EXERCISES 6, QUESTION 1

1. Let K be an algebraic number field of degree n. Let $\theta \in K$ be such that $K = \mathbb{Q}(\theta)$. Let $\theta_1 = \theta, \theta_2, \ldots, \theta_n$ be the conjugates of θ over $\mathbb{Q}$. Let $\alpha \in K$ so there exist unique rational numbers $c_0, c_1, \ldots, c_{n-1}$ such that

$$\alpha = c_0 + c_1\theta + \cdots + c_{n-1}\theta^{n-1}.$$

For $k = 1, 2, \ldots, n$ let

$$\alpha_k = c_0 + c_1\theta_k + \cdots + c_{n-1}\theta_k^{n-1}$$

so that $\alpha_1 = \alpha$. Prove that the set of conjugates $\{\alpha_1, \ldots, \alpha_n\}$ of α relative to K does not depend on the choice of θ.

Solution. Let $K = \mathbb{Q}(\theta)$, $[K : \mathbb{Q}] = n$. Let $\alpha \in K$. Then there exist $c_0, c_1, \ldots, c_{n-1} \in \mathbb{Q}$ such that

$$\alpha = c_0 + c_1\theta + \cdots + c_{n-1}\theta^{n-1}.$$

Let $\sigma_i : K \to \mathbb{C}$ $(i = 1, 2, \ldots, n)$ be the n monomorphisms from K to $\mathbb{C}$. Set

$$\sigma_i(\theta) = \theta_i \ (i = 1, 2, \ldots, n).$$

The K-conjugates of α are

$$\alpha_i = c_0 + c_1\theta_i + \cdots + c_{n-1}\theta_i^{n-1} \ (i = 1, 2, \ldots, n).$$

Let $\phi \in K$ be such that $K = \mathbb{Q}(\phi)$. Set

$$\sigma_i(\phi) = \phi_i \ (i = 1, 2, \ldots, n).$$

As $\alpha \in K$ and $K = \mathbb{Q}(\phi)$ there exist $d_0, d_1, \ldots, d_{n-1} \in \mathbb{Q}$ such that

$$\alpha = d_0 + d_1\phi + \cdots + d_{n-1}\phi^{n-1}.$$

Then, for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
\alpha_i &= c_0 + c_1\theta_i + \cdots + c_{n-1}\theta_i^{n-1} \\
&= c_0 + c_1\sigma_i(\theta) + \cdots + c_{n-1}\sigma_i(\theta^{n-1}) \\
&= \sigma_i(c_0 + c_1\theta + \cdots + c_{n-1}\theta^{n-1}) \\
&= \sigma_i(d_0 + d_1\phi + \cdots + d_{n-1}\phi^{n-1}) \\
&= d_0 + d_1\sigma_i(\phi) + \cdots + d_{n-1}\sigma_i(\phi^{n-1}) \\
&= d_0 + d_1\phi_i + \cdots + d_{n-1}\phi_i^{n-1},
\end{aligned}$$

so that the K-conjugates of α are independent of the choice of θ such that $K = \mathbb{Q}(\theta)$. $\blacksquare$

February 12, 2004