CHAPTER 5, QUESTION 13

13. Prove that $\dfrac{-\theta+\theta^2}{2}$ is an algebraic integer in $K=\mathbb{Q}(\theta)$, where $\theta^3+11\theta-4=0$.

Solution. Let $\phi=\dfrac{-\theta+\theta^2}{2}$. Clearly $\phi\in K=\mathbb{Q}(\theta)$. We have

$$
\begin{aligned}
\phi^2 &= \left(\frac{-\theta+\theta^2}{2}\right)^2 = \frac{\theta^2-2\theta^3+\theta^4}{4} = \frac{\theta^2-2(-11\theta+4)+\theta(-11\theta+4)}{4} \\
&= \frac{-8+26\theta-10\theta^2}{4} = \frac{-4+13\theta-5\theta^2}{2}, \\
\phi^3 &= \phi\phi^2 = \left(\frac{-\theta+\theta^2}{2}\right)\left(\frac{-4+13\theta-5\theta^2}{2}\right) = \frac{4\theta-17\theta^2+18\theta^3-5\theta^4}{4} \\
&= \frac{4\theta-17\theta^2+18(-11\theta+4)-5\theta(-11\theta+4)}{4} \\
&= \frac{72-214\theta+38\theta^2}{4} = \frac{36-107\theta+19\theta^2}{2}.
\end{aligned}
$$

We seek $A,B,C\in\mathbb{Z}$ such that

$$
\begin{aligned}
0 &= \phi^3+A\phi^2+B\phi+C \\
&= \left(\frac{36-107\theta+19\theta^2}{2}\right)+A\left(\frac{-4+13\theta-5\theta^2}{2}\right)+B\left(\frac{-\theta+\theta^2}{2}\right)+C \\
&= \frac{1}{2}\left((36-4A+2C)+(-107+13A-B)\theta+(19-5A+B)\theta^2\right).
\end{aligned}
$$

As $1,\theta,\theta^2$ are linearly independent over $\mathbb{Q}$, we must have

$$
\begin{aligned}
36-4A+2C &= 0, \\
-107+13A-B &= 0, \\
19-5A+B &= 0.
\end{aligned}
$$

Solving these equations, we obtain

$$A=11,\quad B=36,\quad C=4.$$

Hence ϕ is a root of $x^3+11x^2+36x+4\in\mathbb{Z}[x]$. Thus ϕ is an algebraic integer in K. ■

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