EXERCISES 9, QUESTION 5

5. Let K be an algebraic number field. Let n be a given positive integer. Prove that there are only finitely many integral ideals I of O_K such that $N(I) = n$.

Solution. Let the decomposition of the principal ideal $< n >$ of O_K into prime ideals be

$$< n > = P_1^{a_1} \cdots P_k^{a_k}.$$

Let I be an ideal (if any) of O_K such that

$$N(I) = n.$$

Then

$$< N(I) > = P_1^{a_1} \cdots P_k^{a_k}.$$

By Question 4 $I \mid < N(I) >$ so that

$$I \mid P_1^{a_1} \cdots P_k^{a_k}.$$

Hence, by Theorem 8.3.2, we have

$$I = P_1^{b_1} \cdots P_k^{b_k},$$

where $b_1 \in \{0, 1, \ldots, a_1\}, \ldots, b_k \in \{0, 1, \ldots, a_k\}$. Thus there are at most $(a_1 + 1) \cdots (a_k + 1)$ choices for I. ∎

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