CHAPTER 4, QUESTION 8

8. Let θ be a root of $x^3+6x+34$. Prove that the domain $\mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2$ is not integrally closed. [HINT: Consider $\phi = \dfrac{1+\theta}{3}$.]

Solution. Let

$$D = \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2 = \{a+b\theta+c\theta^2 \mid a,b,c \in \mathbb{Z},\ \theta^3+6\theta+34=0\}.$$

The field of quotients of D is

$$K = \mathbb{Q}(\theta) = \{r+s\theta+t\theta^2 \mid r,s,t \in \mathbb{Q}\}.$$

Clearly

$$\phi = \frac{1+\theta}{3} \in K, \quad \phi \notin D.$$

Now

$$\begin{aligned}
\phi^3-\phi^2+\phi+1 &= \left(\frac{1+\theta}{3}\right)^3 - \left(\frac{1+\theta}{3}\right)^2 + \left(\frac{1+\theta}{3}\right) + 1 \\
&= \frac{1}{27}(1+3\theta+3\theta^2+\theta^3-3-6\theta-3\theta^2+9+9\theta+27) \\
&= \frac{1}{27}(\theta^3+6\theta+34) \\
&= 0
\end{aligned}$$

so that ϕ is integral over D. Hence D is not integrally closed. ∎

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