EXERCISES 5, QUESTION 2

2. Prove that $x^4 + 1$ is irreducible in $\mathbb{Q}[x]$, see Example 5.1.1.

Solution. Let $f(x) = x^4 + 1$. As $f(\pm 1) = 2 \neq 0$, ± 1 are not roots of $f(x)$. Thus $f(x)$ has no linear factors in $\mathbb{Z}[x]$. Suppose $f(x)$ has quadratic factors in $\mathbb{Z}[x]$, say,

$$f(x) = (x^2 + ax + b)(x^2 + cx + d),\ \ a, b, c, d \in \mathbb{Z}.$$

Then

$$a + c = 0, \quad (1)$$
$$ac + b + d = 0, \quad (2)$$
$$ad + bc = 0, \quad (3)$$
$$bd = 1. \quad (4)$$

From (4) we deduce that

$$b = d = \epsilon,\ \ \epsilon = \pm 1.$$

Then, from (1) and (2), we obtain

$$a^2 = -ac = b + d = 2\epsilon,$$

which is impossible. Thus $f(x)$ is irreducible in $\mathbb{Z}[x]$. As $f(x) \in \mathbb{Z}[x]$ it is therefore irreducible in $\mathbb{Q}[x]$. ∎

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