CHAPTER 5, QUESTION 8

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8. Find the minimal polynomial of $2^{1/3} + \omega$ over $\mathbb{Q}(2^{1/3})$, where ω is a complex cube root of unity.

Solution. Let $\alpha = 2^{1/3} + \omega$. Then $\omega = \alpha - 2^{1/3}$. As $\omega^2 + \omega + 1 = 0$, we have

$$(\alpha - 2^{1/3})^2 + (\alpha - 2^{1/3}) + 1 = 0.$$

Thus α is a root of

$$f(x) = x^2 + (1 - 2^{4/3})x + (1 - 2^{1/3} + 2^{2/3}) \in \mathbb{Q}(2^{1/3})[x].$$

The other root of $f(x)$ is clearly $2^{1/3} + \omega^2$. As

$$2^{1/3} + \omega \in \mathbb{C}\ \mathbb{R}, \quad 2^{1/3} + \omega^2 \in \mathbb{C} \setminus \mathbb{R},$$

$f(x)$ cannot have roots in $\mathbb{Q}(2^{1/3}) \subset \mathbb{R}$. Thus $f(x)$ is irreducible in $\mathbb{Q}(2^{1/3})[x]$. Hence $f(x)$ is the minimal polynomial of $2^{1/3} + \omega$ over $\mathbb{Q}(2^{1/3})$. ∎

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