A SUM OF FRACTIONAL PARTS

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Let L denote the set of points $x=(x_1, \cdots, x_n)$ with integral coordinates in Euclidean n-space. For any prime $p\geqq 3$, let $C=C(p)$ be the set of points of L in the cube $0\leqq x_i <p$ $(i=1, 2, \cdots, n)$. Suppose that $f(x)$ is any polynomial of degree d in $x_1, \cdots, x_n$ with integral coefficients, which does not vanish identically (mod p). We let $\{a\}$ denote the fractional part of the real number a and consider the problem of estimating

$$\sum_{x\in C} \{f(x)/p\}$$

for large primes p. When $d=1$ this sum is just $\frac{1}{2}(p-1)p^{n-1}=\frac{1}{2}p^n+O(p^{n-1})$. For $d>1$ we prove, using a result of Uchiyama [1], the following theorem.

THEOREM 1.

$$\sum_{x\in C} \{f(x)/p\} = \tfrac{1}{2}p^n + O(p^{n-1/2}\log p), \tag{1}$$

as $p\rightarrow\infty$, where the constant implied in the O-symbol depends only upon n and d.

Proof.

$$\sum_{x\in C} \{f(x)/p\} = \sum_{r=0}^{p-1} \sum_{\substack{x\in C\\ f(x)\equiv r \,(\mathrm{mod}\, p)}} \{r/p\} = 1/p \sum_{r=0}^{p-1} r \sum_{\substack{x\in C\\ f(x)\equiv r \,(\mathrm{mod}\, p)}} 1. \tag{2}$$

Now let $e(t)$ denote $\exp(2\pi i t p^{-1})$ for any real t. Then as

$$(1/p)\sum_{v=0}^{p-1} e(av) = \begin{cases} 1 & \text{if } a\equiv 0 \pmod p\\ 0 & \text{if } a\not\equiv 0 \pmod p\end{cases}$$

we have

$$\begin{aligned}\sum_{x\in C} \{f(x)/p\} &= (1/p^2)\sum_{r=0}^{p-1} r \sum_{x\in C}\sum_{v=0}^{p-1} e(v(f(x)-r))\\ &= (1/p^2)\sum_{v=0}^{p-1}\sum_{x\in C} e(vf(x))\sum_{r=0}^{p-1} re(-rv).\end{aligned}$$

The term corresponding to $v=0$ is just

$$(1/p^2)\sum_{x\in C}\sum_{r=0}^{p-1} r = \tfrac{1}{2}(p-1)p^{n-1}.$$

For $v\neq 0$ we have

$$\sum_{r=0}^{p-1} re(-rv) = -p/(1-e(-v)),$$

whence

$$\left|\sum_{x\in C}\{f(x)/p\} - \tfrac{1}{2}p^{n-1}(p-1)\right| = \left|(1/p)\sum_{v=1}^{p-1} 1/(1-e(-v))\sum_{x\in C} e(vf(x))\right|$$
$$\leqq (1/p)\sum_{v=1}^{p-1} 1/\left|1-e(-v)\right|\left|\sum_{x\in C} e(vf(x))\right|.$$

By a recent result of Uchiyama [1]

$$\left|\sum_{x\in C} e(vf(x))\right| \leqq kp^{n-1/2}$$

where the constant k depends only upon n and d. Hence we have

$$\left|\sum_{x\in C}\left\{\frac{f(x)}{p}\right\} - \frac{1}{2}p^{n-1}(p-1)\right| \leqq \frac{k}{2}p^{n-3/2}\sum_{v=1}^{p-1}\frac{1}{\sin(\pi v/p)},$$

from which the theorem follows in view of the well-known result

$$\sum_{v=1}^{p-1}\frac{1}{\sin(\pi v/p)} \leqq p\sum_{v=1}^{(p-1)/2}\frac{1}{v} \leqq p\log p.$$

We now show that the error term in (1) can be improved for polynomials $f(x)$ of certain special types, namely diagonal and quadratic polynomials.

THEOREM 2. *If* $f(x)=a_1x_1^{k_1}+\cdots+a_nx_n^{k_n}+a_0$, *where each* $k_i\geqq 1$ *and* $p\nmid a_1\cdots a_n$, *then, provided* $n\geqq 3$,

$$\sum_{x\in C}\{f(x)/p\} = (p^n/2) + O(p^{n-1}),$$

as $p\to\infty$, *where the constant implied in the O-symbol depends only upon* $k_1,\cdots,k_n$.

THEOREM 3. *Let* $f(x_1, x_2)$ *be a quadratic polynomial which is not a function of only one integral variable and which has a discriminant not divisible by* p. *Suppose further that the discriminant of* $f_1(x_1, x_2, 0)$ *is also not divisible by* p, *where* $f_1(x_1, x_2, x_3)\equiv x_3^2 f(x_1/x_3, x_2/x_3)$. *Then*

$$\sum_{x_1=0}^{p-1}\sum_{x_2=0}^{p-1}\{f(x_1,x_2)/p\} = \tfrac{1}{2}p^2 + O(p),$$

as $p\to\infty$.

Proof of Theorem 2. By (2)

$$\sum_{x\in C}\{f(x)/p\} = (1/p)\sum_{r=1}^{p-1} rN_r, \tag{3}$$

where N_r denotes the number of $x\in C$ with $f(x)\equiv r \pmod p$. Weil has shown [2] that

$$N_r = p^{n-1} + O(p^{n/2}),$$

as $p \to \infty$, where the constant in the O-symbol depends only on $k_1, \cdots, k_n$. The theorem follows at once.

Proof of Theorem 3. As is well known, under the conditions stated in the theorem, we may reduce $f(x_1, x_2)$ by a linear nonsingular transformation (mod p) to the canonical form

$$ay_1^2 + by_2^2 + c \qquad (p \nmid abc).$$

This transformation will not affect the sum under consideration and so we need only consider

$$\sum_{y_1=0}^{p-1} \sum_{y_2=0}^{p-1} \left\{ \frac{ay_1^2 + by_2^2 + c}{p} \right\} \qquad (p \nmid abc).$$

The theorem then follows from (3), as in this case

$$N_r = \begin{cases} p - (-ab/p) & \text{if } r \not\equiv c \\ p + (p-1)(-ab/p) & \text{if } r \equiv c. \end{cases}$$

CONJECTURE: (1) may be replaced by

$$(1') \qquad \sum_{x \in C} \{ f(x)/p \} = \tfrac{1}{2} p^n + O(p^{n-1/2}), \quad \text{as } p \to \infty.$$

References

1. S. Uchiyama, On a multiple exponential sum, Proc. Japan Acad., 32 (1956) 748–749.

2. A. Weil, Numbers of solutions of equations in finite fields, Bull. Amer. Math. Soc., 55 (1949) 497–508.

TAYLOR'S FORMULA AND THE EXISTENCE OF nTH DERIVATIVES

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A common form of Taylor's theorem states sufficient conditions for a function f to have an expansion of the form

$$(1) \qquad f(a + h) = a_0 + a_1 h + a_2 h^2 + \cdots + a_{n-1} h^{n-1} + R_n(h),$$

valid in some one- or two-sided neighborhood of $h=0$, the remainder term R_n involving a factor of the form $f^{(n)}(X)$, where X is strictly between a and $a+h$. The existence (finite or infinite) of $f^{(n)}$ *throughout* at least some deleted neighborhood (or one-sided neighborhood) of a obviously must be postulated in order to prove any theorem of this kind. On the other hand, $f^{(n)}(a)$ need not exist for the validity of such an expansion.