Chapter 9, Question 6

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6. Let $K = \mathbb{Q}(\theta)$, where $\theta^3 - \theta - 1 = 0$. Prove that $< 23, 3 - \theta >$ is a prime ideal in O_K.

Solution. As

$$(\pm 1)^3 - (\pm 1) - 1 = -1 \neq 0$$

the polynomial $x^3 - x - 1$ is irreducible in $\mathbb{Q}[x]$ so that $[K : \mathbb{Q}] = 3$.

Next

$$D(\theta) = -4(-1)^3 - 27(-1)^2 = 4 - 27 = -23$$

is squarefree, so by Theorem 7.1.8, $\{1, \theta, \theta^2\}$ is an integral basis for K, and $d(K) = D(\theta) = -23$.

Now we determine a basis for the ideal $A =< 23, 3 - \theta >$. We have

$$\begin{aligned}
&< 23, 3 - \theta > \\
&\quad = \{23(a + b\theta + c\theta^2) + (3 - \theta)(d + e\theta + f\theta^2) \mid a, b, c, d, e, f \in \mathbb{Z}\} \\
&\quad = \{(23a + 3d - f) + (23b - d + 3e - f)\theta(23c - e + 3f)\theta^2 \mid a, b, c, d, e, f \in \mathbb{Z}\} \\
&\quad = \{(23a + 3d - f) + (23b + 69c - d + 8f - 3z)\theta + z\theta^2 \mid a, b, c, d, f, z \in \mathbb{Z}\} \\
&\quad = \{23(a + 3b + 9c + f) - 3y - 9z + y\theta + z\theta^2 \mid a, b, c, f, y, z \in \mathbb{Z}\} \\
&\quad = \{23x - 3y - 9z + y\theta + z\theta^2 \mid x, y, z \in \mathbb{Z}\} \\
&\quad = \{23x + y(\theta - 3) + z(\theta^2 - 9) \mid x, y, z \in \mathbb{Z}\} \\
&\quad = 23\mathbb{Z} + (\theta - 3)\mathbb{Z} + (\theta^2 - 9)\mathbb{Z}.
\end{aligned}$$

Hence a basis for $A =< 23, 3 - \theta >$ is $\{23, \theta - 3, \theta^2 - 9\}$.

Next we determine the discriminant $D(A)$ of the ideal A. We have by Definition 6.5.2

$$\begin{aligned}
D(A) &= D(23, \theta - 3, \theta^2 - 9) \\
&= \begin{vmatrix} 23 & \theta - 3 & \theta^2 - 9 \\ 23 & \theta' - 3 & \theta'^2 - 9 \\ 23 & \theta'' - 3 & \theta''^2 - 9 \end{vmatrix}^2
\end{aligned}$$

where θ', θ'' are the two other conjugates of θ. Thus

$$\begin{aligned}
D(A) &= \begin{vmatrix} 23 & \theta - 3 & \theta^2 - 9 \\ 0 & \theta' - \theta & \theta'^2 - \theta^2 \\ 0 & \theta'' - \theta & \theta''^2 - \theta^2 \end{vmatrix}^2 \\
&= 23^2 \begin{vmatrix} \theta' - \theta & \theta'^2 - \theta^2 \\ \theta'' - \theta & \theta''^2 - \theta^2 \end{vmatrix}^2 \\
&= 23^2 \Big((\theta' - \theta)(\theta''^2 - \theta^2) - (\theta'' - \theta)(\theta'^2 - \theta^2) \Big)^2 \\
&= 23^2 (\theta' - \theta)^2 (\theta'' - \theta)^2 ((\theta'' + \theta) - (\theta' + \theta))^2 \\
&= 23^2 (\theta' - \theta)^2 (\theta'' - \theta)^2 (\theta'' - \theta')^2 \\
&= 23^2 D(\theta) \\
&= 23^2 d(K)
\end{aligned}$$

so that

$$N(A) = \sqrt{\frac{D(A)}{d(K)}} = 23.$$

As 23 is a prime, by Question 3, A is a prime ideal. ∎

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