Chapter 6, Question 8

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8. Let θ be a root of the equation $x^3 + 2x + 2 = 0$. Let $K = \mathbb{Q}(\theta)$ and $\alpha = \theta - \theta^2$. Determine the field polynomial of α over K.

Solution. The polynomial $x^3 + 2x + 2 \in \mathbb{Z}[x]$ is 2-Eisenstein and therefore irreducible. Hence

$$\mathrm{irr}_{\mathbb{Q}}(\theta) = x^3 + 2x + 2$$

and

$$[K : \mathbb{Q}] = [\mathbb{Q}(\theta) : \mathbb{Q}] = \deg(x^3 + 2x + 2) = 3.$$

Let $\theta_1 = \theta, \theta_2, \theta_3$ be the three roots of $x^3 + 2x + 2$ so that

$$\begin{aligned}
&\theta_1 + \theta_2 + \theta_3 = 0,\\
&\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1 = 2,\\
&\theta_1\theta_2\theta_3 = -2.
\end{aligned}$$

The conjugates of $\alpha = \theta - \theta^2$ with respect to K are

$$\alpha = \theta - \theta^2 = \theta_1 - \theta_1^2,\ \theta_2 - \theta_2^2,\ \theta_3 - \theta_3^2.$$

Hence the field polynomial of α over K is

$$(x - (\theta_1 - \theta_1^2))(x - (\theta_2 - \theta_2^2))(x - (\theta_3 - \theta_3^2)) = x^3 - Ax^2 + Bx - C,$$

where

$$\begin{aligned}
A &= (\theta_1 - \theta_1^2) + (\theta_2 - \theta_2^2) + (\theta_3 - \theta_3^2),\\
B &= (\theta_1 - \theta_1^2)(\theta_2 - \theta_2^2) + (\theta_2 - \theta_2^2)(\theta_3 - \theta_3^2) + (\theta_3 - \theta_3^2)(\theta_1 - \theta_1^2),\\
C &= (\theta_1 - \theta_1^2)(\theta_2 - \theta_2^2)(\theta_3 - \theta_3^2).
\end{aligned}$$

First we determine A. We have

$$\begin{aligned}
A &= (\theta_1 + \theta_2 + \theta_3) - (\theta_1^2 + \theta_2^2 + \theta_3^2)\\
&= (\theta_1 + \theta_2 + \theta_3) - ((\theta_1 + \theta_2 + \theta_3)^2 - 2(\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1))\\
&= 0 - (0^2 - 2(2)) = 4.
\end{aligned}$$

Next we determine B. We have

$$\begin{aligned}
B &= \theta_1\theta_2(1-\theta_1)(1-\theta_2) + \theta_2\theta_3(1-\theta_2)(1-\theta_3) + \theta_3\theta_1(1-\theta_3)(1-\theta_1) \\
&= \theta_1\theta_2(1-(\theta_1+\theta_2)+\theta_1\theta_2) + \theta_2\theta_3(1-(\theta_2+\theta_3)+\theta_2\theta_3) \\
&\quad +\theta_3\theta_1(1-(\theta_3+\theta_1)+\theta_3\theta_1) \\
&= \theta_1\theta_2(1+\theta_3+\theta_1\theta_2) + \theta_2\theta_3(1+\theta_1+\theta_2\theta_3) + \theta_3\theta_1(1+\theta_2+\theta_3\theta_1) \\
&= (\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1) + 3\theta_1\theta_2\theta_3 + (\theta_1^2\theta_2^2+\theta_2^2\theta_3^2+\theta_3^2\theta_1^2) \\
&= 2 + 3(-2) + (\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1)^2 - 2\theta_1\theta_2\theta_3(\theta_1+\theta_2+\theta_3) \\
&= -4+4-0 = 0.
\end{aligned}$$

Finally we determine C. We have

$$\begin{aligned}
C &= \theta_1\theta_2\theta_3(1-\theta_1)(1-\theta_2)(1-\theta_3) \\
&= -2(1-(\theta_1+\theta_2+\theta_3)+(\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1)-\theta_1\theta_2\theta_3) \\
&= -2(1-0+2+2) = -10.
\end{aligned}$$

Hence

$$\mathrm{fld}_K(\alpha) = x^3 - 4x^2 + 10. \qquad \blacksquare$$

As a check on this calculation we show that $\theta-\theta^2$ is a root of x^3-4x^2+10. We have as $\theta^3 = -2-2\theta$

$$\begin{aligned}
&\left(\theta-\theta^2\right)^3 - \left(\theta-\theta^2\right)^2 + 10 \\
&\quad = \theta^3 - 3\theta^4 + 3\theta^5 - \theta^6 - 4\theta^2 + 8\theta^3 - 4\theta^4 + 10 \\
&\quad = 10 - 4\theta^2 + 9\theta^3 - 7\theta^4 + 3\theta^5 - \theta^6 \\
&\quad = 10 - 4\theta^2 + 9(-2-2\theta) - 7(-2\theta-2\theta^2) + 3(-2\theta^2-2\theta^3) \\
&\qquad -(-2\theta^3-2\theta^4) \\
&\quad = -8 - 4\theta + 4\theta^2 - 4\theta^3 + 2\theta^4 \\
&\quad = -8 - 4\theta + 4\theta^2 - 4(-2-2\theta) + 2(-2\theta-2\theta^2) \\
&\quad = -8 - 4\theta + 4\theta^2 + 8 + 8\theta - 4\theta - 4\theta^2 = 0.
\end{aligned}$$