

EXERCISES 4, QUESTION 2

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2. Prove that

$$\frac{10^{2/3} - 1}{\sqrt{-3}}$$

is an algebraic integer.

Solution. Let

$$\alpha = \frac{10^{2/3} - 1}{\sqrt{-3}}.$$

Then

$$\sqrt{-3}\alpha + 1 = 10^{2/3}.$$

Cubing both sides, we obtain

$$(1 - 9\alpha^2) + 3\sqrt{-3}(\alpha - \alpha^3) = 100.$$

Hence

$$\sqrt{-3}(\alpha - \alpha^3) = 33 + 3\alpha^2.$$

Squaring both sides, we have

$$-3(\alpha^2 - 2\alpha^4 + \alpha^6) = 1089 + 198\alpha^2 + 9\alpha^4$$

and thus

$$\alpha^6 + \alpha^4 + 67\alpha^2 + 363 = 0.$$

As  $\alpha$  is the root of a monic polynomial with integer coefficients,  $\alpha$  is an algebraic integer. ■

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