EXERCISES 4, QUESTION 1

1. Prove that

$$\frac{1}{3}(1 + 10^{1/3} + 10^{2/3})$$

is an algebraic integer.

Solution. Let

$$\alpha = \frac{1}{3}(1 + 10^{1/3} + 10^{2/3}).$$

Then

$$3\alpha - 1 = 10^{1/3} + 10^{2/3}.$$

Cubing both sides, we obtain

$$27\alpha^3 - 27\alpha^2 + 9\alpha - 1 = 110 + 30(10^{1/3} + 10^{2/3}) = 80 + 90\alpha,$$

so that

$$\alpha^3 - \alpha^2 - 3\alpha - 3 = 0.$$

As α is the root of a monic polynomial with integer coefficients, α is an algebraic integer. ∎

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