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On Certain Sums of Fractional Parts

By

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1. Introduction. Let a and b be integers and m an integer > 1. In this note we evaluate the sums

(1.1) $$C_m(a, b) = \sum_{x=0}^{m-1} \left\{ \frac{ax+b}{m} \right\}, \quad R_m(a, b) = \sum_{\substack{x=0 \\ (x, m)=1}}^{m-1} \left\{ \frac{ax+b}{m} \right\},$$

where $\{y\}$ denotes the fractional part of y. Certain special cases of these sums are known, for example (see [1] page 333)

$$C_m(a, 0) = \tfrac{1}{2}(m - (a, m)),$$

and if $(a, m) = 1$ (see [3] page 50)

$$C_m(a, b) = \tfrac{1}{2}(m - 1), \quad R_m(a, 0) = \tfrac{1}{2}\varphi(m).$$

We evaluate $C_m(a, b)$ and $R_m(a, b)$ in general. We let $p_1, \ldots, p_s$ be the s distinct primes dividing both a and m. Then we write

(1.2) $$a = p_1^{a_1} \cdots p_r^{a_r} p_{r+1}^{a_{r+1}} \cdots p_s^{a_s} a', \quad m = p_1^{m_1} \cdots p_r^{m_r} p_{r+1}^{m_{r+1}} \cdots p_s^{m_s} m'$$

where r $(0 \leqq r \leqq s)$ is the unique integer such that

$$a_i \geqq m_i \ (i = 1, \ldots, r), \quad a_i < m_i \ (i = r+1, \ldots, s),$$

and $p_i \nmid a'm'$ $(i = 1, \ldots, s)$, $(a', m') = 1$, and set

$$A = p_{r+1}^{a_{r+1}} \cdots p_s^{a_s}, \quad M = p_1^{m_1} \cdots p_r^{m_r}.$$

We prove

Theorem 1. $C_m(a, b) = \frac{1}{2}(m - (a, m)) + (a, m)\{b/(a, m)\}$.

Theorem 2.

$$R_m(a, b) = \begin{cases} \varphi(m)\{b/m\}, & \text{if } m \mid a, \\ \tfrac{1}{2}\varphi(m) + b\,\varphi(m)/m - A\,\varphi(M)\,\varphi([b/AM], m/M), & \text{if } m \nmid a, \end{cases}$$

where $\varphi(k)$ is Euler's φ-Function and $\varphi(k, l)$ is Minine's generalization (see [1] page 124) of Euler's φ-Function, that is, $\varphi(k, l)$ denotes the number of integers $\leqq k$ which are relatively prime to l.

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2. Proof of theorem 1. Our starting point is the following well-known identity (see for example [2] page 122): if k is any integer $\geqq 1$ and α is any real number then

$$(2.1)\qquad \sum_{x=0}^{k-1}\left[\frac{x}{k}+\alpha\right]=[\alpha k],$$

where $[y]$ denotes the greatest integer $\leqq y$ (so that $y=[y]+\{y\}$). It is clear that (2.1) can be rewritten as

$$(2.2)\qquad \sum_{x=0}^{k-1}\left\{\frac{x}{k}+\alpha\right\}=\tfrac{1}{2}(k-1)+\{\alpha k\}.$$

For fixed k and α, $\{x/k+\alpha\}$ is periodic in x with period k, and if c is an integer such that $(c,k)=1$ the mapping $x\to cx$ is a bijection on a complete residue system modulo k. Applying this bijection to (2.2) we obtain

$$(2.3)\qquad \sum_{x=0}^{k-1}\left\{\frac{cx}{k}+\alpha\right\}=\tfrac{1}{2}(k-1)+\{\alpha k\}.$$

Setting $c=a/(a,m)$ and $k=m/(a,m)$ (so that $(c,k)=1$) in (2.3) we get

$$\sum_{x=0}^{m/(a,m)-1}\left\{\frac{a}{m}x+\alpha\right\}=\frac{1}{2}\left(\frac{m}{(a,m)}-1\right)+\left\{\frac{\alpha m}{(a,m)}\right\},$$

and so

$$\sum_{x=0}^{m-1}\left\{\frac{a}{m}x+\alpha\right\}=\sum_{z=0}^{(a,m)-1}\ \sum_{y=0}^{m/(a,m)-1}\left\{\frac{a}{m}\left(y+\frac{m}{(a,m)}z\right)+\alpha\right\}=$$
$$=\sum_{z=0}^{(a,m)-1}\ \sum_{y=0}^{m/(a,m)-1}\left\{\frac{a}{m}y+\alpha\right\}$$

giving

$$(2.4)\qquad \sum_{x=0}^{m-1}\left\{\frac{a}{m}x+\alpha\right\}=\tfrac{1}{2}(m-(a,m))+(a,m)\left\{\frac{\alpha m}{a,m)}\right\}.$$

Theorem 1 follows by taking $\alpha=b/m$ in (2.4).

3. Proof of theorem 2. As

$$\sum_{d\mid k}\mu(d)=\begin{cases}1, & \text{if } k=1,\\ 0, & \text{if } k>1,\end{cases}$$

we have

$$R_m(a,b)=\sum_{x=0}^{m-1}\left\{\frac{ax+b}{m}\right\}\sum_{d\mid(x,m)}\mu(d)=\sum_{d\mid m}\mu(d)\sum_{t=0}^{m/d-1}\left\{\frac{at}{m/d}+\frac{b}{m}\right\},$$

and so applying (2.4) with m/d, b/m replacing m, α respectively, and recalling that $\sum_{d\mid m}\frac{\mu(d)}{d}=\frac{\varphi(m)}{m}$, we obtain

$$R_m(a,b)=\sum_{d\mid m}\mu(d)\left(\tfrac{1}{2}(m/d-(a,m/d))+(a,m/d)\left\{\frac{b}{d(a,m/d)}\right\}\right)=$$

$$= \tfrac{1}{2}\varphi(m) + b\,\frac{\varphi(m)}{m} - \tfrac{1}{2}\sum_{d|m}\mu(d)\,(a, m/d) -$$

$$- \sum_{d|m}\mu(d)\,(a, m/d)\left[\frac{b}{d\,(a, m/d)}\right].$$

Now $\sum_{d|m}\mu(d)\,(a, m/d)$ is a multiplicative function of m and so we have

$$\sum_{d|m}\mu(d)\,(a, m/d) = \prod_{p^s \| m}\Big(\sum_{d|p^s}\mu(d)\,(a, p^s/d)\Big) = \prod_{p^s\|m}((a, p^s) - (a, p^{s-1}))\,.$$

If $m\,|\,a$ we have

$$\prod_{p^s\|m}((a, p^s) - (a, p^{s-1})) = \prod_{p^s\|m}(p^s - p^{s-1}) = m\prod_{p|m}(1 - 1/p) = \varphi(m)\,.$$

If $m \nmid a$ then either there exists a prime p_1 such that $p_1\,|\,m$ but $p_1 \nmid a$ or every prime $p\,|\,m$ divides a but there exists a prime p_2 with $p_2^u \| m$, $p_2^v \| a$ and $u > v$, so that

$$(a, p_i^s) - (a, p_i^{s-1}) = 0 \quad (i = 1, 2)\,.$$

Putting the two possibilities together we have

$$\sum_{d|m}\mu(d)\,(a, m/d) = \begin{cases}\varphi(m), & \text{if } m\,|\,a\,,\\ 0\,, & \text{if } m\nmid a\,.\end{cases}$$

Finally in the sum

$$\sum_{d|m}\mu(d)\,(a, m/d)\left[\frac{b}{d\,(a, m/d)}\right]$$

we sum over $d\,|\,m$ by summing over d_1, d_2 with $d_1\,|\,p_1^{m_1}\cdots p_r^{m_r}$, $d_2\,|\,p_{r+1}^{m_{r+1}}\cdots p_s^{m_s}m'$. Clearly d_1 and d_2 are coprime so that $\mu(d_1d_2) = \mu(d_1)\,\mu(d_2)$ and the only d_1, d_2 contributing to the sum are those for which d_1, d_2 are both squarefree. For d_2 squarefree we have

$$(a, m/d) = \left(p_1^{a_1}\cdots p_r^{a_r}\,p_{r+1}^{a_{r+1}}\cdots p_s^{a_s}a',\ \frac{p_1^{m_1}\cdots p_r^{m_r}}{d_1}\cdot\frac{p_{r+1}^{m_{r+1}}\cdots p_s^{m_s}m'}{d_2}\right) =$$

$$= \left(p_1^{a_1}\cdots p_r^{a_r}\,p_{r+1}^{a_{r+1}}\cdots p_s^{a_s},\ \frac{p_1^{m_1}\cdots p_r^{m_r}}{d_1}\cdot\frac{p_{r+1}^{m_{r+1}}\cdots p_s^{m_s}m'}{d_2}\right) =$$

$$= \frac{p_1^{m_1}\cdots p_r^{m_r}}{d_1}\left(p_{r+1}^{a_{r+1}}\cdots p_s^{a_s},\ \frac{p_{r+1}^{m_{r+1}}\cdots p_s^{m_s}m'}{d_2}\right) =$$

$$= \frac{p_1^{m_1}\cdots p_r^{m_r}}{d_1}\cdot p_{r+1}^{a_{r+1}}\cdots p_s^{a_s} = \frac{M}{d_1}A\,,$$

and so the sum becomes

$$MA\sum_{d_1|M}\frac{\mu(d_1)}{d_1}\sum_{d_2|m/M}\mu(d_2)\left[\frac{b}{AM\,d_2}\right] = A\,\varphi(M)\sum_{d_2|m/M}\mu(d_2)\left[\frac{[b/A\,M]}{d_2}\right],$$

that is

$$\text{(3.1)}\qquad \sum_{d|m}\mu(d)\,(a, m/d)\left[\frac{b}{d\,(a, m/b)}\right] = A\,\varphi(M)\,\varphi\left(\left[\frac{b}{AM}\right],\ \frac{m}{M}\right),$$

as (see for example [2] page 123)

$$\varphi(k, l) = \sum_{d \mid l} \mu(d)\,[k/d]\,.$$

We note that when $m \mid a$ (so that $A = 1$, $M = m$, $m' = 1$, $r = s$) (3.1) gives $\varphi(m)\,\{b/m\}$. Putting these results together we obtain theorem 2.

References

[1] L. E. DICKSON, History of the Theory of Numbers, Vol. 1. New York 1966.
[2] C. T. LONG, Number Theory. Boston 1965.
[3] I. M. VINOGRADOV, Elements of Number Theory. New York 1954.

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