10. If K and L are submodules of an R-module M with $K \subseteq L$, prove that

$$M/K \;/\; L/K \simeq M/L.$$

Solution. As K and L are submodules of the R-module M, the factor modules M/K and M/L are R-modules. Define a map $\theta : M/K \longrightarrow M/L$ by

$$\begin{aligned}\theta(m+K) &= m+L, \quad m \in M,\\ \theta(r(m+K)) &= r(m+L), \quad r \in R,\ m \in M.\end{aligned}$$

We first show that θ is well defined. If $m+K = m'+K$, where $m, m' \in M$, then $m - m' \in K$. But $K \subseteq L$. Thus $m - m' \in L$. Hence $m + L = m' + L$ and so $\theta(m+K) = \theta(m'+K)$. If $r(m+K) = r'(m'+K)$, where $r, r' \in R$ and $m, m' \in K$, then $rm + K = r'm' + K$ so $rm - r'm' \in K$. But $K \subseteq L$ so $rm - r'm' \in L$. Thus $rm + L = r'm' + L$ and so $\theta(r(m+K)) = r(m+L) = rm + L = r'm' + L = r'(m'+L) = \theta(r'(m'+K))$.

Next we show that θ is a module homomorphism. Let $m_1, m_2 \in M$ so that $m_1 + K$, $m_2 + K \in M/K$. Then,

$$\theta((m_1+K)+(m_2+K)) = \theta(m_1+m_2+K) = m_1+m_2+L = (m_1+L)+(m_2+L)$$

so that

$$\theta((m_1+K)+(m_2+K)) = \theta(m_1+K) + \theta(m_2+K).$$

Now let $r \in R$ and $m \in M$. Then

$$\theta(r(m+K)) = r(m+L) = r\theta(m+K).$$

This completes the proof that θ is a module homomorphism.

Next we determine $\ker\theta$. We have

$$\begin{aligned}\ker\theta &= \{m+K \in M/K \mid \theta(m+K) = 0+L\}\\ &= \{m+K \in M/K \mid m+L = 0+L\}\\ &= \{m+K \in M/K \mid m \in L\}\\ &= L/K.\end{aligned}$$

Now we determine im θ. As $\theta : M/K \longrightarrow M/L$ we have im $\theta \subseteq M/L$. As $m+L = \theta(m+K)$ for all $m \in M$ we have $M/L \subseteq$ im θ. Hence im $\theta = M/L$.

Finally by Question 9 (iii) we have

$$M/K \;/\; L/K = M/K \;/\; \ker\theta \simeq \operatorname{im}\theta = M/L. \qquad \blacksquare$$

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