CHAPTER 3, QUESTION 5

5. Prove that the ideal $< 2, X >$ in $\mathbb{Z}[X]$ is not principal (Example 3.3.6).

Solution. Suppose that the ideal $I =< 2, X >$ in $\mathbb{Z}[X]$ is principal. Then there exists $f(X) \in \mathbb{Z}[X]$ such that $I =< f(X) >$. As $2 \in I$ there exists $g(X) \in \mathbb{Z}[X]$ such that

$$2 = f(X)g(X).$$

Hence

$$\deg f(X) + \deg g(X) = \deg f(X)g(X) = \deg\ 2 = 0$$

so

$$\deg f(X) = \deg g(X) = 0.$$

Thus

$$f(X) = a,\ \ g(X) = b,\ \ a, b \in \mathbb{Z}.$$

As

$$2 = ab$$

we have

$$a = \pm 1 \text{ or } \pm 2.$$

If $a = \pm 1$ then $< 2, X >=< 1 >$. Hence there exist $r(X), s(X) \in \mathbb{Z}[X]$ such that

$$1 = 2r(X) + Xs(X).$$

Equating constant terms in these polynomials we obtain

$$1 = 2r(0) \text{ so that } r(0) = \frac{1}{2},$$

contradicting $r(0) \in \mathbb{Z}$.

If $a = \pm 2$ then $< 2, X >=< 2 >$. Hence there exists $v(X) \in \mathbb{Z}[X]$ such that

$$X = 2v(X).$$

Then $v(X) = \frac{1}{2}X$, contradicting $v(X) \in \mathbb{Z}[X]$. This proves that $< 2, X >$ is nonprincipal. $\blacksquare$

June 20, 2004