CHAPTER 5, QUESTION 4

4. Determine

$$\operatorname{irr}_{\mathbb{Q}(i)}\left(\frac{1+i}{\sqrt{2}}\right)$$

and

$$\operatorname{irr}_{\mathbb{Q}(\sqrt{-2})}\left(\frac{1+i}{\sqrt{2}}\right).$$

Solution. Let $\alpha = \frac{1+i}{\sqrt{2}}$. α is a root of $x^2 - i \in \mathbb{Q}(i)[x]$. Clearly $x^2 - i$ is irreducible in $\mathbb{Q}(i)[x]$ as the two roots $x = \pm\frac{1+i}{\sqrt{2}}$ of $x^2 - i = 0$ do not belong in $\mathbb{Q}(i)$ (as $\sqrt{2} \notin \mathbb{Q}(i)$). Hence

$$\operatorname{irr}_{\mathbb{Q}(i)}\left(\frac{1+i}{\sqrt{2}}\right) = x^2 - i.$$

α is also a root of $x^2 - \sqrt{-2}x - 1 \in \mathbb{Q}(\sqrt{-2})[x]$. The two roots $x = \frac{\sqrt{-2}\pm\sqrt{2}}{2}$ of $x^2 - \sqrt{-2}x - 1 = 0$ do not belong in $\mathbb{Q}(\sqrt{-2})$ (as $\sqrt{2} \notin \mathbb{Q}(\sqrt{-2})$) so that $x^2 - \sqrt{-2}x - 1$ is irreducible in $\mathbb{Q}(\sqrt{-2})[x]$. Hence

$$\operatorname{irr}_{\mathbb{Q}(\sqrt{-2})}\left(\frac{1+i}{\sqrt{2}}\right) = x^2 - \sqrt{-2}x - 1. \qquad \blacksquare$$

June 21, 2004