SIAM J. CONTROL OPTIM.
Vol. 59, No. 6, pp. 4373–4399

GRAPHON MEAN FIELD GAMES AND THEIR EQUATIONS*

PETER E. CAINES[†] AND MINYI HUANG[‡]

Abstract. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks. Furthermore, the study of the decentralized control of such systems has been initiated in which graphon mean field games (GMFG) and the GMFG equations have been formulated for the analysis of noncooperative dynamic games on unbounded networks; in that work, existence and uniqueness results have been introduced for the GMFG equations, together with an ϵ-Nash theory for GMFG systems which relates infinite population equilibria on infinite networks to finite population equilibria on finite networks. Those results are rigorously established in this paper.

Key words. mean field games, networks, graphons

AMS subject classifications. 49N80, 91A16, 91A43, 93E20

DOI. 10.1137/20M136373X

1. Introduction.

One response to the problems arising in the analysis of systems of great complexity is to pass to an appropriately formulated infinite limit. This approach has a distinguished history since it is the conceptual principle underlying the celebrated Boltzmann equation of statistical mechanics and that of the fundamental Navier–Stokes equation of fluid mechanics (see, e.g., [38, 23, 15, 16]). Similarly the Fokker–Planck–Kolmogorov (FPK) equation for the macroscopic flow of probabilities [13, 28] is used to describe a vast range of phenomena which at a micro or mezzo level are modeled via the random interactions of discrete entities.

The work in this paper is formulated within two recent theories which were developed with an analogous motive to that above, namely, the mean field game (MFG) theory for the analysis of equilibria in very large populations of noncooperative agents (see [26, 24, 31, 32, 10, 11, 9]) and the graphon theory of the infinite limits of graphs and networks (see [34, 2, 3, 4, 33]).

A mathematically rigorous study of MFG systems with state values in finite graphs is provided in [22], and MFG systems where the agent subsystems are defined at the nodes (vertices) of finite random Erdös–Rényi graphs are treated in [12]. The system behavior in [22] is subject to a fixed underlying network. The random graphs in [12] have unbounded growth but do not create spatial distinction of the agents due to symmetry properties of the interactions. However, graphon theory gives a rigorous formulation of the notion of limits for infinite sequences of networks of increasing size, and the first application of graphon theory in dynamics appears to be in the work of Medvedev [35, 36] and Kaliuzhnyi-Verbovetskyi and Medvedev [27]. The law of large numbers for graphon mean field systems is proven in [1] as a generalization of results

*Received by the editors September 1, 2020; accepted for publication (in revised form) June 1, 2021; published electronically November 18, 2021. Preliminary results have been presented in conference papers [6, 7].
https://doi.org/10.1137/20M136373X
Funding: The work of the first author was funded by NSERC and AFOSR. The research of the second author was funded by NSERC.
[†]Department of Electrical and Computer Engineering, McGill University, Montreal, QC, H3A 2A7, Canada (peterc@cim.mcgill.ca).
[‡]School of Mathematics and Statistics, Carleton University, Ottawa, ON, K1S 5B6, Canada (mhuang@math.carleton.ca).

TABLE 1
Notation.

G_k	the kth graph in a sequence of graphs
g^k	weights of G_k as a step function
M_k	the number of nodes in G_k
$\mathcal{C}_i$	the cluster of agents residing at node i of G_k
$\mathcal{C}(i)$	the cluster that agent i belongs to
I_i^*, $I^*(i)$	the midpoint of an interval of length $1/M_k$
g	the graphon function
$\mu_\alpha(t)$	the local mean field generated by agents at vertex $\alpha \in [0,1]$
$\mu_G(t)$	an ensemble of local mean fields $(\mu_\alpha(t))_{0\le\alpha\le 1}$
$\mathcal{M}_{[0,T]}$	a class of $\mu_G(\cdot)$ satisfying a Hölder continuity condition
C_T	the space of continuous functions on $[0,T]$
$\mathcal{F}_T$	σ-algebra induced by cylindrical sets in C_T
$(C_T, \mathcal{F}_T, m_\alpha)$	probability measure space for the path space at vertex α
$\mathbf{M}_T$	the set of probability measures on $(C_T, \mathcal{F}_T)$
D_T	Wasserstein metric on $\mathbf{M}_T$
$\mathbf{M}_T^G$	the product space $\prod_{\alpha\in[0,1]} \mathbf{M}_T$
$\mathbf{M}_T^{G0}$, $\mathbf{M}_T^{G1}$	subsets of $\mathbf{M}_T^G$
m_G	an ensemble of measures $(m_\alpha)_{0\le\alpha\le 1} \in \mathbf{M}_T^G$
$\mathrm{Proj}_\alpha(m_G)$	the component m_α at vertex α
$\mathrm{Marg}_t(m_\alpha)$	the time t-marginal of m_α
x_α	the state of a generic agent at vertex α
w_α	the standard Brownian motion of a generic agent at vertex α
$\varphi(t, x_\alpha \vert \mu_G(\cdot); g_\alpha)$	the best response at vertex α with $\mu_G(\cdot)$ given by the GMFG system; abbreviated as $\varphi(t, x_\alpha, g_\alpha)$ or φ_α
$\phi(t, x_\alpha \vert \mu_G(\cdot); g_\alpha)$	the best response at vertex α with respect to an arbitrary $\mu_G(\cdot)$; abbreviated as $\phi_\alpha(t, x_\alpha \vert \mu_G(\cdot))$ or ϕ_α

for standard interacting particle systems. Furthermore, the work in [39] derives the McKean–Vlasov limit for a network of agents described by delay stochastic differential equations that are coupled by randomly generated connections.

The first applications of graphon theory in systems and control theory are those in [18, 19, 17, 20, 21] which treat the centralized and distributed control of arbitrarily large networks of linear dynamical control systems for which a direct solution would be intractable. Approximate control is achieved by solving control problems on the infinite limit graphon and then applying control laws derived from those solutions on the finite network of interest. The analogy with the strategies for finding feedback laws resulting in ϵ-Nash equilibria in the MFG framework is obvious. In this connection we note that work on static game theoretic equilibria for infinite populations on graphons was reported in [37].

A natural framework for the formulation of game theoretic problems involving large populations of agents distributed over large networks is given by the MFG theory defined on graphons. The resulting basic idea and the associated fundamental equations for what we term graphon MFG (GMFG) systems and the GMFG equations are the subject of the current paper and its predecessors [6, 7]. The GMFG equations are of significant generality since they permit the study, in the limit, of both dense and sparse infinite networks of noncooperative dynamical agents. Moreover the classical MFG equations are retrieved as a special case. We observe that an early analysis of linear quadratic Gaussian (LQG) models in MFGs on networks with nonuniform edge weightings can be found in [25]. However, in that work there was no application of graphon theory, and in the uniform system parameter case there is one agent per node and a single mean field, whereas in the present work there is a subpopulation with its own mean field at each node.

The basic ϵ-Nash equilibrium result in MFG theory and its corresponding form in GMFG theory are vital for the application of MFG-derived control laws. This is the case since the solution of the MFG and GMFG equations is necessarily simpler than the effectively intractable task of finding the solution to the game problems for the large finite population systems. Indeed, this was one of the original motives for the creation of MFG theory, and it is a basic feature of graphon systems control theory [18].

The paper is organized as follows. Section 2 provides preliminary materials on graphons. Section 3 introduces the GMFG equation system and proves the existence and uniqueness of a solution. For the decentralized strategies determined by the GMFG equations, an ϵ-Nash equilibrium theorem is proven in section 4. The GMFG equations are illustrated by an LQG example in section 5.

For the reader's convenience, a list of key notation is provided in Table 1.

2. The concept of a graphon. The basic idea of the theory of graphons is that the edge structure of each finite cardinality network is represented by a step function density on the unit square in $\mathbb{R}^2$ on which the so-called cut norm and cut metrics are defined. The set of finite graphs endowed with the cut metric then gives rise to a metric space, and the completion of this space is the space of graphons. Let $\mathbf{G_0^{sp}}$ denote the linear space of bounded symmetric Lebesgue measurable functions $W : [0,1]^2 \to \mathbb{R}$, which are called kernels. The space $\mathbf{G^{sp}}$ of graphons is a subset of $\mathbf{G_0^{sp}}$ and consists of kernels $W : [0,1]^2 \to [0,1]$ which can be interpreted as weighted graphs on the vertex set $[0,1]$. We note that functions $W \in \mathbf{G^{sp}}$ taking values in finite sets satisfy this definition and so, in particular, graphons are defined on finite graphs.

The cut norm of a kernel $W \in \mathbf{G_0^{sp}}$ then has the expression

$$\|W\|_\square = \sup_{M,T\subset[0,1]} \left| \int_{M\times T} W(x,y)dxdy \right|,$$

with the supremum taking over all measurable subsets M and T of $[0,1]$. Denote the set of measure preserving bijections $[0,1] \to [0,1]$ by $S_{[0,1]}$. The *cut metric* between two graphons V and W is then given by $\delta_\square(W,V) = \inf_{\phi\in S_{[0,1]}} \|W^\phi - V\|_\square$, where $W^\phi(x,y) := W(\phi(x),\phi(y))$ and any pair of graphons at zero distance are identified with each other. The space $(\mathbf{G^{sp}}, \delta_\square)$ is compact in the topology given by the cut metric [33]. Furthermore, sets in $(\mathbf{G^{sp}}, \delta_\square)$ which are compact with respect to the L^2 metric are compact with respect to the cut metric. Since $\mathbf{G^{sp}}$ is compact in the cut metric all sequences of graphons have subsequential limits.

In this paper, we start with the modeling of the game of a finite population based on a finite graph. Specifically, the population resides on a weighted finite graph G_k with a set of nodes (or vertices) $\mathcal{V}_k = \{1,\ldots,M_k\}$ and weights $g_{ij}^k \in [0,1]$ for $(i,j) \in \mathcal{V}_k \times \mathcal{V}_k$, where a value g_{ii}^k is assigned in the case $i=j$. We call $g_i^k := (g_{i1}^k,\ldots,g_{iM_k}^k)$ a section of g^k at i. Each node l is occupied by a set of agents which is called a cluster of the population, and hence the number of clusters is M_k. We list the clusters as $\mathcal{C}_1,\ldots,\mathcal{C}_{M_k}$. Without loss of generality, we assume the lth cluster occupies node l. Let $\mathcal{C}(i)$ denote the cluster that agent i belongs to. So $i \in \mathcal{C}(i)$. Our further analysis in the paper is based on the convergence of g^k to a graphon limit g. We may naturally identify $(g_{ij}^k)_{1\le i,j\le M_k}$ with a graphon $g^k(\alpha,\beta)$ as a step function defined on $[0,1]\times[0,1]$ (see [33]). However, convergence in the cut norm or the cut metric is inadequate for the analysis in this paper as it does not capture sufficiently strong sectional information of the difference $g^k - g$. We will adopt a different convergence notion strengthening the sectional requirement as in assumption (H11) below. To indicate its arguments, we may write $g(\alpha, \beta)$ or alternatively $g_{\alpha,\beta}$. We define the section of g at α by $g_\alpha : \beta \mapsto g_{\alpha,\beta}$, $\beta \in [0, 1]$.

Since clusters $\mathcal{C}_{i_1}$ and $\mathcal{C}_{i_2}$ reside on nodes i_1 and i_2 of G_k, respectively, we define $g^k_{\mathcal{C}_{i_1}\mathcal{C}_{i_2}} = g^k_{i_1 i_2}$. Similarly, we define the section $g^k_{\mathcal{C}_i} = g^k_i$.

We partition $[0, 1]$ into M_k subintervals of equal length. Here $I^k_l = [(l-1)/M_k, l/M_k]$ for $1 \le l \le M_k$. When it is clear from the context, we omit the superscript k and write I_l. To relate the clusters of agents to the vertex set $[0, 1]$, we let the cluster $\mathcal{C}_l$ correspond to I_l.

Throughout this paper, $C, C_0, C_1, \dots$ denote generic constants, which do not depend on the graph index k and population size N and may vary from place to place.

3. GMFG systems and the GMFG equations.

3.1. The standard MFG model and its graphon generalization. In the diffusion-based models of large population games the state evolution of N agents $\mathcal{A}_i, 1 \le i \le N$, is specified by a set of N controlled stochastic differential equations (SDEs). A simplified form of the general case is given by the following set of controlled SDEs:

$$dx_i(t) = \frac{1}{N} \sum_{j=1}^{N} f(x_i(t), u_i(t), x_j(t))dt + \sigma dw_i(t), \tag{3.1}$$

where $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^{n_u}$ the control input, and $w_i \in \mathbb{R}^{n_w}$ a standard Brownian motion, and where $\{w_i, 1 \le i \le N\}$ are independent processes. All initial states are taken to be independent and have finite second moment. The cost of agent $\mathcal{A}_i$ is given by

$$J_i^N(u_i, u_{-i}) = E \int_0^T \frac{1}{N} \sum_{j=1}^{N} l(x_i(t), u_i(t), x_j(t))dt, \tag{3.2}$$

where $l(\cdot)$ is the pairwise running cost and u_{-i} denotes the controls of all other agents.

The dynamics of a generic agent $\mathcal{A}_i$ in the infinite population limit of this system is then described by the controlled McKean–Vlasov (MV) equation

$$dx_i = f[x_i, u_i, \mu_t]dt + \sigma dw_i, \quad 0 \le t \le T, \tag{3.3}$$

where μ_t is the distribution of $x_i(t)$, $f[x, u, \mu_t] := \int_{\mathbb{R}^n} f(x, u, y)\mu_t(dy)$ and where the initial distribution μ_0^x of $x_i(0)$ is specified. Setting $l[x, u, \mu_t] = \int_{\mathbb{R}^n} l(x, u, y)\mu_t(dy)$, the corresponding infinite population cost for $\mathcal{A}_i$ takes the form

$$J_i(u_i; \mu(\cdot)) := E \int_0^T l[x_i(t), u_i(t), \mu_t]dt. \tag{3.4}$$

For notational simplicity, we present the GMFG framework with scalar individual states and controls; i.e., $n = n_u = n_w = 1$. Its extension to the vector case is evident.

Now we consider a finite population distributed over the finite graph G_k. Let $\boldsymbol{x}_{G_k} = \bigoplus_{l=1}^{M_k} \{x_i | i \in \mathcal{C}_l\}$ denote the states of all agents in the total set of clusters of the population. This gives a total of $N = \sum_{l=1}^{M_k} |\mathcal{C}_l|$ individual states. The key feature of the GMFG construction beyond the standard MFG scheme is that at any agent in a network the averaged dynamics (3.1) and cost function (3.2) decompose into averages of subpopulations distributed at that agent's neighboring nodes plus an average term for the local cluster. In the limit, the summed subpopulation averages are given by an integral over the local mean fields of the neighboring agents.

For $\mathcal{A}_i$ in the cluster $\mathcal{C}(i)$, two coupling terms in the dynamics take the form

$$f_0(x_i, u_i, \mathcal{C}(i)) = \frac{1}{|\mathcal{C}(i)|} \sum_{j \in \mathcal{C}(i)} f_0(x_i, u_i, x_j), \tag{3.5}$$

$$f_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}) = \frac{1}{M_k} \sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \frac{1}{|\mathcal{C}_l|} \sum_{j \in \mathcal{C}_l} f(x_i, u_i, x_j). \tag{3.6}$$

They model intra- and intercluster couplings, respectively. The specification of f_{G_k} relies on the sectional information $g^k_{\mathcal{C}(i)\bullet}$. Concerning the coupling structure in (3.6) we observe that with respect to $\mathcal{A}_i$, all individuals residing in cluster $\mathcal{C}_l$ are symmetric and their state average generates the overall impact of that cluster on $\mathcal{A}_i$ mediated by the graphon weighting $g^k_{\mathcal{C}(i)\bullet}$. The two coupling terms are combined additively resulting in the local dynamics

$$\tilde{f}_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}) = f_0(x_i, u_i, \mathcal{C}(i)) + f_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}).$$

Note that $\mathcal{A}_i$ interacts with the overall population through a function of the complete system state $\boldsymbol{x}_{G_k}$ and the cluster sizes. These details shall be suppressed in this paper, and we only indicate the graph G_k and the section $g^k_{\mathcal{C}(i)}$. The state process of $\mathcal{A}_i$ is then given by the SDE

$$dx_i(t) = \tilde{f}_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)})dt + \sigma dw_i, \quad 1 \le i \le N,$$

where $\sigma > 0$ and the initial states $\{x_i(0), 1 \le i \le N\}$ are independent and identically distributed (i.i.d.) with distribution $\mu^x_0 \in \mathcal{P}_1(\mathbb{R})$, the set of probability measures on $\mathbb{R}$ with finite mean.

The limit of the two dynamic coupling terms of an agent at a node α (called an α-agent), as the number of nodes of the graph G_k and the subpopulation at each node tend to infinity, is described by the expressions

$$f_0[x_\alpha, u_\alpha, \mu_\alpha] := \int_{\mathbb{R}} f_0(x_\alpha, u_\alpha, z)\mu_\alpha(dz), \tag{3.7}$$

$$f[x_\alpha, u_\alpha, \mu_G; g_\alpha] := \int_0^1 \int_{\mathbb{R}} f(x_\alpha, u_\alpha, z) g(\alpha, \beta)\mu_\beta(dz)d\beta, \tag{3.8}$$

which give the complete local graphon dynamics via

$$\widetilde{f}[x_\alpha, u_\alpha, \mu_G; g_\alpha] := f_0[x_\alpha, u_\alpha, \mu_\alpha] + f[x_\alpha, u_\alpha, \mu_G; g_\alpha]. \tag{3.9}$$

We call μ_β the local mean field at node β, which is interpreted as the limit of the empirical distributions of agents at node β. $\mu_G = \{\mu_\beta, 0 \le \beta \le 1\}$ is the ensemble of local mean fields. Due to the integration with respect to β, the dependence of $\widetilde{f}$ on the graphon limit g is through the section g_α. Since μ_G contains μ_α, we do not list μ_α as an argument of $\widetilde{f}$.

Parallel to the standard MFG case, in the graphon case the SDE

$$\begin{aligned}&[\text{MV-SDE}](\alpha) \quad dx_\alpha(t) = \widetilde{f}[x_\alpha(t), u_\alpha(t), \mu_G(t); g_\alpha]dt + \sigma dw_\alpha(t),\\ &0 \le t \le T, \quad \alpha \in [0,1],\end{aligned} \tag{3.10}$$

generalizes the standard controlled MV equation (3.3). We note that in a parallel development of graphon-based stochastic dynamical populations [1] the system disturbance intensity σ is also a function of graphon-weighted state functions at other clusters. For simplicity, we consider a constant σ, and our analysis may be generalized to the case of a state and mean field dependent diffusion term. Similarly, for simplicity our dynamics and cost do not include a separate parametrization by α.

Analogously, in the GMFG case, we define the cost coupling terms for $\mathcal{A}_i$ to be

$$l_0(x_i, u_i, \mathcal{C}(i)) = \frac{1}{|\mathcal{C}(i)|} \sum_{j \in \mathcal{C}(i)} l_0(x_i, u_i, x_j),$$

$$l_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}) = \frac{1}{M_k} \sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \frac{1}{|\mathcal{C}_l|} \sum_{j \in \mathcal{C}_l} l(x_i, u_i, x_j).$$

Define $\tilde{l}_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}) = l_0(x_i, u_i, \mathcal{C}(i)) + l_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)})$. The cost of $\mathcal{A}_i$ in a finite population on a finite graph G_k is given in the form

$$J_i = E \int_0^T \tilde{l}_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}) dt. \tag{3.11}$$

Denote

$$l_0[x_\alpha, u_\alpha, \mu_\alpha] = \int_{\mathbb{R}} l_0(x_\alpha, u_\alpha, z) \mu_\alpha(dz),$$

$$l[x_\alpha, u_\alpha, \mu_G; g_\alpha] = \int_0^1 \int_{\mathbb{R}} l(x_\alpha, u_\alpha, z) g(\alpha, \beta) \mu_\beta(dz) d\beta,$$

$$\widetilde{l}[x_\alpha, u_\alpha, \mu_G; g_\alpha] = l_0[x_\alpha, u_\alpha, \mu_\alpha] + l[x_\alpha, u_\alpha, \mu_G; g_\alpha].$$

In the infinite population graphon case, the individual α-agent has the cost function

$$J_\alpha(u_\alpha; \mu_G(\cdot)) = E \int_0^T \widetilde{l}[x_\alpha(t), u_\alpha(t), \mu_G(t); g_\alpha] dt. \tag{3.12}$$

3.2. The GMFG model and its equations. In this section the standard MFG equations (see, e.g., [5, 9]) will be generalized so that they subsume the standard (implicitly uniform totally connected) dense network case and cover the fully general graphon limit network case. Specifically, agent $\mathcal{A}_i$ in a population of N agents will be located at the lth node in an M_k node network (identified with its graphon), and in the infinite population graphon limit that node will be taken to map to $\alpha \in [0,1]$. It is important to note here that *although the limit network is assumed dense it is not assumed to be uniformly totally connected*; indeed, the connection structure of the infinite network is represented precisely by its graphon $g(\alpha, \beta)$, $0 \le \alpha, \beta \le 1$.

The generalized GMFG scheme below on $[0,T]$ is given for each α by (i) the Hamilton–Jacobi–Bellman (HJB) equation generating the value function V^α when all other agents' control laws and the ensemble μ_G of local mean fields are given, (ii) the

FPK equation generating the local mean field μ_α given μ_G, and (iii) the specification of the best response (BR) feedback law.

Suppressing the time index on the measures for simplicity of notation, we have the *GMFG equations*:

$$
\begin{aligned}
&[\mathrm{HJB}](\alpha) \quad -\frac{\partial V^\alpha(t,x)}{\partial t} = \inf_{u\in U}\Bigg\{\widetilde{f}[x,u,\mu_G;g_\alpha]\frac{\partial V^\alpha(t,x)}{\partial x} \\
&\qquad\qquad + \widetilde{l}[x,u,\mu_G;g_\alpha]\Bigg\} + \frac{\sigma^2}{2}\frac{\partial^2 V^\alpha(t,x)}{\partial x^2}, \qquad (3.13)\\
&V^\alpha(T,x)=0, \quad (t,x)\in[0,T]\times\mathbb{R}, \quad \alpha\in[0,1],
\end{aligned}
$$

$$
\begin{aligned}
&[\mathrm{FPK}](\alpha) \quad \frac{\partial p_\alpha(t,x)}{\partial t} = -\frac{\partial\{\widetilde{f}[x,u^0,\mu_G;g_\alpha]p_\alpha(t,x)\}}{\partial x} \\
&\qquad\qquad + \frac{\sigma^2}{2}\frac{\partial^2 p_\alpha(t,x)}{\partial x^2}, \qquad (3.14)
\end{aligned}
$$

$$
[\mathrm{BR}](\alpha) \quad u^0 := \varphi(t,x|\mu_G;g_\alpha).
$$

Here $p_\alpha(t,x)$ with initial condition $p_\alpha(0)$ is used to denote the density of the measure $\mu_\alpha(t)$ whenever a density is assumed to exist. In this paper, the FPK equation will be replaced by the following closed-loop MV-SDE:

$$
[\mathrm{MV}](\alpha) \quad dx_\alpha(t) = \widetilde{f}[x_\alpha(t),\varphi(t,x_\alpha(t)|\mu_G;g_\alpha),\mu_G(t);g_\alpha]dt + \sigma dw_\alpha(t), \qquad (3.15)
$$

where $x_\alpha(0)$ has initial distribution μ_0^x. Our subsequent analysis will directly treat the pair $(V^\alpha(t,x),\mu_\alpha(t))$, where $\mu_\alpha(t)$ is specified as the law of $x_\alpha(t)$ in (3.15).

If a solution exists for the GMFG equations, the resulting BR $\varphi(t,x|\mu_G;g_\alpha)$ depends upon the ensemble μ_G of local mean fields and the individual agent's state. This is a natural generalization of the standard case. The standard MFG case is simply obtained by setting $g(\alpha,\beta)\equiv 0, 0\le\alpha,\beta\le 1$, which results in $\widetilde{f}[x,u,\mu_G;g_\alpha] = f_0[x,u,\mu]$ and $\widetilde{l}[x,u,\mu_G;g_\alpha] = l_0[x,u,\mu]$ [5, 9].

A collection of measures on some measurable space which are indexed by the vertex set $[0,1]$ is called a measure ensemble. Thus, for each fixed t, $\mu_G(t)$ is a measure ensemble.

On $\mathcal{P}_1(\mathbb{R})$ we endow the Wasserstein metric W_1: for any $\mu,\nu\in\mathcal{P}_1(\mathbb{R})$, $W_1(\mu,\nu) = \inf_{\widehat{\gamma}}\int|x-y|\widehat{\gamma}(dx,dy)$, where $\widehat{\gamma}$ is a probability measure on $\mathbb{R}^2$ with marginals μ,ν.

Let $C([0,1],\mathcal{P}_1(\mathbb{R}))$ be the set of measure ensembles $\nu_G = (\nu_\beta)_{\beta\in[0,1]}$ satisfying $\nu_\beta\in\mathcal{P}_1(\mathbb{R})$, and $\lim_{\beta'\to\beta}W_1(\nu_{\beta'},\nu_\beta)=0$ for any $\beta\in[0,1]$.

In order to analyze the solvability of the GMFG equations, we need to restrict $\mu_G(\cdot)$ to a certain class. We say $\{\mu_G(t), 0\le t\le T\}$ is from the admissible set $\mathcal{M}_{[0,T]}$ if and only if the following apply:

(C1) For each fixed t, $\mu_G(t)$ is in $C([0,1],\mathcal{P}_1(\mathbb{R}))$.

(C2) There exists $\eta\in(0,1]$ such that for any bounded and Lipschitz continuous function ϕ on $\mathbb{R}$,

$$
\sup_{\beta\in[0,1]}\left|\int_{\mathbb{R}}\phi(y)\mu_\beta(t_1,dy) - \int_{\mathbb{R}}\phi(y)\mu_\beta(t_2,dy)\right| \le C_h|t_1-t_2|^\eta,
$$

where C_h may be selected to depend only on the Lipschitz constant $\mathrm{Lip}(\phi)$ for ϕ.

Condition (C1) ensures that integration with respect to $d\beta$ in (3.8) is well defined. Condition (C2) ensures that the drift term in the HJB equation (3.13) has a certain time continuity, which facilitates the subsequent existence analysis of the BR.

3.3. Existence analysis. We introduce the following assumptions:

(H1) U is a compact set.

(H2) $f_0(x,u,y)$, $f(x,u,y)$, $l_0(x,u,y)$, and $l(x,u,y)$ are continuous and bounded functions on $\mathbb{R}\times U\times\mathbb{R}$ and are Lipschitz continuous in (x,y), uniformly with respect to u.

(H3) $f_0(x,u,y)$ and $f(x,u,y)$ are Lipschitz continuous in u, uniformly with respect to (x,y).

(H4) For any $q\in\mathbb{R}$, $\alpha\in[0,1]$, and probability measure ensemble $\nu_G\in C([0,1],\mathcal{P}_1(\mathbb{R}))$, the set

$$S_\alpha^{\nu_G}(x,q)=\arg\min_{u\in U}\{q(\widetilde{f}[x,u,\nu_G;g_\alpha])+\widetilde{l}[x,u,\nu_G;g_\alpha]\}$$

is a singleton, and for any given compact interval $\mathcal{I}=[\underline{q},\bar{q}]$, the resulting u as a function of $(x,q)\in\mathbb{R}\times\mathcal{I}$ is Lipschitz continuous in (x,q), uniformly with respect to ν_G and g_α, $0\le\alpha\le1$.

The next two assumptions will be used to ensure that the BRs have continuous dependence on α. In particular, (H5) is a continuity assumption on the graphon function $g(\alpha,\beta)$. Under (H5), $\widetilde{f}$ and $\widetilde{l}$ have continuity in α.

(H5) For any bounded and measurable function $h(\beta)$, the function $\int_0^1 g(\alpha,\beta)h(\beta)d\beta$ is continuous in $\alpha\in[0,1]$.

(H6) For given $\nu_G\in C([0,1],\mathcal{P}_1(\mathbb{R}))$, $S_\alpha^{\nu_G}(x,q)$ is continuous in (α,x,q).

Although the GMFG equation system only involves $\{\mu_G(t),0\le t\le T\}$, which may be viewed as a collection of marginals at different vertices, it is necessary to develop the existence analysis in the underlying probability spaces (see related discussions in [26, p. 240]).

We begin by introducing some analytic preliminaries. For the space $C_T=C([0,T],\mathbb{R})$, we specify a σ-algebra $\mathcal{F}_T$ induced by all cylindrical sets of the form $\{x(\cdot)\in C_T: x(t_i)\in B_i, 1\le i\le j \text{ for some } j\}$, where B_i is a Borel set. Let $\mathbf{M}_T$ denote the space of probability measures on $(C_T,\mathcal{F}_T)$. The canonical process X is defined by $X_t(\omega)=\omega_t$ for $\omega\in C_T$. On C_T, we define the metric $\rho(x,y)=\sup_t|x(t)-y(t)|\wedge 1$. Then (C_T,ρ) is a complete metric space. Based on ρ, we introduce the Wasserstein metric on $\mathbf{M}_T$. For $m_1,m_2\in\mathbf{M}_T$, denote

$$D_T(m_1,m_2)=\inf_{\widehat{m}}\int_{C_T\times C_T}\left(\sup_{s\le T}|X_s(\omega_1)-X_s(\omega_2)|\wedge 1\right)d\widehat{m}(\omega_1,\omega_2),$$

where $\widehat{m}$ is called a coupling as a probability measure on $(C_T,\mathcal{F}_T)\times(C_T,\mathcal{F}_T)$ with the pair of marginals m_1 and m_2, respectively. Then $(\mathbf{M}_T,D_T)$ is a complete metric space [41].

We introduce the product of probability measure spaces $\prod_{\alpha\in[0,1]}(C_T,\mathcal{F}_T,m_\alpha)$, where each individual space is interpreted as the path space of the agent at vertex α with a corresponding probability measure m_α. Denote the product of spaces of probability measures $\mathbf{M}_T^G=\prod_{\alpha\in[0,1]}\mathbf{M}_T$. An element in $\mathbf{M}_T^G$ is a measure ensemble. Given $m_G\in\mathbf{M}_T^G$, the projection operator Proj_α picks out its component m_α associated with $\alpha\in[0,1]$. Let $\mathbf{M}_T^{G0}$ consist of all $(m_\alpha)_{\alpha\in[0,1]}\in\mathbf{M}_T^G$ such that for any $\alpha\in[0,1]$, $D_T(m_{\alpha'},m_\alpha)\to0$ as $\alpha'\to\alpha$.

For two measure ensembles $m_G:=(m_\alpha)_{\alpha\in[0,1]}$ and $\bar{m}_G:=(\bar{m}_\alpha)_{\alpha\in[0,1]}$ in $\mathbf{M}_T^G$, define $d(m_G,\bar{m}_G)=\sup_{\alpha\in[0,1]}D_T(m_\alpha,\bar{m}_\alpha)$.

LEMMA 3.1. *$(\mathbf{M}_T^G, d)$ is a complete metric space.*

Proof. If $\{m_G^k, k \geq 1\}$ is a Cauchy sequence in $\mathbf{M}_T^G$, then for each given α, the sequence $\{\text{Proj}_\alpha(m_G^k), k \geq 1\}$ (of probability measures) is a Cauchy sequence in the complete metric space $\mathbf{M}_T$, and so it contains a limit. This in turn determines a limit in $\mathbf{M}_T^G$. ◻

Given the probability measure $m_\alpha \in \mathbf{M}_T$, we define the t-marginal $\mu_\alpha(t)$ by $\mu_\alpha(t, B) = m_\alpha(\{x(\cdot) \in C_T : x(t) \in B\})$ for any Borel set $B \subset \mathbb{R}$ and denote the mapping from $\mathbf{M}_T$ to $\mathcal{P}(\mathbb{R})$ (the set of probability measures on $\mathbb{R}$):

$$\mu_\alpha(t) = \text{Marg}_t(m_\alpha). \tag{3.16}$$

Consider the measure ensemble $m_G = (m_\alpha)_{\alpha\in[0,1]} \in \mathbf{M}_T^G$ with $\mu_\alpha(t)$ given by (3.16). Define the time t-marginals by the following mapping:

$$\text{Marg}_t(m_G) = (\mu_\alpha(t))_{\alpha\in[0,1]}, \tag{3.17}$$

where the right-hand side is simply written as $\mu_G(t)$. For a given t, $\mu_G(t)$ may be interpreted as a measure valued function defined on the vertex set $[0, 1]$. Further denote the mapping $\text{Marg}(m_G) = (\mu_G(t))_{t\in[0,T]} = \mu_G(\cdot)$.

Take a fixed $\mu_G(\cdot) \in \mathcal{M}_{[0,T]}$ with its associated Hölder parameter η in (C2), and denote

$$\widetilde{f}_\alpha^*(t, x, u) = \widetilde{f}[x, u, \mu_G(t); g_\alpha], \quad \widetilde{l}_\alpha^*(t, x, u) = \widetilde{l}[x, u, \mu_G(t); g_\alpha].$$

LEMMA 3.2. *Assume* (H1) and (H2). *For $h_\alpha = \widetilde{f}_\alpha^*(t, x, u)$ or $\widetilde{l}_\alpha^*(t, x, u)$, there exist constants C and C_{μ_G}, where the latter depends on $\mu_G(\cdot)$, such that*

$$\sup_{t\leq T, u\in U, \alpha\in[0,1]} |h_\alpha(t, x, u) - h_\alpha(t, y, u)| \leq C|x - y|,$$
$$\sup_{x\in\mathbb{R}, u\in U, \alpha\in[0,1]} |h_\alpha(t, x, u) - h_\alpha(s, x, u)| \leq C_{\mu_G}|t - s|^\eta.$$

Proof. The Lipschitz continuity of $\widetilde{f}_\alpha^*$ with respect to x follows from (H2), (3.7), and (3.8). For $t_1, t_2 \in [0, T]$, we estimate $|\widetilde{f}[x, u, \mu_G(t_1); g_\alpha] - \widetilde{f}[x, u, \mu_G(t_2); g_\alpha]|$ by using the Lipschitz condition of f_0, f and condition (C2) for $\mathcal{M}_{[0,T]}$. This establishes the Hölder continuity of $\widetilde{f}_\alpha^*$ in t. The other cases can be similarly checked. ◻

In order to analyze the BR of the α-agent, we introduce the HJB equation

$$-V_t^\alpha(t, x) = \inf_{u\in U}\{\widetilde{f}_\alpha^*(t, x, u)V_x^\alpha(t, x) + \widetilde{l}_\alpha^*(t, x, u)\} + \frac{\sigma^2}{2}V_{xx}^\alpha(t, x), \tag{3.18}$$

where $V^\alpha(T, 0) = 0$. It differs from (3.13) by allowing an arbitrary $\mu_G(\cdot) \in \mathcal{M}_{[0,T]}$.

For studying (3.18), we introduce some standard definitions. Denote $Q_T = (0, T) \times \mathbb{R}$ and $\overline{Q}_T = [0, T] \times \mathbb{R}$. Let $C^{1,2}(\overline{Q}_T)$ (resp., $C^{1,2}(Q_T)$) denote the set of functions with continuous derivatives v_t, v_x, v_{xx} on $\overline{Q}_T$ (resp., Q_T). Let $C_b^{1,2}(\overline{Q}_T)$ be the set of bounded functions in $C^{1,2}(\overline{Q}_T)$, and let the open (or closed) set Q_b be a bounded subset of Q_T. $W_\lambda^{1,2}(Q_b)$, $1 \leq \lambda < \infty$, shall denote the Sobolev space consisting of functions v such that each v and its generalized derivatives v_t, v_x, v_{xx} are in $L^\lambda(Q_b)$; further we have the norm

$$\|v\|_{\lambda,Q_b}^{(2)} = \|v\|_{\lambda,Q_b} + \|v_t\|_{\lambda,Q_b} + \|v_x\|_{\lambda,Q_b} + \|v_{xx}\|_{\lambda,Q_b}, \tag{3.19}$$

where $\|v\|_{\lambda,Q_b} = (\int_{Q_b} |v(t,x)|^\lambda dtdx)^{1/\lambda}$. Set $|v|_{Q_b} = \sup_{(t,x)\in Q_b} |v(t,x)|$. For $Q_b = (T_1,T_2)\times\mathcal{I}$, where $\mathcal{I}$ is a bounded open subset of $\mathbb{R}$ and $\beta\in(0,1)$, define the Hölder norms

$$|v|^\beta_{Q_b} = |v|_{Q_b} + \sup_{t\in(T_1,T_2),x,y\in\mathcal{I}} |v(t,x)-v(t,y)|\cdot|x-y|^{-\beta} + \sup_{s,t\in(T_1,T_2),x\in\mathcal{I}} |v(s,x)-v(t,x)|\cdot|s-t|^{-\beta/2},$$

$$|v|^{1+\beta}_{Q_b} = |v|^\beta_{Q_b} + |v_x|^\beta_{Q_b}, \qquad |v|^{2+\beta}_{Q_b} = |v|^{1+\beta}_{Q_b} + |v_t|^\beta_{Q_b} + |v_{xx}|^\beta_{Q_b}.$$

LEMMA 3.3. *Under* (H1)–(H4) *and for fixed* $\mu_G(\cdot)\in\mathcal{M}_{[0,T]}$, *the following holds:*

(i) *Equation* (3.18) *has a unique solution* V^α *in* $C^{1,2}_b(\overline{Q}_T)$, *and moreover* $\sup_{\overline{Q}_T} |V^\alpha_{xx}| \le C$.

(ii) *The BR*

$$u_\alpha = \phi_\alpha(t,x|\mu_G(\cdot)), \quad \alpha\in[0,1] \tag{3.20}$$

as the optimal control law solved from (3.18) *is bounded and Borel measurable on* $[0,T]\times\mathbb{R}$, *and Lipschitz continuous in* x, *uniformly with respect to* α *for the given* $\mu_G(\cdot)$.

Proof. (i) Denote $\boldsymbol{H}_\alpha(t,x,q) = \min_{u\in U}\{q\widetilde{f}^*_\alpha(t,x,u) + \widetilde{l}^*_\alpha(t,x,u)\}$. Then (3.18) may be rewritten as

$$-V^\alpha_t(t,x) = \boldsymbol{H}_\alpha(t,x,V^\alpha_x) + \frac{\sigma^2}{2}V^\alpha_{xx}, \qquad V^\alpha(T,x)=0. \tag{3.21}$$

As in the proof of [26, Thm. 5], we use Hölder and Lipschitz continuity (with respect to t and x, respectively) of $\widetilde{f}^*_\alpha$ and $\widetilde{l}^*_\alpha$ in Lemma 3.2 and follow the method in the proof of Theorem VI.6.2 of [14, p. 210] to show that (3.18) has a unique solution $V^\alpha\in C^{1,2}_b(\overline{Q}_T)$, where uniqueness follows from a verification theorem using the closed-loop state process.

Next we show that V^α_{xx} is bounded on $\overline{Q}_T$. Take any $x_0\in\mathbb{R}$. Denote $B_r(x_0) = (x_0-r,x_0+r)$ for $r>0$, and $Q^{x_0,r}_T = (0,T)\times B_r(x_0)$. We use two steps involving local estimates. Each step gets refined information about V^α in a region based on available bound information in a larger region. It suffices to obtain a bound of V^α_{xx} on $Q^{x_0,1}_T$ as long as this bound does not change with x_0.

Step 1. First, there exists a constant C_1 such that

$$\sup_{t,x,\alpha}|V^\alpha|\le C_1, \quad \sup_{t,x,\alpha}|V^\alpha_x|\le C_1. \tag{3.22}$$

The first inequality is obtained using (H1) and (H2) and the fact that V^α is the value function of the associated optimal control problem. The second inequality is proven by the difference estimate of $|V^\alpha(t,x)-V^\alpha(t,y)|$ as in [14, p. 209].

By (H1), (H2), and (3.22), we have $\sup_\alpha\sup_{(t,x)\in\overline{Q}_T}|\boldsymbol{H}_\alpha(t,x,V^\alpha_x(t,x))|\le C_2$.

We use a typical method for analyzing semilinear parabolic equations. Once V^α is known to be a solution of (3.21), we view V^α as the solution of a linear equation with the free term $\boldsymbol{H}_\alpha(t,x,V^\alpha_x)$. For further estimates, we need $\lambda>n+2$ when using the norm (3.19). Fix $\lambda=n+3=4$. This yields the bound $\|V^\alpha\|^{(2)}_{\lambda,Q^{x_0,2}_T}\le C_3$, where C_3 depends on (C_2,T,σ) and the bound of (f,f_0,l,l_0) but not on x_0, α; see [14, p. 207] and also [30, p. 342] for local estimates of the Sobolev norm of solutions defined on unbounded domain using a cutoff function. Take $\beta = 1-(n+2)/\lambda = 1/4$. Subsequently, since $\lambda > n+2$, we have the Hölder estimate

$$|V^\alpha|^{1+\beta}_{Q_T^{x_0,2}} \le C_4 \|V^\alpha\|^{(2)}_{\lambda, Q_T^{x_0,2}} \le C_3 C_4, \tag{3.23}$$

where C_4 is determined by $\lambda = 4$ without depending on x_0, α; see [14, p. 207], [30, p. 343].

Step 2. On $[0,T]\times\mathbb{R}\times[-C_1, C_1]$, we can show $\boldsymbol{H}_\alpha(t,x,q)$ is Hölder continuous in t and Lipschitz continuous in (x,q). Denote $\beta_1 = \min\{\eta, \beta\}$. Next we view $\boldsymbol{H}_\alpha(t,x,V_x^\alpha(t,x))$ as a function of (t,x). Then by use of (3.23) we further obtain a bound on the Hölder norm:

$$\sup_\alpha \sup_{x_0} |\boldsymbol{H}_\alpha(\cdot,\cdot,V_x^\alpha)|^{\beta_1}_{Q_T^{x_0,2}} \le C_5. \tag{3.24}$$

Subsequently, by the method in [14, pp. 207–208] with its cutoff function technique and [30, pp. 351–352], we use (3.24) and local Hölder estimates of (3.21) to obtain

$$|V^\alpha|^{2+\beta_1}_{Q_T^{x_0,1}} \le C_6, \tag{3.25}$$

where C_6 depends on C_5 but not on x_0, α. Since x_0 is arbitrary, it follows that

$$\sup_\alpha \sup_{\overline{Q}_T} |V^\alpha_{xx}| \le C_6. \tag{3.26}$$

(ii) By (H4), the optimal control law (3.20) as a function of (t,x) is well defined and is bounded on $[0,T]\times\mathbb{R}$ by compactness of U. It is Borel measurable on $\overline{Q}_T$; see [14, p. 168]. Since $S_\alpha^{\nu_G}(x,q)$ is Lipschitz continuous in $(x,q)\in\mathbb{R}\times[-C_1,C_1]$ and $V_x^\alpha(t,x)$ is Lipschitz continuous in $x\in\mathbb{R}$ by (3.26), uniformly with respect to α in each case, ϕ_α is uniformly Lipschitz continuous in x. ∎

Denote

$$\Psi^\alpha(t,x) = (V^\alpha(t,x), V_t^\alpha(t,x), V_x^\alpha(t,x), V_{xx}^\alpha(t,x)), \quad (t,x)\in\overline{Q}_T.$$

We prove the following continuity lemma for the solution of (3.18). For $\overline{Q}_T$, define the compact subsets $B_j = \{(t,x) | 0\le t\le T, |x|\le j\}$, $j\in\mathbb{N}$.

LEMMA 3.4. *Assume* (H1)–(H5), *and let* $\mu_G(\cdot)\in\mathcal{M}_{[0,T]}$ *in* (3.18) *be fixed. Then the following holds:*

(i) *For all compact set* B_j, $\lim_{\alpha'\to\alpha} |\Psi^{\alpha'} - \Psi^\alpha|_{B_j} = 0$.

(ii) $\lim_{\alpha'\to\alpha} V_x^{\alpha'}(t,x) = V_x^\alpha(t,x)$ *for all* $(t,x)\in[0,T]\times\mathbb{R}$.

Proof. It suffices to show (i) as (ii) follows immediately from (i).

Step 1. By (3.25) and the fact that the constant C_6 can be selected without depending on α, there exists a constant C such that $\sup_\alpha |V^\alpha|^{2+\beta_1}_{B_j} \le C$, which implies that $\{\Psi^\alpha, \alpha\in[0,1]\}$ is uniformly bounded and equicontinuous on B_j. For any sequence $\{\alpha_k, k\ge 1\}$ converging to α, by Ascoli–Arzela's lemma, for $j=1$, there exists a subsequence denoted by $\{\bar\alpha_k, k\ge1\}$ such that $\Psi^{\bar\alpha_k}$ converges uniformly on B_1. By a diagonal argument, we may further extract a subsequence of $\{\bar\alpha_k, k\ge1\}$, denoted by $\{\hat\alpha_k, k\ge 1\}$, such that $\Psi^{\hat\alpha_k}$ converges uniformly on each set B_j, $j\ge1$. Hence there exists a function V^* with continuous derivatives V_t^*, V_x^*, V_{xx}^* on $\overline{Q}_T$ such that

$$\lim_{k\to\infty} \Psi^{\hat{\alpha}_k}(t,x) = \Psi^*(t,x) \qquad \text{for all } (t,x) \in \overline{Q}_T, \tag{3.27}$$

where $\Psi^* = (V^*, V_t^*, V_x^*, V_{xx}^*)$. Since

$$-V_t^{\hat{\alpha}_k}(t,x) = \boldsymbol{H}_{\alpha_k}(t,x,V_x^{\hat{\alpha}_k}) + \frac{\sigma^2}{2}V_{xx}^{\hat{\alpha}_k}, \quad V^{\alpha_k}(T,x) = 0,$$

it follows from (3.27) that

$$-V_t^*(t,x) = \boldsymbol{H}_{\alpha}(t,x,V_x^*) + \frac{\sigma^2}{2}V_{xx}^*, \qquad V^*(T,x) = 0.$$

We have used the fact that $\boldsymbol{H}_\alpha(t,x,q)$ is continuous in α due to (H5) and condition (C1) of $\mathcal{M}_{[0,T]}$. It is clear that $V^* = V^\alpha$ by uniqueness of the solution of (3.21). So $\Psi^* = \Psi^\alpha$. Now it follows that $\lim_{k\to\infty} |\Psi^{\hat{\alpha}_k} - \Psi^\alpha|_{B_j} = 0$ for all $j \geq 1$.

Step 2. Suppose (i) does not hold so that for some $\hat{j}$ we have that $|\Psi^{\alpha'} - \Psi^\alpha|_{B_{\hat{j}}}$ does not converge to 0 as $\alpha' \to \alpha$, which implies that there exist some $\epsilon_0 > 0$ and a sequence $\{\alpha_k^0\}$ converging to α such that for each k,

$$|\Psi^{\alpha_k^0} - \Psi^\alpha|_{B_{\hat{j}}} \geq \epsilon_0. \tag{3.28}$$

Step 3. Recall that $\{\alpha_k\}$ in Step 1 is arbitrary as long as it converges to α. Now we just take $\{\alpha_k\}$ in Step 1 as $\{\alpha_k^0\}$. By Step 1, there exists a subsequence of $\{\alpha_k^0\}$, denoted by $\{\hat{\alpha}_k^0\}$, such that $\lim_{k\to\infty} |\Psi^{\hat{\alpha}_k^0} - \Psi^\alpha|_{B_{\hat{j}}} = 0$, which contradicts (3.28). Hence (i) holds. ∎

LEMMA 3.5. *Assume* (H1)–(H6). *For $\mu_G(\cdot) \in \mathcal{M}_{[0,T]}$, the BR $\phi_\alpha(t,x|\mu_G(\cdot))$ in* (3.20) *continuously depends on α. Specifically, for any $\alpha \in [0,1]$,*

$$\lim_{\alpha'\to\alpha} \phi_{\alpha'}(t,x|\mu_G(\cdot)) = \phi_\alpha(t,x|\mu_G(\cdot)) \quad \textit{for all } t,x. \tag{3.29}$$

Proof. The BRs can be written as

$$\phi_\alpha(t,x|\mu_G(\cdot)) = S_\alpha^{\mu_G(t)}(x, V_x^\alpha(t,x)), \quad \phi_{\alpha'}(t,x|\mu_G(\cdot)) = S_{\alpha'}^{\mu_G(t)}(x, V_x^{\alpha'}(t,x)).$$

It follows that

$$\begin{aligned}
&|S_\alpha^{\mu_G(t)}(x, V_x^\alpha(t,x)) - S_{\alpha'}^{\mu_G(t)}(x, V_x^{\alpha'}(t,x))| \\
&\quad \leq |S_\alpha^{\mu_G(t)}(x, V_x^\alpha(t,x)) - S_\alpha^{\mu_G(t)}(x, V_x^{\alpha'}(t,x))| \\
&\qquad + |S_\alpha^{\mu_G(t)}(x, V_x^{\alpha'}(t,x)) - S_{\alpha'}^{\mu_G(t)}(x, V_x^{\alpha'}(t,x))|.
\end{aligned}$$

Given $\mu_G(\cdot)$ we have the prior upper bound $\sup_{\alpha,t,x} |V_x^\alpha(t,x)| \leq C$. It suffices to show that (3.29) holds for any given $C_0 > 0$ and $t \in [0,T]$, $|x| \leq C_0$. By (H6), for the given $\mu_G(t)$, $S_\alpha^{\mu_G(t)}(x,q)$ is uniformly continuous in $\alpha \in [0,1]$, $|x| \leq C_0$, $q \in [-C, C]$. For any $\epsilon > 0$, there exists $\delta > 0$ such that $|\alpha - \alpha'| < \delta$ implies $\sup_{|x|\leq C_0, |q|\leq C} |S_\alpha^{\mu_G(t)}(x,q) - S_{\alpha'}^{\mu_G(t)}(x,q)| \leq \epsilon/2$, and moreover,

$$\sup_{|x|\leq C_0} |S_\alpha^{\mu_G(t)}(x, V_x^\alpha(t,x)) - S_\alpha^{\mu_G(t)}(x, V_x^{\alpha'}(t,x))| \leq \frac{\epsilon}{2}$$

in view of Lemma 3.4 (i). Therefore (3.29) holds. ∎

We proceed to show the existence of a solution to the GMFG equations (3.13) and (3.15) in terms of $\{(V^\alpha, \mu_\alpha(\cdot))|\alpha \in [0,1]\}$. For $\mu_G \in \mathcal{M}_{[0,T]}$, denote the mapping

$$(\phi_\alpha)_{\alpha\in[0,1]} = \mathit{\Gamma}(\mu_G(\cdot)),$$

where $(\phi_\alpha)_{\alpha\in[0,1]}$ is given by (3.20) as the set of BRs with respect to $\mu_G(\cdot)$. Next, we combine $(\phi_\alpha)_{\alpha\in[0,1]}$ with $\mu_G(\cdot)$ to determine the law m_α of the closed-loop state process:

$$dx_\alpha(t) = \widetilde{f}[x_\alpha(t), \phi_\alpha(t, x_\alpha(t)|\mu_G(\cdot)), \mu_G(t); g_\alpha]dt + \sigma dw_\alpha(t),$$

where $x_\alpha(0)$ has distribution μ_0^x. The choice of the Brownian motion for x_α is immaterial. For m_α above, denote the mapping from $\mathcal{M}_{[0,T]}$ to $\mathbf{M}_T^G$: $(m_\alpha)_{\alpha\in[0,1]} = \widehat{\mathit{\Gamma}}(\mu_G(\cdot))$.

Define the set $\mathbf{M}_T^{G1} := \widehat{\mathit{\Gamma}}(\mathcal{M}_{[0,T]}) \subset \mathbf{M}_T^G$. Now the existence analysis may be formulated as the problem of finding a fixed point of the form

$$m_G = \widehat{\mathit{\Gamma}} \circ \mathrm{Marg}(m_G), \tag{3.30}$$

in case $m_G \in \mathbf{M}_T^{G1}$. Note that $\mathrm{Marg}(m_G) = \{(\mathrm{Marg}_t(m_\alpha))_{\alpha\in[0,1]}, 0 \le t \le T\}$.

Remark 3.6. The fixed point problem requires m_G to be from the subset $\mathbf{M}_T^{G1}$ of $\mathbf{M}_T^G$. If one simply looks for $m_G \in \mathbf{M}_T^G$, the resulting $\mu_G(\cdot) = \mathrm{Marg}(m_G)$ lacks the Hölder continuity in (C2), and this will cause difficulties in establishing Lemma 3.3 for the HJB equation.

LEMMA 3.7. *Under* (H1)–(H6)*, the following assertions hold:*
(i) $\mathbf{M}_T^{G1} \subset \mathbf{M}_T^{G0}$.
(ii) *For any* $m_G \in \mathbf{M}_T^{G1}$, $\mu_G(\cdot) := \mathrm{Marg}(m_G) \in \mathcal{M}_{[0,T]}$.
(iii) *The BR* $\phi_\alpha(t, x|\mu_G(\cdot))$ *with* $\mu_G(\cdot)$ *given in* (ii) *is Lipschitz continuous in* x*, uniformly with respect to* $\alpha \in [0,1]$ *and* $m_G \in \mathbf{M}_T^{G1}$.

Proof. (i) and (ii) For $m_G \in \mathbf{M}_T^{G1}$, there exists $\mu'_G \in \mathcal{M}_{[0,T]}$ such that $m_G = \widehat{\mathit{\Gamma}}(\mu'_G(\cdot))$. To estimate $D_T(m_\alpha, m_{\bar\alpha})$ and $W_1(\mu_\alpha(t), \mu_{\bar\alpha}(t))$, let x_α and $x_{\bar\alpha}$ be state processes generated by (3.10) with μ'_G, the same initial state and Brownian motion under the control laws $\phi_\alpha(t, x|\mu'_G(\cdot))$ and $\phi_{\bar\alpha}(t, x|\mu'_G(\cdot))$, respectively. Then $D_T(m_\alpha, m_{\bar\alpha}) \le E\sup_{t\le T}|x_\alpha(t) - x_{\bar\alpha}(t)|$, and $W_1(\mu_\alpha(t), \mu_{\bar\alpha}(t)) \le E|x_\alpha(t) - x_{\bar\alpha}(t)|$. Fixing $\bar\alpha$, we have

$$\begin{aligned}|x_\alpha(t) - x_{\bar\alpha}(t)| \le \int_0^t &|\widetilde{f}[x_\alpha(s), \phi_\alpha(s, x_\alpha(s)|\mu'_G(\cdot)), \mu'_G(s); g_\alpha] \\ &- \widetilde{f}[x_{\bar\alpha}(s), \phi_{\bar\alpha}(s, x_{\bar\alpha}(s)|\mu'_G(\cdot)), \mu'_G(s); g_{\bar\alpha}]|ds.\end{aligned} \tag{3.31}$$

Denote

$$\delta_1 = |f_0[x_{\bar\alpha}(s), \phi_{\bar\alpha}(s, x_{\bar\alpha}(s)|\mu'_G(\cdot)), \mu'_\alpha(s)] - f_0[x_{\bar\alpha}(s), \phi_{\bar\alpha}(s, x_{\bar\alpha}(s)|\mu'_G(\cdot)), \mu'_{\bar\alpha}(s)]|,$$
$$\delta_2 = |f[x_{\bar\alpha}(s), \phi_{\bar\alpha}(s, x_{\bar\alpha}(s)|\mu'_G(\cdot)), \mu'_G(s); g_\alpha] - f[x_{\bar\alpha}(s), \phi_{\bar\alpha}(s, x_{\bar\alpha}(s)|\mu'_G(\cdot)), \mu'_G(s); g_{\bar\alpha}]|.$$

Then by (3.31) and the Lipschitz continuity in x of ϕ_α in Lemma 3.3 (ii), we obtain

$$\begin{aligned}|x_\alpha(t) - x_{\bar\alpha}(t)| &\le C_1 \int_0^t |x_\alpha(s) - x_{\bar\alpha}(s)|ds \\ &+ C_2 \int_0^t \Big\{|\phi_\alpha(s, x_{\bar\alpha}(s)|\mu'_G(\cdot)) - \phi_{\bar\alpha}(s, x_{\bar\alpha}(s)|\mu'_G(\cdot))| + \delta_1(s) + \delta_2(s)\Big\}ds,\end{aligned} \tag{3.32}$$

where C_2 depends only on the Lipschitz constants of f_0, f and C_1 does not change with α for the fixed μ'_G. Since $W_1(\mu'_\alpha(s), \mu'_{\bar\alpha}(s)) \to 0$ as $\alpha \to \bar\alpha$, by (H2) $E\delta_1(s) \to 0$ as $\alpha \to \bar\alpha$. By (H5), we have $E\delta_2(s) \to 0$ as $\alpha \to \bar\alpha$. Then using Lemma 3.5 and boundedness of the integrand below, we obtain

$$\lim_{\alpha\to\bar\alpha} E\int_0^T \Big\{|\phi_\alpha(s, x_{\bar\alpha}(s)|\mu'_G(\cdot)) - \phi_{\bar\alpha}(s, x_{\bar\alpha}(s)|\mu'_G(\cdot))| + \delta_1(s) + \delta_2(s)\Big\}ds = 0.$$

By Gronwall's lemma and (3.32), it follows that $\lim_{\alpha\to\bar\alpha} E\sup_{0\le t\le T}|x_\alpha(t) - x_{\bar\alpha}(t)| = 0$. Subsequently, as $\alpha \to \bar\alpha$, we obtain $D_T(m_\alpha, m_{\bar\alpha}) \to 0$, which implies (i); in addition, $W_1(\mu_\alpha(t), \mu_{\bar\alpha}(t)) \to 0$, which verifies condition (C1) of $\mathcal{M}_{[0,T]}$ for μ_G. Since each m_α is the distribution of x_α, for $\mu_G(\cdot)$ we take the Hölder parameter $\eta = 1/2$ and a constant C_h independent of μ'_G for (C2). So (ii) holds.

(iii) Due to the choice of η and C_h for $\mu_G(\cdot)$ in (ii), we may select a fixed constant C_5 in (3.24), which does not change with $(\alpha, \mu_G(\cdot))$. Subsequently, the upper bound C_6 in (3.26) for $|V^\alpha_{xx}|$ does not change with $\alpha \in [0,1], \mu_G(\cdot) \in \mathrm{Marg}(\widehat{\Gamma}(\mathcal{M}_{[0,T]}))$. This ensures a uniform bound for the Lipschitz constant for x in ϕ_α. ☐

We introduce the sensitivity condition.

(H7) For $m_G, \bar m_G \in \mathbf{M}_T^{G1} = \widehat{\Gamma}(\mathcal{M}_{[0,T]})$, there exists a constant c_1 such that

$$\sup_{t,x,\alpha} |\phi_\alpha(t, x|\mu_G(\cdot)) - \bar\phi_\alpha(t, x|\bar\mu_G(\cdot))| \le c_1 d(m_G, \bar m_G), \tag{3.33}$$

where the set of control laws $\{\phi_\alpha(t, x|\mu_G(\cdot)), \alpha \in [0,1]\}$ (resp., $\{\bar\phi_\alpha(t, x|\bar\mu_G(\cdot)), \alpha \in [0,1]\}$) is determined by use of $\mu_G = \mathrm{Marg}(m_G)$ (resp., $\bar\mu_G = \mathrm{Marg}(\bar m_G)$) in the optimal control problem specified by (3.10) and (3.12) with the graphon section g_α.

Assumption (H7) is a generalization from the finite type model in [26] where an illustration via a linear model is presented. Related sensitivity conditions are studied in [29].

Let $(\phi_\alpha)_{\alpha\in[0,1]}$ in (3.20) be applied by all agents, where $\mu_G(\cdot) \in \mathcal{M}_{[0,T]}$. We consider the following generalized MV equation

$$dx_\alpha(t) = \widetilde{f}[x_\alpha(t), \phi_\alpha(t, x_\alpha(t)|\mu_G), \nu_G(t); g_\alpha]dt + \sigma dw_\alpha(t), \tag{3.34}$$

where $x_\alpha(0)$ is given with distribution μ_0^x. For this equation, ν_G is part of the solution. If ν_G is determined, we have a unique solution x_α on $[0,T]$ which further determines its law as the measure m_α on $(C_T, \mathcal{F}_T)$. Note that m_α does not depend on the choice of the standard Brownian motion w_α. We look for $\nu_G \in \mathcal{M}_{[0,T]}$ to satisfy the condition

$$\mathrm{Marg}_t(m_\alpha) = \nu_\alpha(t) \quad \text{for all } \alpha \in [0,1],\ t \in [0,T]; \tag{3.35}$$

i.e., $\nu_\alpha(t)$ is the law of $x_\alpha(t)$ for all α, t (and we say $(x_\alpha)_{0\le\alpha\le1}$ is consistent with ν_G).

Lemma 3.8. *Assume* (H1)–(H6). *For the BR* $\phi_\alpha(t, x_\alpha|\mu_G(\cdot))$ *in* (3.20), *where* $\mu_G(\cdot) \in \mathcal{M}_{[0,T]}$, *there exists a unique* $\nu_G(\cdot)$ *for* (3.34) *satisfying* (3.35).

Proof. In order to solve (x_α, ν_G) in (3.34), we specify the law of the process x_α instead of just its marginal $\nu_\alpha(t)$. This extends the fixed point idea for treating standard MV equations [41].

For $(m_\alpha)_{\alpha\in[0,1]} \in \mathbf{M}_T^{G0}$, we determine ν_G^1 according to $\nu_\alpha^1(t) = \mathrm{Marg}_t(m_\alpha)$, which is used in (3.34) by taking $\nu_G = \nu_G^1$ to solve x_α on $[0,T]$. Let m_α^{new} denote the law of x_α. It in general does not satisfy $\mathrm{Marg}_t(m_\alpha^{\mathrm{new}}) = \nu_\alpha(t)$ for all t. Denote the mapping

$$(m_\alpha^{\mathrm{new}})_{\alpha\in[0,1]} = \varPhi_{\mathbf{M}_T^{G0}}((m_\alpha)_{\alpha\in[0,1]}).$$

By (H5) and Lemma 3.5, $\Phi_{\mathbf{M}_T^{G0}}$ is a mapping from $\mathbf{M}_T^{G0}$ to itself. Similarly, for $(\bar{m}_\alpha)_{\alpha\in[0,1]} \in \mathbf{M}_T^{G0}$ we determine $\bar{\nu}_G^1$ for (3.34) and solve $\bar{x}_\alpha$ with its law $\bar{m}_\alpha^{\text{new}}$. Denote $(\bar{m}_\alpha^{\text{new}})_{\alpha\in[0,1]} = \Phi_{\mathbf{M}_T^{G0}}((\bar{m}_\alpha)_{\alpha\in[0,1]})$.

If $h(x,y)$ is a bounded Lipschitz continuous function with $|h(x,y) - h(\bar{x},\bar{y})| \le C_1|x-\bar{x}| + C_2(|y-\bar{y}|\wedge 1)$, we have

$$\begin{aligned}
&\left|\int h(x,y)g(\alpha,\beta)\nu_\beta^1(t,dy)d\beta - \int h(\bar{x},\bar{y})g(\alpha,\beta)\nu_\beta^2(t,d\bar{y})d\beta\right| \\
&\quad\le C_1|x-\bar{x}| + \sup_\beta\left|\int h(\bar{x},y)\nu_\beta^1(t,dy) - \int h(\bar{x},\bar{y})\nu_\beta^2(t,d\bar{y})\right| \\
&\quad= C_1|x-\bar{x}| + \sup_\beta\left|\int_{C_T} h(\bar{x},X_t(\omega))dm_\beta(\omega) - \int_{C_T} h(\bar{x},X_t(\bar{\omega}))d\bar{m}_\beta(\bar{\omega})\right| \\
&\quad\le C_1|x-\bar{x}| + C_2\sup_\beta\int_{C_T\times C_T}(|X_t(\omega)-X_t(\bar{\omega})|\wedge 1)d\widehat{m}_\beta(\omega,\bar{\omega}),
\end{aligned}$$

where X is the canonical process, $\omega,\bar{\omega}\in C_T$, and $\widehat{m}_\beta$ is any coupling of m_β and $\bar{m}_\beta$. Hence

$$\begin{aligned}
&\left|\int h(x,y)g(\alpha,\beta)\nu_\beta^1(t,dy)d\beta - \int h(\bar{x},\bar{y})g(\alpha,\beta)\nu_\beta^2(t,d\bar{y})d\beta\right| \\
&\quad\le C_1|x-\bar{x}| + C_2\sup_\beta D_t(m_\beta,\bar{m}_\beta).
\end{aligned}\tag{3.36}$$

By (H2), (H3), the uniform Lipschitz continuity of ϕ_α in x by Lemma 3.3 (ii), and (3.36),

$$\begin{aligned}
&|\widetilde{f}[x_\alpha,\phi_\alpha(t,x_\alpha|\mu_G),\nu_G^1(t);g_\alpha] - \widetilde{f}[\bar{x}_\alpha,\phi_\alpha(t,\bar{x}_\alpha|\mu_G),\nu_G^2(t);g_\alpha]| \\
&\quad\le C_1(|x_\alpha-\bar{x}_\alpha|\wedge 1) + C_2\sup_\beta D_t(m_\beta,\bar{m}_\beta).
\end{aligned}$$

Hence by (3.34),

$$\sup_{s\le t}|x_\alpha(s)-\bar{x}_\alpha(s)| \le C_1\int_0^t |x_\alpha(s)-\bar{x}_\alpha(s)|\wedge 1ds + C_3\int_0^t\sup_\beta|D_s(m_\beta,\bar{m}_\beta)|ds.$$

Therefore, by Gronwall's lemma, $\sup_{s\le t}|x_\alpha(s)-\bar{x}_\alpha(s)|\wedge 1 \le C_4\int_0^t\sup_\beta|D_s(m_\beta, \bar{m}_\beta)|ds$, which combined with the definition of the Wasserstein metric $D_t(\cdot,\cdot)$ implies that

$$\sup_\beta|D_t(m_\beta^{\text{new}},\bar{m}_\beta^{\text{new}})| \le C_4\int_0^t\sup_\beta|D_s(m_\beta,\bar{m}_\beta)|ds.\tag{3.37}$$

By iterating (3.37) as in [41, p. 174], we can show that for a sufficiently large k_0, $\Phi_{\mathbf{M}_T^{G0}}^{k_0}$ is a contraction. We can further show that $\{\Phi_{\mathbf{M}_T^{G0}}^k(m_G), k\ge 1\}$ is a Cauchy sequence, and we obtain a unique fixed point m_G^* for $\Phi_{\mathbf{M}_T^{G0}}$. Then we obtain a solution of (3.34) by taking $\nu_\alpha(t) = \text{Marg}_t(m_\alpha^*)$. If there are two different solutions with $\nu_G \ne \nu_G'$, we can derive a contradiction by using uniqueness of the fixed point of $\Phi_{\mathbf{M}_T^{G0}}$. ∎

Consider two sets of BRs $(\phi_\alpha(t,x_\alpha|\mu_G))_{\alpha\in[0,1]}$ and $(\bar{\phi}_\alpha(t,x_\alpha|\bar{\mu}_G))_{\alpha\in[0,1]}$, where $\mu_G = \text{Marg}(m_G), \bar{\mu}_G = \text{Marg}(\bar{m}_G)$ for $m_G,\bar{m}_G\in\mathbf{M}_T^{G1}$ (then $\mu_G,\bar{\mu}_G\in\mathcal{M}_{[0,T]}$), and use Lemma 3.8 to solve (x_α,ν_G) and $(x_\alpha',\bar{\nu}_G)$ from the generalized MV-SDEs:

$$dx_\alpha = \widetilde{f}[x_\alpha, \phi_\alpha(t, x_\alpha|\mu_G), \nu_G(t); g_\alpha]dt + \sigma dw_\alpha, \tag{3.38}$$

$$dx'_\alpha = \widetilde{f}[x'_\alpha, \bar{\phi}_\alpha(t, x'_\alpha|\bar{\mu}_G), \bar{\nu}_G(t); g_\alpha]dt + \sigma dw_\alpha, \tag{3.39}$$

where $x'_\alpha(0) = x_\alpha(0)$ is given. Let m^{mv}_α (resp., $\bar{m}^{\mathrm{mv}}_\alpha$) denote the law of x_α (resp., x'_α). The following lemma is a generalization of [26, Lem. 9] to the graphon network case.

LEMMA 3.9. *For* (3.38) *and* (3.39) *there exists a constant* c_2 *independent of* $(m_G, \bar{m}_G)$ *such that*

$$\sup_\alpha D_T(m^{\mathrm{mv}}_\alpha, \bar{m}^{\mathrm{mv}}_\alpha) \le c_2 \sup_{t,x,\alpha} |\phi_\alpha(t, x|\mu_G(\cdot)) - \bar{\phi}_\alpha(t, x|\bar{\mu}_G(\cdot))|.$$

Proof. For (3.38) and (3.39), denote

$$\varDelta_s = \widetilde{f}[x_\alpha(s), \phi_\alpha(s, x_\alpha(s)|\mu_G), \nu_G(s); g_\alpha] - \widetilde{f}[x'_\alpha(s), \bar{\phi}_\alpha(s, x'_\alpha(s)|\bar{\mu}_G), \bar{\nu}_G(s); g_\alpha].$$

Then $x_\alpha(t) - x'_\alpha(t) = \int_0^t \varDelta_s ds$. Noting $\nu_\alpha(t) = \mathrm{Marg}_t(m^{\mathrm{mv}}_\alpha)$ and $\bar{\nu}_\alpha(t) = \mathrm{Marg}_t(\bar{m}^{\mathrm{mv}}_\alpha)$, we have

$$\begin{aligned} |\varDelta_s| &\le |\widetilde{f}[x_\alpha(s), \phi_\alpha(s, x_\alpha(s)|\mu_G), \nu_G(s); g_\alpha] - \widetilde{f}[x'_\alpha(s), \phi_\alpha(s, x'_\alpha(s)|\mu_G), \bar{\nu}_G(s); g_\alpha]| \\ &\quad + |\widetilde{f}[x'_\alpha(s), \phi_\alpha(s, x'_\alpha(s)|\mu_G), \bar{\nu}_G(s); g_\alpha] - \widetilde{f}[x'_\alpha(s), \bar{\phi}_\alpha(s, x'_\alpha(s)|\bar{\mu}_G), \bar{\nu}_G(s); g_\alpha]| \\ &\le C_1|x_\alpha(s) - x'_\alpha(s)| + C_2 \sup_\beta D_s(m^{\mathrm{mv}}_\beta, \bar{m}^{\mathrm{mv}}_\beta) \\ &\quad + C_3 \sup_{t,x} |\phi_\alpha(t, x|\mu_G(\cdot)) - \bar{\phi}_\alpha(t, x|\bar{\mu}_G(\cdot))|, \end{aligned} \tag{3.40}$$

where C_1, C_2, and C_3 do not depend on $(\alpha, m_G, \bar{m}_G)$. The difference term on the first line is estimated by the method in (3.36). We have used the fact that ϕ_α is uniformly Lipschitz continuous in x by Lemma 3.7 (iii). Therefore, by (3.40),

$$\begin{aligned} |x_\alpha(t) - x'_\alpha(t)| &\le \int_0^t \Big[C_1|x_\alpha(s) - x'_\alpha(s)| + C_2 \sup_\beta D_s(m^{\mathrm{mv}}_\beta, \bar{m}^{\mathrm{mv}}_\beta)\Big]ds \\ &\quad + C_3 t \sup_{t,x} |\phi_\alpha(t, x|\mu_G(\cdot)) - \bar{\phi}_\alpha(t, x|\bar{\mu}_G(\cdot))|. \end{aligned} \tag{3.41}$$

Applying Gronwall's lemma to (3.41) and next using the definition of $D_t(\cdot, \cdot)$, we obtain

$$\begin{aligned} &D_t(m^{\mathrm{mv}}_\alpha, \bar{m}^{\mathrm{mv}}_\alpha) \le E\left(\sup_{0\le s\le t} |x_\alpha(s) - x'_\alpha(s)| \wedge 1\right) \\ &\quad \le e^{C_1 t} C_2 \int_0^t \sup_\beta D_s(m^{\mathrm{mv}}_\beta, \bar{m}^{\mathrm{mv}}_\beta) ds + e^{C_1 t} C_3 t \sup_{t,x,\alpha} |\phi_\alpha(t, x|\mu_G(\cdot)) - \bar{\phi}_\alpha(t, x|\bar{\mu}_G(\cdot))|. \end{aligned}$$

The lemma follows from applying Gronwall's lemma again to $\sup_\alpha D_t(m^{\mathrm{mv}}_\alpha, \bar{m}^{\mathrm{mv}}_\alpha)$. ∎

3.4. Existence theorem. We state the main result on the existence and uniqueness of solutions to the GMFG equation system. We introduce a contraction condition:

(H8) $c_1 c_2 < 1$, where c_1 is the constant in the sensitivity condition (H7) and c_2 is specified in Lemma 3.9.

Remark 3.10. Under weak coupling effect or small T, a small c_2 can be obtained.

Remark 3.11. For linear models, a verification of the contraction condition can be done under reasonable model parameters, as in [26].

THEOREM 3.12. *Under* (H1)–(H8)*, there exists a unique solution* $(V^\alpha, \mu_\alpha(\cdot))_{\alpha\in[0,1]}$ *to the GMFG equations* (3.13) *and* (3.15)*, which* (i) *gives the feedback control BR strategy* $\varphi(t, x_\alpha|\mu_G(\cdot); g_\alpha)$*,* $\alpha \in [0, 1]$*, depending only upon the agent's state and the ensemble* μ_G *of local mean fields (i.e.,* (x_α, μ_G)*), and* (ii) *generates a Nash equilibrium.*

Proof. Step 1. We return to the fixed point equation (3.30), which is redisplayed below:

$$m_G = \widehat{\Gamma} \circ \mathrm{Marg}(m_G), \tag{3.42}$$

where $m_G = (m_\alpha)_{\alpha\in[0,1]} \in \mathbf{M}_T^{G1}$. For $m_G \in \mathbf{M}_T^{G1}$, the Hölder continuity in t of the regenerated $\mu_G(\cdot) = \mathrm{Marg}(m_G)$ can be checked by elementary SDE estimates by adapting the proof of [26, Lem. 7].

Step 2. Take any $m_G \in \mathbf{M}_T^{G1}$ to determine $\mu_G = \mathrm{Marg}(m_G)$ and $\phi_\alpha(t, x_\alpha|\mu_G(\cdot))$. When $\bar{m}_G \in \mathbf{M}_T^{G1}$ is used, we determine $\bar{\mu}_G$ and $\bar{\phi}_\alpha(t, x_\alpha|\bar{\mu}_G(\cdot))$. Once the set of strategies $(\phi_\alpha)_{\alpha\in[0,1]}$ is applied to the generalized MV equation (3.34), by Lemma 3.8, we may solve for $(x_\alpha, \nu_G(\cdot))$ such that x_α has the law m_α^{new} and $\mathrm{Marg}_t(m_\alpha^{\mathrm{new}}) = \nu_\alpha(t)$. This is done in parallel for $\bar{m}_G$ to generate $\bar{m}_\alpha^{\mathrm{new}}$. We accordingly determine m_G^{new} and $\bar{m}_G^{\mathrm{new}}$.

Step 3. By (3.33) and Lemma 3.9, we obtain $d(m_G^{\mathrm{new}}, \bar{m}_G^{\mathrm{new}}) \le c_1c_2 d(m_G, \bar{m}_G)$. Based on the above contraction property, we construct a Cauchy sequence in the complete metric space $\mathbf{M}_T^G$ by iterating with m_G and establish existence of a solution to the GMFG equation system. To show uniqueness, suppose m_G and $\tilde{m}_G$ are two fixed points to (3.42). We obtain $d(m_G, \tilde{m}_G) \le c_1c_2 d(m_G, \tilde{m}_G)$, which implies $m_G = \tilde{m}_G$.

The Nash equilibrium property follows from the BR property of φ_α for given α. ∎

3.5. An example on Lipschitz feedback. The main analysis in section 3 relies on (H4) to ensure Lipschitz feedback. We provide a concrete model to check this assumption.

Example 3.13. The dynamics and cost have

$$f_0(x, u, y) = f_0(x, y)u, \quad f(x, u, y) = f(x, y)u,$$
$$l_0(x, u, y) = l_1(x, y) + l_2(x, y)u^2, \quad l(x, u, y) = l_3(x, y) + l_4(x, y)u^2,$$

where $x, y \in \mathbb{R}$ and $u \in U = [a, b]$. The functions f_0, f, l_1, l_2, l_3, l_4 satisfy (H1)–(H3), and there exists $c_0 > 0$ such that $l_2, l_4 \ge c_0$ for all x, y.

Given $\nu_G \in C([0, 1], \mathcal{P}_1(\mathbb{R}))$, for $x, q \in \mathbb{R}$, we check the minimizer

$$S_\alpha^{\nu_G}(x, q) = \arg\min_{u\in U}\{q(f_0[x, \nu_\alpha] + f[x, \nu_G; g_\alpha])u + (l_2[x, \nu_\alpha] + l_4[x, \nu_G; g_\alpha])u^2\}.$$

PROPOSITION 3.14. *Given any compact interval* $\mathcal{I}$*,* $S_\alpha^{\nu_G}(x, q)$ *in Example* 3.13 *is a singleton and Lipschitz continuous in* (x, q)*, where* $x \in \mathbb{R}$ *and* $q \in \mathcal{I}$*, uniformly with respect to* (ν_G, α)*.*

Proof. Consider the function $\Phi(u) = u^2 - 2su$, where $u \in U$ and s is a parameter. Its minimum is attained at the unique point $\hat{u} = \Theta(s)$ which is defined to be equal to (i) a if $s \le a$, (ii) s if $a < s < b$, and (iii) b if $s \ge b$. Denote the function

$$h_{\alpha,\nu_G}(x) = -\frac{f_0[x, \mu_\alpha] + f[x, \nu_G; g_\alpha]}{2(l_2[x, \mu_\alpha] + l_4[x, \nu_G; g_\alpha])}.$$

By elementary estimates we obtain $\sup_{\alpha,\nu_G} |h_{\alpha,\nu_G}(x) - h_{\alpha,\nu_G}(y)| \le C_0|x-y|$. We have

$$S_\alpha^{\nu_G}(x,q) = \arg\min_u (u^2 - 2qh_{\alpha,\nu_G}(x)u) = \Theta(qh_{\alpha,\nu_G}(x)).$$

It is clear that $S_\alpha^{\nu_G}(x,q)$ is a continuous function of (x,q). For $(x_i,q_i) \in \mathbb{R}\times\mathcal{I}$, $i=1,2$,

$$\begin{aligned}|S_\alpha^{\nu_G}(x_1,q_1) - S_\alpha^{\nu_G}(x_2,q_2)| &\le \mathrm{Lip}(\Theta)|q_1h_{\alpha,\nu_G}(x_1) - q_2h_{\alpha,\nu_G}(x_2)| \\ &\le \mathrm{Lip}(\Theta)\Big(|q_1-q_2|\sup_x|h_{\alpha,\nu_G}(x)| + C_0|x_1-x_2||q_2|\Big).\end{aligned}$$

In fact, the Lipschitz constant $\mathrm{Lip}(\Theta)=1$. Note that $\sup_{x,\alpha,\nu_G}|h_{\alpha,\nu_G}(x)| \le C$ for some constant C. This proves the proposition. ◻

If (H1), (H2), (H3), and (H5) hold for Example 3.13, they further imply (H4) and (H6) so that the BR is Lipschitz continuous in x by Lemma 3.3 and Proposition 3.14.

4. Performance analysis. In the MFG case it is shown [26, 9] that the joint strategy $\{u_i^o(t) = \varphi_i(t, x_i(t)|\mu_\bullet), 1 \le i \le N\}$ yields an ϵ-Nash equilibrium; i.e., for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \ge N(\epsilon)$

$$J_i^N(u_i^\circ, u_{-i}^\circ) - \epsilon \le \inf_{u_i\in\mathcal{U}_i} J_i^N(u_i, u_{-i}^\circ) \le J_i^N(u_i^\circ, u_{-i}^\circ). \tag{4.1}$$

This form of approximate Nash equilibrium is a principal result of the MFG analyses in the sequence [26, 9, 40] and in many other studies. The importance of (4.1) is that it states that the cost function of any agent in a finite population can be reduced by at most ϵ *if it changes unilaterally from the infinite population MFG feedback law* while all other agents remain with the infinite population based control strategies. The main result of this section is that the same property holds for GMFG systems.

Throughout this section, let $\mu_G(\cdot)$ be solved from the GMFG equations (3.13) and (3.15).

4.1. The ϵ-Nash equilibrium. The analysis of GMFG systems as limits of finite objects necessarily involves the consideration of graph limits and double limits in population and graph order. A corresponding set of assumptions is given below.

(H9) $M_k \to \infty$ and $\min_{1\le l\le M_k}|\mathcal{C}_l| \to \infty$ as $k\to\infty$.

(H10) All agents have i.i.d. initial states with distribution μ_0^x and $E|x_i(0)| \le C_0$.

Remark 4.1. (H10) is a simplifying assumption to keep further notation light. It may be generalized to α dependent initial distributions.

(H11) The sequence $\{G_k; 1\le k<\infty\}$ and the graphon limit satisfy

$$\lim_{k\to\infty}\max_i \sum_{j=1}^{M_k}\left|\frac{g^k_{\mathcal{C}_i\mathcal{C}_j}}{M_k} - \int_{\beta\in I_j} g_{I_i^*,\beta}d\beta\right| = 0,$$

where I_i^* is the midpoint of the subinterval $I_i \in \{I_1,\ldots,I_{M_k}\}$ of length $1/M_k$.

Remark 4.2. Assumption (H11) specifies the nature of the approximation error between g^k for the finite graph and the graphon function g.

Remark 4.3. Given $\{g^k, k\ge 1\}$ under (H9), if there exists a graphon function g satisfying (H5) and (H11), it is unique. This can be proven by showing that the cut norm $\|g-\hat g\|_\square = 0$ if $\hat g$ also satisfies (H5) and (H11). A key step is to show that $\lim_{k\to\infty}|\int_{\mathcal{S}\times\mathcal{T}}(g^k - g)dxdy| = 0$ for any fixed measurable sets $\mathcal{S},\mathcal{T}\subset[0,1]$. See [8] for details.

For the ϵ-Nash equilibrium analysis, we consider a sequence of games each defined on a finite graph G_k. Recall that there is a total of $N = \sum_{l=1}^{M_k} |\mathcal{C}_l|$ agents. Suppose the cluster $\mathcal{C}(i)$ of agent $\mathcal{A}_i$ corresponds to the subinterval $I(i) \in \{I_1, \dots, I_{M_k}\}$. The agent $\mathcal{A}_i$ takes the midpoint $I^*(i)$ of $I(i)$ and uses the GMFG system-based control law

$$\hat{u}_i = \varphi(t, x_i | \mu_G(\cdot); g_{I^*(i)}), \quad 1 \le i \le N, \tag{4.2}$$

which we simply write as $\varphi(t, x_i, g_{I^*(i)})$.

Recall f_0 and f_{G_k} in (3.5) and (3.6). The closed-loop system of N agents on the finite graph G_k under the set of strategies (4.2) is given by

$$\begin{aligned} \textit{System A:} \quad d\hat{x}_i^N &= f_0(\hat{x}_i^N, \varphi(t, \hat{x}_i^N, g_{I^*(i)}), \mathcal{C}(i))dt \\ &\quad + f_{G_k}(\hat{x}_i^N, \varphi(t, \hat{x}_i^N, g_{I^*(i)}), g^k_{\mathcal{C}(i)})dt + \sigma dw_i, \end{aligned} \tag{4.3}$$

where $1 \le i \le N$ and $\hat{x}_i^N(0) = x_i^N(0)$. The superscript N is added to indicate the population size. We state the following main result.

THEOREM 4.4 (ϵ-Nash equilibrium). *Assume* (H1)–(H11) *hold. Then when the strategies* (4.2) *determined by the GMFG equations* (3.13) *and* (3.15) *are applied to a sequence of finite graph systems* $\{G_k; 1 \le k < \infty\}$*, the ϵ-Nash equilibrium property holds, where $\epsilon \to 0$ as $k \to \infty$ and where the unilateral agent $\mathcal{A}_i$ uses a centralized Lipschitz feedback strategy $\psi(t, x_i, x_{-i})$, where x_{-i} denotes the set of states of all other agents.*

We first explain the basic idea for the demonstration of the ϵ-Nash equilibrium property. Suppose all other players, except agent $\mathcal{A}_\iota$, employ the strategies in (4.2). When $\mathcal{A}_\iota$ employs a different strategy, the resulting change in its performance can be measured using a limiting stochastic control problem where both the system dynamics and the cost are subject to small perturbation due to the mean field approximation of the effects of all other agents. The proof is technical and preceded by some lemmas.

4.2. Proof of Theorem 4.4. Suppose $\mathcal{A}_\iota$ applies a general feedback control law u_ι^N instead of (4.2) while all other agents $\mathcal{A}_j$, $j \ne \iota$, still adopt strategies in (4.2). Consider

$$\textit{System B:} \quad \begin{cases} dx_\iota^N = f_0(x_\iota^N, u_\iota^N, \mathcal{C}(\iota))dt + f_{G_k}(x_\iota^N, u_\iota^N, g^k_{\mathcal{C}(\iota)})dt + \sigma dw_\iota, \\ dx_j^N = f_0(x_j^N, \varphi(t, x_j^N, g_{I^*(j)}), \mathcal{C}(j))dt \\ \qquad\qquad + f_{G_k}(x_j^N, \varphi(t, x_j^N, g_{I^*(j)}), g^k_{\mathcal{C}(j)})dt + \sigma dw_j, \\ \qquad\qquad j \ne \iota, \quad 1 \le j \le N. \end{cases} \tag{4.4}$$

We note that x_j^N is affected by the unilateral choice of strategy by $\mathcal{A}_\iota$ due to the coupling in f_0 and f_{G_k}. For this reason, x_j^N differs from $\hat{x}_j^N$ in (4.3) although the control law of $\mathcal{A}_j$, $j \ne \iota$, remains the same. The central task is to estimate by how much $\mathcal{A}_\iota$ can reduce its cost.

For the performance estimate in System B, we introduce two auxiliary systems below. Consider

$$\begin{aligned} \textit{System C:} \quad dy_i^N = &\int_{\mathbb{R}} f_0(y_i^N, \varphi(t, y_i^N, g_{I^*(i)}), z) m_{y_i^N}(dz)dt \\ &+ \frac{1}{M_k} \sum_{l=1}^{M_k} \frac{g^k_{\mathcal{C}(i)\mathcal{C}_l}}{|\mathcal{C}_l|} \sum_{j \in \mathcal{C}_l} \int_{\mathbb{R}} f(y_i^N, \varphi(t, y_i^N, g_{I^*(i)}), z) m_{y_j^N}(dz)dt \end{aligned}$$

$$
\begin{aligned}
&+\sigma dw_i \\
&= \int_{\mathbb{R}} f_0(y_i^N, \varphi(t, y_i^N, g_{I^*(i)}), z) m_{y_i^N}(dz) dt \\
&+ \frac{1}{M_k} \sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \int f(y_i^N, \varphi(t, y_i^N, g_{I^*(i)}), z) m_l^N(t, dz) dt \\
&+ \sigma dw_i,
\end{aligned}
\tag{4.5}
$$

where $1 \le i \le N$ and $y_i^N(0) = x_i^N(0)$, and $m_{y_j^N(t)}$ denotes the law of $y_j^N(t)$. Each Brownian motion w_i is the same as in (4.3). The second equality holds since all processes in cluster $\mathcal{C}_l$ have the same distribution denoted by $m_l^N(t, dz)$ at time t. It is clear that the processes $y_1^N, \dots, y_N^N$ are independent, and $\{y_j^N, j \in \mathcal{C}_l\}$ are i.i.d. for any given l.

Next we introduce

$$
\text{System D:} \quad dy_i^\infty(t) = \widetilde{f}[y_i^\infty(t), \varphi(t, y_i^\infty(t), g_{I^*(i)}), \mu_G(t); g_{I^*(i)}\ ]dt + \sigma dw_i(t), \tag{4.6}
$$

where $1 \le i \le N$ and $y_i^\infty(0) = x_i^N(0)$. Here w_i is the same as in (4.3). The process y_i^∞ is generated by the closed-loop dynamics for an agent at vertex $I^*(i)$ using the GMFG-based control law (4.2) while situated in an infinite population represented by the ensemble $\mu_G(\cdot)$ of local mean fields. We view (4.6) as an instance of the generic equation (3.10) under the control law (4.2). By Theorem 3.12, $y_i^\infty(t)$ has the law $\mu_{I^*(i)}(t)$. If $j \in \mathcal{C}(i)$, y_i^∞ and y_j^∞ are two processes of the same distribution.

We shall denote the A to C system deviation by $\epsilon_{1,N}$, the C to D deviation by $\epsilon_{2,N}$, and the (nonunilateral agent) B to D deviation by $\epsilon_{3,N}$. Specifically, we set

$$
\epsilon_{1,N} = \sup_{i \le N, t} E|\hat{x}_i^N(t) - y_i^N(t)|, \qquad \epsilon_{2,N} = \sup_{i \le N, t} E|y_i^N(t) - y_i^\infty(t)|,
$$
$$
\epsilon_{3,N} = \sup_{u_\iota^N, t, \iota \ne j \le N} E|x_j^N(t) - y_j^\infty(t)|,
$$

where x_j^N is given by (4.4).

LEMMA 4.5. *The SDE system* (4.5) *has a unique solution* $(y_1^N, \dots, y_N^N)$.

Proof. The proof is similar to [26, Thm. 6]. ∎

LEMMA 4.6. $\epsilon_{1,N} \to 0$ *as* $N \to \infty$ *(due to* $k \to \infty$*).*

Proof. By the Lipschitz property of the SDE of $\hat{x}_i^N - y_i^N$, we derive an integral inequality for $E|\hat{x}_i^N(t) - y_i^N(t)|$ and apply Gronwall's inequality under (H9); see [8] for details. ∎

LEMMA 4.7. *We have* $\epsilon_{2,N} \to 0$ *as* $N \to \infty$.

Proof. In the integral equation of y_i^∞, we approximate $(\mu_\beta(t))_{\beta \in [0,1]}$ by discrete points of β and use Gronwall's lemma and Lemma A.1 to estimate $E|y_i^\infty(t) - y_i^N(t)|$ under (H11). See [8] for details. ∎

LEMMA 4.8. $\lim_{N\to\infty} \sup_{t, i \le N} E|\hat{x}_i^N - y_i^\infty| = 0$.

Proof. The lemma follows from Lemmas 4.6 and 4.7. ∎

LEMMA 4.9. $\lim_{N\to\infty} \epsilon_{3,N} = 0$.

Proof. For $(\hat{x}_1^N, \dots, \hat{x}_N^N)$ in System A and $(x_1^N, \dots, x_N^N)$ in System B, we compare the SDEs of $\hat{x}_j^N$ and x_j^N and apply Gronwall's lemma to obtain $\sup_{u_\iota^N, t, j \ne \iota} |x_j^N - \hat{x}_j^N| \le C/\min_l |\mathcal{C}_l|$. Next by Lemma 4.8, we obtain the desired estimate. ∎

Consider the limiting optimal control problem with dynamics and cost

$$dx_\iota^\infty = \widetilde{f}[x_\iota^\infty, u_\iota, \mu_G; g_{I^*(\iota)}]dt + \sigma dw_\iota, \tag{4.7}$$

$$J_\iota^* = E\int_0^T \widetilde{l}[x_\iota^\infty, u_\iota, \mu_G; g_{I^*(\iota)}]dt, \tag{4.8}$$

where $x_\iota^\infty(0) = x_\iota^N(0)$ and $\mu_G(\cdot)$ is given by the GMFG equation system.

To establish the ϵ-Nash equilibrium property, the dynamics and cost of $\mathcal{A}_\iota$ in System B can be written using the mean field limit dynamics and cost up to small error terms that can be bounded uniformly with respect to u_ι^N. We rewrite the first equation in (4.4) of System B as

$$dx_\iota^N = \widetilde{f}[x_\iota^N, u_\iota^N, \mu_G; g_{I^*(\iota)}]dt + (\delta_{f_0}^k(t) + \delta_f^k(t))dt + \sigma dw_\iota, \tag{4.9}$$

where we denote $\delta_{f_0}^k = f_0(x_\iota^N, u_\iota^N, \mathcal{C}(\iota)) - f_0[x_\iota^N, u_\iota^N, \mu_{I^*(\iota)}]$, $\delta_f^k = f_{G_k}(x_\iota^N, u_\iota^N, g_{\mathcal{C}(\iota)}^k) - f[x_\iota^N, u_\iota^N, \mu_G; g_{I^*(\iota)}]$. Similarly the cost of $\mathcal{A}_\iota$ in System B is written as

$$J_\iota^N(u_\iota^N) = E\int_0^T \Big(\widetilde{l}[x_\iota^N, u_\iota^N, \mu_G; g_{I^*(\iota)}] + \delta_{l_0}^k(t) + \delta_l^k(t)\Big)dt,$$

where we have $\delta_{l_0}^k = l_0(x_\iota^N, u_\iota^N, \mathcal{C}(\iota)) - l_0[x_\iota^N, u_\iota^N, \mu_{I^*(\iota)}]$ and $\delta_l^k = l_{G_k}(x_\iota^N, u_\iota^N, g_{\mathcal{C}(\iota)}^k) - l[x_\iota^N, u_\iota^N, \mu_G; g_{I^*(\iota)}]$. Note that all other agents have applied the strategies $\varphi(t, x_j^N, g_{I^*(j)})$, $j \neq \iota$. So we only indicate u_ι^N within J_ι^N. It is clear that $\delta_{f_0}^k$, δ_f^k, $\delta_{l_0}^k$, and δ_l^k are all affected by the control law u_ι^N. Let $\boldsymbol{y}_t^\infty = (y_1^\infty(t), \dots, y_N^\infty(t))$ for System D. Our next step is to derive a uniform small upper bounded for $E|\delta_f^k|$ and $E|\delta_l^k|$ with respect to u_ι^N.

Let $z \in \mathbb{R}$ and $u \in U$ be deterministic and fixed; define the two random variables

$$\varDelta_f^k(z, u, \boldsymbol{y}_t^\infty) = \frac{1}{M_k}\sum_{l=1}^{M_k} g_{\mathcal{C}(\iota)\mathcal{C}_l}^k \frac{1}{|\mathcal{C}_l|}\sum_{j\in\mathcal{C}_l} f(z, u, y_j^\infty(t)) - f[z, u, \mu_G(t); g_{I^*(\iota)}],$$

$$\varDelta_l^k(z, u, \boldsymbol{y}_t^\infty) = \frac{1}{M_k}\sum_{l=1}^{M_k} g_{\mathcal{C}(\iota)\mathcal{C}_l}^k \frac{1}{|\mathcal{C}_l|}\sum_{j\in\mathcal{C}_l} l(z, u, y_j^\infty(t)) - l[z, u, \mu_G(t); g_{I^*(\iota)}].$$

LEMMA 4.10. *We have* $\lim_{k\to\infty}\sup_{z,u,t} E(|\varDelta_f^k(z,u,\boldsymbol{y}_t^\infty)|^2 + |\varDelta_l^k(z,u,\boldsymbol{y}_t^\infty)|^2) = 0$.

Proof. As in the proof of Lemma 4.7, we approximate μ_β, $\beta \in [0,1]$, by using a finite number of points of β and next expand the two quadratic terms $|\varDelta_f^k|^2$ and $|\varDelta_l^k|^2$. The estimate is carried out using (H11) and Lemma A.1. ◻

LEMMA 4.11. *For any given constant* $C_z > 0$ *and any* $\epsilon \in (0,1)$,

$$\lim_{k\to\infty}\inf_t P\big(\cap_{(z,u)\in[-C_z,C_z]\times U}\ \{|\varDelta_f^k(z,u,\boldsymbol{y}_t^\infty)| \le \epsilon\}\big) = 1, \tag{4.10}$$

$$\lim_{k\to\infty}\inf_t P\big(\cap_{(z,u)\in[-C_z,C_z]\times U}\ \{|\varDelta_l^k(z,u,\boldsymbol{y}_t^\infty)| \le \epsilon\}\big) = 1. \tag{4.11}$$

Proof. We establish (4.10) and may deal with (4.11) in the same way. The event

$$\mathcal{E}_{fC_z}^k \coloneqq \cap_{(z,u)\in[-C_z,C_z]\times U}\{|\varDelta_f^k(z,u,\boldsymbol{y}_t^\infty)| \le \epsilon\} \tag{4.12}$$

is well defined since $\varDelta_f^k$ is continuous in (z,u) and the intersection may be equivalently expressed using only a countable number of values of (z,u) in $[-C_z, C_z]\times U$.

Take any $\epsilon \in (0,1)$. By (H2) and (H3), we can find $\delta_\epsilon > 0$ such that $|\Delta_f^k(z,u,\boldsymbol{y}_t^\infty) - \Delta_f^k(z',u',\boldsymbol{y}_t^\infty)| \le \epsilon/2$ whenever $|z-z'| + |u-u'| \le \delta_\epsilon$. For the selected δ_ϵ, we can find a fixed p_0 and $(z^j,u^j) \in [-C_z, C_z] \times U$, $j = 1,\dots,p_0$ such that for any $(z,u) \in [-C_z,C_z] \times U$, there exists some j_0 ensuring $|z - z^{j_0}| + |u - u^{j_0}| \le \delta_\epsilon$.

By Lemma 4.10 and Markov's inequality, for any $\delta > 0$, there exists K_{δ,p_0} such that for all $k \ge K_{\delta,p_0}$, we have

$$P\big(\{|\Delta_f^k(z^j,u^j,\boldsymbol{y}_t^\infty)| \le \epsilon/2\}\big) \ge 1 - \delta/p_0 \quad \text{for all } j,t. \tag{4.13}$$

Let $\mathcal{E}_j^k$ denote the event $\{|\Delta_f^k(z^j,u^j,\boldsymbol{y}_t^\infty)| \le \epsilon/2\}$. By (4.13), $P(\cap_{j=1}^{p_0}\mathcal{E}_j^k) \ge 1-\delta$ for $k \ge K_{\delta,p_0}$. Now if $\omega \in \mathcal{E}^k := \cap_{j=1}^{p_0}\mathcal{E}_j^k$, $k \ge K_{\delta,p_0}$, then for any $(z,u) \in [-C_z,C_z]\times U$, we have $|\Delta_f^k(z,u,\boldsymbol{y}_t^\infty)| \le \epsilon$. Hence $\mathcal{E}^k \subset \mathcal{E}_{fC_z}^k$. It follows that for all $k \ge K_{\delta,p_0}$, $P(\mathcal{E}_{fC_z}^k) \ge 1-\delta$. Since $\delta \in (0,1)$ is arbitrary and K_{δ,p_0} does not depend on t, the first limit follows. ∎

LEMMA 4.12. *We have*

$$\lim_{k\to\infty} \sup_{t,u_\iota^N} E\Big(|\Delta_f^k(x_\iota^N(t),u_\iota^N(t),\boldsymbol{y}_t^\infty)| + |\Delta_l^k(x_\iota^N(t),u_\iota^N(t),\boldsymbol{y}_t^\infty)|\Big) = 0.$$

Proof. Fix any $\epsilon \in (0,1)$. By (H1) and (H2) we can find a sufficiently large C_z, independent of (k,N), such that for all $u_\iota^N(\cdot)$, $P(\mathcal{E}_x) \ge 1-\epsilon$, where $\mathcal{E}_x := \{\sup_{0\le t\le T}|x_\iota^N(t)| \le C_z\}$. By Lemma 4.11, for the above ϵ and $\mathcal{E}_{fC_z}^k$ given by (4.12), there exists K_0 independent of t such that for all $k \ge K_0$, $P(\mathcal{E}_{fC_z}^k) \ge 1-\epsilon$. Now if $\omega \in \mathcal{E}_x \cap \mathcal{E}_{fC_z}^k$, then $|\Delta_f^k(x_\iota^N(t),u_\iota^N(t),\boldsymbol{y}_t^\infty)| \le \epsilon$. We have $P(\mathcal{E}_x \cap \mathcal{E}_{fC_z}^k) \ge 1-2\epsilon$, and so

$$P\big(|\Delta_f^k(x_\iota^N(t),u_\iota^N(t),\boldsymbol{y}_t^\infty)| \le \epsilon\big) \ge P(\mathcal{E}_x \cap \mathcal{E}_{fC_z}^k) \ge 1-2\epsilon.$$

It follows that for all $k \ge K_0$, $E|\Delta_f^k(x_\iota^N(t),u_\iota^N(t),\boldsymbol{y}_t^\infty)| \le \epsilon + 2\epsilon C$, where C does not depend on $(u_\iota^N(\cdot),t)$. The bound for $E|\Delta_l^k(x_\iota^N(t),u_\iota^N(t),\boldsymbol{y}_t^\infty)|$ is similarly obtained. ∎

LEMMA 4.13. $\lim_{k\to\infty}\sup_{t,u_\iota^N(\cdot)} E(|\delta_f^k| + |\delta_l^k|) = 0$.

Proof. By Lipschitz continuity of (f,l), we estimate $\sup_{t,u_\iota^N} E|\delta_f^k - \Delta_f^k(x_\iota^N,u_\iota^N,\boldsymbol{y}_t^\infty)|$ and $\sup_{t,u_\iota^N} E|\delta_l^k - \Delta_l^k(x_\iota^N,u_\iota^N,\boldsymbol{y}_t^\infty)|$ and next apply Lemma 4.9 to show that they converge to zero as $k \to \infty$. Recalling Lemma 4.12, we complete the proof. ∎

LEMMA 4.14. $\lim_{k\to\infty}\sup_{t,u_\iota^N(\cdot)} E(|\delta_{f_0}^k| + |\delta_{l_0}^k|) = 0$.

Proof. The proof is similar to that of Lemma 4.13, and the details are omitted. ∎

Denote $\epsilon_{fl}^k = \sup_{t,u_\iota^N(\cdot)} E(|\delta_{f_0}^k| + |\delta_{l_0}^k| + |\delta_f^k| + |\delta_l^k|)$.

LEMMA 4.15. *For any admissible control u_ι^N in System B and J_ι^* in (4.8),*

$$J_\iota^N(u_\iota^N) \ge \inf_{u_\iota} J_\iota^*(u_\iota) - C\epsilon_{fl}^k,$$

where the constant C does not depend on u_ι^N.

Proof. Take any full state-based Lipschitz feedback control u_ι^N. It together with the other agents' control laws generates the closed-loop state processes $x_1^N(t),\dots, x_N^N(t)$. Let $u_\iota^N(t,\omega)$ denote the realization as a nonanticipative process. Now we take $\check{u}_\iota = u_\iota^N(t,\omega)$ in (4.7), and let $\check{x}_\iota^\infty$ be the resulting state process. It is clear from (4.8) that

$$J_\iota^*(\check{u}_\iota) \geq \inf_{u_\iota} J_i^*(u_\iota). \tag{4.14}$$

Recalling (4.9) and applying Gronwall's lemma to estimate the difference $\check{x}_\iota^\infty - x_\iota^N$, we can show there exists C independent of u_ι^N such that $|J_\iota^N(u_\iota^N) - J_\iota^*(\check{u}_\iota)| \leq C\epsilon_{fl}^k$, which combined with (4.14) completes the proof. ∎

LEMMA 4.16. *Let $\varphi_{I^*(\iota)} = \varphi(t, x, g_{I^*(\iota)})$ be the GMFG-based control law* (4.2). *We have $J_\iota^N(\varphi_{I^*(\iota)}) \leq \inf_{u_\iota} J_\iota^*(u_\iota) + C\epsilon_{fl}^k$.*

Proof. Let $\varphi_{I^*(\iota)}$ be applied to the two systems (4.7) and (4.9). We further use Gronwall's lemma to estimate $E|x_\iota^\infty - x_\iota^N|$. We obtain

$$|J_\iota^N(\varphi_{I^*(\iota)}) - J_\iota^*(\varphi_{I^*(\iota)})| \leq C\epsilon_{fl}^k.$$

Note that $J_\iota^*(\varphi_{I^*(\iota)}) = \inf_{u_\iota} J_\iota^*(u_\iota)$. This completes the proof. ∎

Proof of Theorem 4.4. The theorem follows from Lemmas 4.13, 4.14, 4.15, and 4.16. ∎

5. The LQG case. This section considers a special class of LQG GMFG models. Consider the graph G_k with vertices $\mathcal{V}_k = \{1, \ldots, M_k\}$ and graph adjacency matrix $g^k = [g_{jl}^k]$. For agent $\mathcal{A}_i$ in subpopulation cluster $\mathcal{C}_q$ situated at node q, let the intra- and intercluster coupling terms be denoted by $z_{0,i}$ and z_i, respectively, where

$$z_{0,i} = \frac{1}{|\mathcal{C}_q|}\sum_{j\in\mathcal{C}_q} x_j, \quad z_i = \frac{1}{|M_k|}\sum_{l\in\mathcal{V}_k} g_{ql}^k \frac{1}{|\mathcal{C}_l|}\sum_{j\in\mathcal{C}_l} x_j, \quad x_j,\ z_{0,i},\ z_i \in \mathbb{R}^n.$$

The dynamics of $\mathcal{A}_i$ are given by the linear system

$$dx_i = (Ax_i + D_0 z_{0,i} + Dz_i + Bu_i)dt + \Sigma dw_i, \quad 1 \leq i \leq N,$$

where $u_i \in \mathbb{R}^{n_u}$ is the control input, $w_i \in \mathbb{R}^{n_w}$ is a standard Brownian motion, and A, B, D_0, D, Σ are conformally dimensioned matrices. Assume $Ex_i(0) = x_0$ for all i.

The individual agent's cost function takes the form

$$\begin{aligned} J_i(u_i;\nu_i) = E\int_0^T &\big[(x_i-\nu_i)^T Q(x_i-\nu_i) + u_i^T R u_i\big]dt \\ &+ E\big[(x_i(T)-\nu_i(T))^T Q_T (x_i(T)-\nu_i(T))\big], \quad 1 \leq i \leq N, \end{aligned}$$

where Q, $Q_T \geq 0$, $R > 0$, and $\nu_i = \gamma_0 z_{0,i} + \gamma z_i + \eta$ is the process tracked by $\mathcal{A}_i$. Here $\eta \in \mathbb{R}^n$ and $\gamma_0, \gamma \in \mathbb{R}$.

In the infinite population and graphon limit case, denote the local mean $\int_{\mathbb{R}^n} x\mu_\alpha(dx)$ at t for an α-agent situated at vertex α by $\bar{x}_\alpha$ and the graphon weighted mean $\int_0^1 g(\alpha,\beta)\bar{x}_\beta d\beta$ by z_α. The α-agent's state equation is given by

$$dx_\alpha = (Ax_\alpha + D_0\bar{x}_\alpha + Dz_\alpha + Bu_\alpha)dt + \Sigma dw_\alpha, \quad \alpha \in [0,1].$$

The α-agent's cost function is

$$\begin{aligned} J_\alpha(u_\alpha;\nu_\alpha) = E\int_0^T &\big[(x_\alpha-\nu_\alpha)^T Q(x_\alpha-\nu_\alpha) + u_\alpha^T R u_\alpha\big]dt \\ &+ E\big[(x_\alpha(T)-\nu_\alpha(T))^T Q_T (x_\alpha(T)-\nu_\alpha(T))\big], \end{aligned}$$

where $\nu_\alpha = \gamma_0\bar{x}_\alpha + \gamma z_\alpha + \eta$.

Consider the Riccati equation

$$0 = \dot{\mathit{\Pi}}_t + A^T \mathit{\Pi}_t + \mathit{\Pi}_t A - \mathit{\Pi}_t B R^{-1} B^T \mathit{\Pi}_t + Q,$$

where $\mathit{\Pi}_T = Q_T$, and

$$0 = \dot{s}_\alpha(t) + (A - BR^{-1}B^T \mathit{\Pi}_t)^T s_\alpha(t) + \mathit{\Pi}_t(D_0 \bar{x}_\alpha(t) + D z_\alpha(t)) - Q\nu_\alpha(t),$$

where $s_\alpha(T) = -Q_T \nu_\alpha(T)$. The BR for the α-agent is given by

$$u_\alpha(t) = -R^{-1}B^T[\mathit{\Pi}_t x_\alpha(t) + s_\alpha(t)].$$

Now the mean state process of x_α is

$$\dot{\bar{x}}_\alpha = (A - BR^{-1}B^T \mathit{\Pi}_t + D_0)\bar{x}_\alpha + D z_\alpha - BR^{-1}B^T s_\alpha, \quad \alpha \in [0,1].$$

The existence analysis reduces to verifying the existence and uniqueness of solutions for the equation system:

$$\dot{\bar{x}}_\alpha = (A - BR^{-1}B^T \mathit{\Pi}_t + D_0)\bar{x}_\alpha - BR^{-1}B^T s_\alpha + D\int_0^1 g(\alpha,\beta)\bar{x}_\beta d\beta, \tag{5.1}$$

$$\begin{aligned}\dot{s}_\alpha = &-(A - BR^{-1}B^T \mathit{\Pi}_t)^T s_\alpha + (\gamma_0 Q - \mathit{\Pi}_t D_0)\bar{x}_\alpha \\ &+ (\gamma Q - \mathit{\Pi}_t D)\int_0^1 g(\alpha,\beta)\bar{x}_\beta d\beta + Q\eta,\end{aligned} \tag{5.2}$$

where $\bar{x}_\alpha(0) = x_0$ and $s_\alpha(T) = -Q_T[\gamma_0 \bar{x}_\alpha(T) + \gamma \int_0^1 g(\alpha,\beta)\bar{x}_\beta(T)d\beta + \eta]$.

To analyze (5.1) and (5.2), let $\mathit{\Phi}(t,s)$ and $\mathit{\Psi}(t,s)$ be the fundamental solution matrix of

$$\dot{x} = (A - BR^{-1}B^T \mathit{\Pi}_t + D_0)x, \qquad \dot{y} = -(A - BR^{-1}B^T \mathit{\Pi}_t)^T y$$

for $x(t), y(t) \in \mathbb{R}^n$. For the special case with $D_0 = 0$, $\mathit{\Psi}(t,s) = \mathit{\Phi}^T(s,t)$ holds. We convert the existence analysis into a fixed point problem. We view $\bar{x}_\beta(t) = \bar{x}(\beta,t)$ as a function of (β,t). Below we derive an equation for $\bar{x}_\alpha(t)$ by eliminating $s_\alpha(t)$. Denote the function space $D_\mathit{\Lambda}$ consisting of continuous $\mathbb{R}^n$-valued functions on $[0,1]\times[0,T]$ with norm $\|\check{x}\| = \sup_{\alpha,t}|\check{x}(\alpha,t)|$. We use $|\cdot|$ to denote the Frobenius norm of a vector or matrix. Define the operator $\mathit{\Lambda}$ as follows: for $\check{x} \in D_\mathit{\Lambda}$,

$$\begin{aligned}(\mathit{\Lambda}\check{x})(\alpha,t) = &\int_0^t \mathit{\Phi}(t,r)BR^{-1}B^T \left\{ \int_r^T \mathit{\Psi}(r,\tau)\left[(\gamma_0 Q - \mathit{\Pi}_\tau D_0)\check{x}(\alpha,\tau)\right.\right. \\ &\left. + (\gamma Q - \mathit{\Pi}_\tau D)\int_0^1 g(\alpha,\beta)\check{x}(\beta,\tau)d\beta\right] d\tau \\ &\left. + \mathit{\Psi}(r,T)Q_T\left[\gamma_0 \check{x}(\alpha,T) + \gamma\int_0^1 g(\alpha,\beta)\check{x}(\beta,T)d\beta\right]\right\} dr \\ &+ \int_0^t \mathit{\Phi}(t,r)D\int_0^1 g(\alpha,\beta)\check{x}(\beta,r)d\beta dr.\end{aligned}$$

If (H5) holds, $\mathit{\Lambda}$ is from $D_\mathit{\Lambda}$ to itself.

The solution of the LQG GMFG reduces to finding a fixed point $\check{x}$ to the equation

$$\check{x}(\alpha,t) = (\Lambda\check{x})(\alpha,t) + \Phi(t,0)x_0 + \int_0^t \Phi(t,r)BR^{-1}B^T\left[\int_r^T \Psi(r,\tau)Qd\tau + \Psi(r,T)Q_T\right]\eta dr.$$

Denote $c_g = \max_\alpha \int_0^1 g(\alpha,\beta)d\beta$. We have the bound for the operator norm:

$$\|\Lambda\| \le c_\Lambda := \sup_{t\in[0,T]}\Bigg\{\int_0^t\int_r^T |\Phi(t,r)BR^{-1}B^T\Psi(r,\tau)|\cdot\big(|\gamma_0 Q - \Pi_\tau D_0| + c_g|\gamma Q - \Pi_\tau D|\big)d\tau dr + \int_0^t \left[|\Phi(t,r)BR^{-1}B^T\Psi(r,T)Q_T|\cdot(|\gamma_0| + c_g|\gamma|) + c_g|\Phi(t,r)D|\right]dr\Bigg\}.$$

If $c_\Lambda < 1$, Λ is a contraction and (5.1) and (5.2) have a unique solution.

As an example for illustration, we assume the graphon weighted mean at vertex α arises from an underlying *uniform attachment graphon*, and consequently

$$z_\alpha = \int_0^1 (1-\max(\alpha,\beta))\int_{\mathbb{R}^n} x\mu_\beta(dx)d\beta, \quad \alpha,\beta\in[0,1],$$

where it is readily verified that the uniform attachment graphon satisfies (H5).

Appendix A.

LEMMA A.1. *Assume* (H1)–(H8). *Let φ_α be the GMFG-based BR* (4.2) *and $\mu_\alpha(t)$ the distribution of the closed-loop process $x_\alpha(t)$, $\alpha\in[0,1]$, in* (3.15) *with initial distribution μ_0^x. Then we have*

$$\lim_{r\to 0}\sup_{|t-t^*|+|\beta-\beta^*|<r} W_1(\mu_\beta(t),\mu_{\beta^*}(t^*)) = 0,$$

where $t,t^*\in[0,T]$, and $\beta,\beta^*\in[0,1]$.

Proof. Due to limited space, we only give a sketch; see [8] for a more detailed proof.

Step 1. Take any $\beta,\beta^*\in[0,1]$. For $\mu_G(\cdot)$ determined from the GMFG equations (3.13) and (3.15), define two processes:

$$dy_{\beta^*} = \widetilde{f}[y_{\beta^*},\varphi(t,y_{\beta^*},g_{\beta^*}),\mu_G;g_{\beta^*}]dt + \sigma dw_{\beta^*},$$
$$dy_\beta = \widetilde{f}[y_\beta,\varphi(t,y_\beta,g_\beta),\mu_G;g_\beta]dt + \sigma dw_{\beta^*},$$

where $y_{\beta^*}(0) = y_\beta(0) = x_i^N(0)$ and the same Brownian motion is used. Then the distributions of $y_{\beta^*}(t)$ and $y_\beta(t)$ are $\mu_{\beta^*}(t)$ and $\mu_\beta(t)$, respectively.

By comparing the two SDEs, we estimate $\sup_{0\le t\le T} E|y_\beta(t)-y_{\beta^*}(t)|$, and next by $W_1(\mu_\beta(t),\mu_{\beta^*}(t)) \le E|y_\beta(t)-y_{\beta^*}(t)|$, we obtain $\lim_{\beta\to\beta^*}\sup_t W_1(\mu_\beta(t),\mu_{\beta^*}(t)) = 0$.

Step 2. Now we consider a given $(\beta^*,t^*)\in[0,1]\times[0,T]$. By use of the SDE of y_β and elementary estimates, we obtain $\lim_{|t-t^*|\to 0}\sup_\beta W_1(\mu_\beta(t^*),\mu_\beta(t)) = 0$. Since $W_1(\mu_\beta(t),\mu_{\beta^*}(t^*)) \le W_1(\mu_\beta(t),\mu_\beta(t^*)) + W_1(\mu_\beta(t^*),\mu_{\beta^*}(t^*))$, we conclude that $\mu_\beta(t)$ as a mapping from the compact space $[0,1]\times[0,T]$ to $\mathcal{P}_1(\mathbb{R})$ with the metric $W_1(\cdot,\cdot)$ is continuous and hence must be uniformly continuous. The lemma follows. ∎

REFERENCES

[1] E. Bayraktar, S. Chakraborty, and R. Wu, *Graphon Mean Field Systems*, preprint, arXiv:2003.13180, 2020.

[2] C. Borgs, J. Chayes, L. Lovász, V. T. Sós, B. Szegedy, and K. Vesztergombi, *Graph limits and parameter testing*, in Proceedings of the 38th Annual ACM Symposium on Theory of Computing, New York, 2006, pp. 261–270.

[3] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi, *Convergent sequences of dense graphs* I*: Subgraph frequencies, metric properties and testing*, Adv. Math., 219 (2008), pp. 1801–1851.

[4] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi, *Convergent sequences of dense graphs* II*. Multiway cuts and statistical physics*, Ann. Math., 176 (2012), pp. 151–219.

[5] P. E. Caines, *Mean field games*, in Encyclopedia of Systems and Control, J. Baillieul and T. Samad, eds., Springer, London, 2015, pp. 706–712.

[6] P. E. Caines and M. Huang, *Graphon mean field games and the GMFG equations*, in Proceedings of the 57th IEEE Conference on Decision and Control, Miami Beach, FL, 2018, pp. 4129–4134.

[7] P. E. Caines and M. Huang, *Graphon mean field games and the GMFG equations: ϵ-Nash equilibria*, in Proceedings of the 58th IEEE Conference on Decision and Control, Nice, France, 2019, pp. 286–292.

[8] P. E. Caines and M. Huang, *Graphon Mean Field Games and the GMFG Equations*, preprint, arXiv:2008.10216v3, 2021.

[9] P. E. Caines, M. Huang, and R. P. Malhamé, *Mean field games*, in Handbook of Dynamic Game Theory, T. Başar and G. Zaccour, eds., Springer, Berlin, 2017, pp. 345–372.

[10] R. Carmona and F. Delarue, *Probabilistic Theory of Mean Field Games with Applications* I, Prob. Theory Stoch. Model. 83, Springer, Cham, Switzerland, 2018.

[11] R. Carmona and F. Delarue, *Probabilistic Theory of Mean Field Games with Applications* II, Prob. Theory Stoch. Model. 84, Springer, Cham, Switzerland, 2018.

[12] F. Delarue, *Mean field games: A toy model on an Erdös–Renyi graph*, ESAIM Proc. Surv., 60 (2017), pp. 1–26.

[13] J. L. Doob, *Stochastic Processes*, Wiley, New York, 1953.

[14] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer, New York, 1975.

[15] I. Gallagher, *From Newton to Navier–Stokes, or how to connect fluid mechanics equations from microscopic to macroscopic scales*, Bull. Amer. Math. Soc., 56 (2013), pp. 65–85.

[16] I. Gallagher, L. Saint-Raymond, and B. Texier, *From Newton to Boltzmann: Hard Spheres and Short-Range Potentials*, Zur. Lect. Adv. Math. 18, European Mathematical Society, 2013.

[17] S. Gao and P. E. Caines, *The control of arbitrary size networks of linear systems via graphon limits: An initial investigation*, in Proceedings of the 56th IEEE Conference on Decision and Control, Melbourne, Australia, 2017, pp. 1052–1057.

[18] S. Gao and P. E. Caines, *Controlling complex networks of linear systems via graphon limits*, in Proceedings of the John Templeton Foundation Symposium of Controlling Complex Networks of NetSci17, Indianapolis, IN, 2017.

[19] S. Gao and P. E. Caines, *Minimum energy control of arbitrary size networks of linear systems via graphon limits*, in Proceedings of the SIAM Workshop on Network Science, Pittsburgh, PA, 2017.

[20] S. Gao and P. E. Caines, *Graphon linear quadratic regulation of large-scale networks of linear systems*, in Proceedings of the 57th IEEE Conference on Decision and Control, Miami Beach, FL, 2018, pp. 5892–5897.

[21] S. Gao and P. E. Caines, *Graphon control of large-scale networks of linear systems*, IEEE Trans. Autom. Control, 65 (2020), pp. 4090–4105.

[22] O. Guéant, *Existence and uniqueness result for mean field games with congestion effect on graphs*, Appl. Math. Optim., 72 (2015), pp. 291–303.

[23] I. H. Herron and M. R. Foster, *Partial Differential Equations in Fluid Dynamics*, Cambridge University Press, Cambridge, UK, 2008.

[24] M. Huang, P. E. Caines, and R. P. Malhamé, *Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized ε-Nash equilibria*, IEEE Trans. Automat. Control, 52 (2007), pp. 1560–1571.

[25] M. Huang, P. E. Caines, and R. P. Malhamé, *The NCE (mean field) principle with locality dependent cost interactions*, IEEE Trans. Automat. Control, 55 (2010), pp. 2799–2805.

[26] M. Huang, R. P. Malhamé, and P. E. Caines, *Large population stochastic dynamic games: Closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle*, Commun. Inf. Syst., 6 (2006), pp. 221–252.

[27] D. Kaliuzhnyi-Verbovetskyi and G. S. Medvedev, *The semilinear heat equation on sparse random graphs*, SIAM J. Math. Anal., 49 (2017), pp. 1333–1355.

[28] I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus*, Grad. Texts in Math. 113, Springer, New York, 2012.

[29] V. Kolokoltsov and W. Yang, *Sensitivity Analysis for HJB Equations with an Application to Coupled Backward-Forward Systems*, preprint, arXiv:1303.6234v2, 2015.

[30] O. A. Ladyzhenskaya, N. Ural'ceva, and V. Solonnikov, *Linear and Quasi-linear Equations of Parabolic Type*, American Mathematical Society, Providence, RI, 1968.

[31] J.-M. Lasry and P.-L. Lions, *Jeux à champ moyen.* I - *Le cas stationnaire*, C. R. Acad. Sci. Paris Ser. I, 343 (2006), pp. 619–625.

[32] J.-M. Lasry and P.-L. Lions, *Jeux à champ moyen.* II-*Horizon fini et controle optimal*, C. R. Acad. Sci. Paris Ser. I, 343 (2006), pp. 679–684.

[33] L. Lovász, *Large Networks and Graph Limits*, vol. 60, American Mathematical Society, Providence, RI, 2012.

[34] L. Lovász and B. Szegedy, *Limits of dense graph sequences*, J. Combin. Theory Ser. B, 96 (2006), pp. 933–957.

[35] G. S. Medvedev, *The nonlinear heat equation on dense graphs and graph limits*, SIAM J. Math. Anal., 46 (2014), pp. 2743–2766.

[36] G. S. Medvedev, *The nonlinear heat equation on w-random graphs*, Arch. Ration. Mech. Anal., 212 (2014), pp. 781–803.

[37] F. Parise and A. Ozdaglar, *Graphon Games: A Statistical Framework for Network Games and Interventions*, preprint, arXiv:1802.00080, 2018.

[38] W. Pauli and C. P. Enz, *Thermodynamics and the Kinetic Theory of Gases*, vol. 3, Courier Corporation, Chelmsford, MA, 2000.

[39] C. Quininao and J. Touboul, *Limits and dynamics of randomly connected neuronal networks*, Acta Appl. Math., 136 (2015), pp. 167–192.

[40] N. Sen and P. E. Caines, *Mean field game theory with a partially observed major agent*, SIAM J. Control Optim., 54 (2016), pp. 3174–3224.

[41] A.-S. Sznitman, *Topics in propagation of chaos*, in Ecole d'Été de Probabilités de Saint-Flour XIX-1989, P.-L. Hennequin, ed., Springer, Berlin, 1991, pp. 165–251.