599921

[27] M. Huang, P. E. Caines and R. P. Malhamé (2007), *Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized ε-Nash equilibria*, IEEE Trans. Autom. Contr., **52**(9):1560–1571. MR2352434

[28] M. Huang, R. P. Malhamé and P.E. Caines (2006), *Large population stochastic dynamic games: Closed-loop McKean-Vlasov systems and*

the Nash certainty equivalence principle, Commun. Inform. Systems, **6**(3):221–252. MR2346927

[29] M. Huang and X. Yang (2020), *Mean field Stackelberg games: State feedback equilibrium*, Proc. 21st IFAC World Congress, Berlin.

[30] M. Huang and M. Zhou (2020), *Linear quadratic mean field games: Asymptotic solvability and relation to the fixed point approach*, IEEE Transactions on Automatic Control, **65**(4):1397–1412. MR4085517

[31] X. Huang, S. Jaimungal and M. Nourian (2019), *Mean-field game strategies for optimal execution*, Applied Mathematical Finance, **26**(2):153–185. MR3953622

[32] V. Kolokoltsov (2017), *The evolutionary game of pressure (or interference), resistance and collaboration*, Math. Oper. Res., **42**(4):915–944. MR3722421

[33] I. Kordonis and G. P. Papavassilopoulos (2015), *LQ Nash games with random entrance: an infinite horizon major player and minor players of finite horizons*, IEEE Trans. Automat. Control, **60**(6):1486–1500. MR3353388

[34] D. Lacker (2016), *A general characterization of the mean field limit for stochastic differential games*, Probab. Theory Relat. Fields, **165**:581–648. MR3520014

[35] J.-M. Lasry and P.-L. Lions (2018), *Mean-field games with a major player*, C. R. Math. Acad. Sci. Paris, **356**(8):886–890. MR3851543

[36] Y. Ma and M. Huang (2020), *Linear quadratic mean field games with a major player: The multi-scale approach*, Automatica, **113**(3). MR4046086

[37] J. W. Milnor and L. S. Shapley (1978), *Values of large games II: Oceanic games*, Math. Oper. Res., **3**:290–307. MR0509666

[38] J. Moon and T. Basar (2018), *Linear-quadratic stochastic differential Stackelberg games with a high population of followers*, Automatica, **97**:200–213. MR3857463

[39] S. L. Nguyen and M. Huang (2012), *Linear-quadratic-Gaussian mixed games with continuum-parametrized minor players*, SIAM J. Control Optim., **50**(5):2907–2937. MR3022092

[40] M. Nourian and P.E. Caines (2013), *ϵ-Nash mean field game theory for nonlinear stochastic dynamical systems with major and minor*

agents, SIAM Journal on Control and Optimization, **51**(4):3302–3331. MR3092738

[41] N. Sen and P.E. Caines (2016), *Mean field game theory with a partially observed major agent*, SIAM J. Control Optim., **54**(6):3174–3224. MR3579694

[42] E. D. Sontag (1998), *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, 2nd ed., New York: Springer. MR1640001

[43] R. F. Tchuendom (2018), *Uniqueness for linear-quadratic mean field games with common noise*, Dyn. Games Appl., **8**(1):199–210. MR3764697

[44] B.C. Wang and J.F. Zhang (2014), *Hierarchical mean field games for multiagent systems with tracking-type costs: Distributed ϵ-Stackelberg equilibria*, IEEE Trans. Autom. Control, **59**(8):2241–2247. MR3245268

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