CHAPTER 5, QUESTION 16

16. Let $\omega = e^{2\pi i/5}$. Show that $\sqrt{5} \in \mathbb{Q}(\omega)$ by expressing $\sqrt{5}$ in the form

$$\sqrt{5} = a\omega + b\omega^2 + c\omega^3 + d\omega^4$$

for suitable integers a, b, c, d.
Solution. By Question 15 we have

$$\omega = \frac{1}{4}(\sqrt{5} - 1 + i\sqrt{10 + 2\sqrt{5}}).$$

Thus

$$\begin{aligned}
\omega^2 &= \frac{1}{4}(-1 - \sqrt{5} + i\sqrt{10 - 2\sqrt{5}}),\\
\omega^3 &= \frac{1}{4}(-1 - \sqrt{5} - i\sqrt{10 - 2\sqrt{5}}),\\
\omega^4 &= \frac{1}{4}(\sqrt{5} - 1 - i\sqrt{10 + 2\sqrt{5}}).
\end{aligned}$$

We seek a, b, c, d such that

$$\begin{aligned}
&a\left(\frac{1}{4}(-1 + \sqrt{5} + i\sqrt{10 + 2\sqrt{5}})\right) + b\left(\frac{1}{4}(-1 - \sqrt{5} + i\sqrt{10 - 2\sqrt{5}})\right)\\
&+c\left(\frac{1}{4}(-1 - \sqrt{5} - i\sqrt{10 - 2\sqrt{5}})\right) + d\left(\frac{1}{4}(-1 + \sqrt{5} - i\sqrt{10 + 2\sqrt{5}})\right) = 5,
\end{aligned}$$

that is we want a, b, c, d to satisfy

$$\begin{aligned}
-\frac{1}{4}a - \frac{1}{4}b - \frac{1}{4}c - \frac{1}{4}d &= 0,\\
\frac{1}{4}a - \frac{1}{4}b - \frac{1}{4}c + \frac{1}{4}d &= 1,\\
\frac{1}{4}a \qquad\qquad - \frac{1}{4}d &= 0,\\
\frac{1}{4}b - \frac{1}{4}c \qquad &= 0.
\end{aligned}$$

Solving these linear equations,we obtain

$$a = 1, \quad b = -1, \quad c = -1, \quad d = 1.$$

Thus

$$\sqrt{5} = \omega - \omega^2 - \omega^3 + \omega^4 \in \mathbb{Q}(\omega). \qquad \blacksquare$$

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