CHAPTER 2, QUESTION 27

27. Let m be a positive integer with $m \equiv 1 \pmod 4$ which possesses a prime divisor $q \equiv 3 \pmod 4$. Prove that there are no integers T and U such that $T^2 + TU + \frac{1}{4}(1-m)U^2 = -1$.

Solution. Let m be a positive integer with $m \equiv 1 \pmod 4$, which possesses a prime divisor $q \equiv 3 \pmod 4$. Suppose that there exist integers T and U such that

$$T^2 + TU + \frac{1}{4}(1-m)U^2 = -1.$$

By Question 26 there exist integers X and Y such that

$$X^2 - mY^2 = -4.$$

As $q \mid m$ and q is odd, we see that $q \nmid X$. Then, as $q \equiv 3 \pmod 4$, we obtain

$$-1 = \left(\frac{-1}{q}\right) = \left(\frac{-4}{q}\right) = \left(\frac{X^2 - mY^2}{q}\right) = \left(\frac{X^2}{q}\right) = 1,$$

a contradiction. Hence no such integers T and U exist. ∎

June 20, 2004