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ON EULER'S CRITERION FOR CUBIC NONRESIDUES[1]

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ABSTRACT. If p is a prime $\equiv 1 \pmod 3$ there are integers L and M such that $4p = L^2 + 27M^2$, $L \equiv 1 \pmod 3$. Indeed L and M^2 are unique. If D is a cubic nonresidue (mod p) it is shown how to choose the sign of M so that

$$D^{(p-1)/3} \equiv (L + 9M)/(L - 9M) \pmod p.$$

The case $D = 2$ has been treated by Emma Lehmer.

1. **Introduction.** If p is a prime $\equiv 1 \pmod 3$ there are integers L and M such that

$$4p = L^2 + 27M^2, \qquad L \equiv 1 \pmod 3. \tag{1.1}$$

Indeed L and M^2 are unique. Moreover, $L, M \not\equiv 0 \pmod p$ so that 1, $(L + 9M)/(L - 9M)$ and $(L - 9M)/(L + 9M)$ are the three distinct cube roots of unity (mod p). Thus, if D is an integer not divisible by p, by Euler's criterion we have

$$D^{(p-1)/3} \equiv \begin{cases} 1, & \text{if } D \text{ is a cubic residue (mod } p), \\ (L \pm 9M)/(L \mp 9M), & \text{if } D \text{ is a cubic nonresidue (mod } p). \end{cases}$$

It is the purpose of this paper to show how the sign of M in (1.1) should be chosen so that if D is a cubic nonresidue (mod p) then

$$D^{(p-1)/3} \equiv (L + 9M)/(L - 9M) \pmod p. \tag{1.2}$$

Clearly there is no loss of generality in restricting D to be a prime ≥ 2, and we consider two cases according as $D = 2, 3$ or $D \geq 5$.

The case $D = 2, 3$ is treated in §2 using the theory of cyclotomy. In this case it is well known that D is a cubic residue (mod p) if and only if

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$M \equiv 0 \pmod{D}$. In Lemma 1 explicit expressions are given for the cyclotomic numbers of order 3 (compare [2, p. 397]). These are used in conjunction with known results in the theory of cyclotomy (see Lemma 2) to show how M must be specified uniquely so that (1.2) holds (Theorem 1). In Theorem 1, (a) is due to Emma Lehmer [5], and (b) is new. Her approach is different to ours.

The case $D \geq 5$ is treated in §3. In this case it is well known that if D is a cubic nonresidue (mod p) then $LM \not\equiv 0 \pmod{D}$, and use of this fact is made from time to time in the proofs. A congruence modulo D (see (3.1)) for the cubic Gauss sum proved by Ankeny [1], and a criterion for cubic residuacity given by Lehmer [4], are used to show how M must be specified uniquely in terms of a certain set $\mathfrak{L}_1(D)$ (see (3.6) and Lemma 5) so that (1.2) holds (Theorem 2). The set $\mathfrak{L}_1(D)$ is easy to calculate for any particular value of D and the values of $\mathfrak{L}_1(D)$ are given for $D = 5, 7, 11, 13, 17, 19$.

2. $D = 2, 3$. Let $w = \exp(2\pi i/3) = \frac{1}{2}(-1 + \sqrt{-3})$, so that $1 + w + w^2 = 0$. If p is a prime $\equiv 1 \pmod 3$ we set, for any L, M satisfying (1.1),

$$\pi = \tfrac{1}{2}(L + 3M) + 3Mw, \tag{2.1}$$

so that π is a prime factor of p in the Eisenstein domain $Z[w]$. We define a cubic residue character $\chi_\pi \pmod{\pi}$ by setting for any $\alpha \in Z[w]$,

$$\chi_\pi(\alpha) = \begin{cases} w^r, & \text{if } \alpha \not\equiv 0 \pmod{\pi} \text{ and } \alpha^{(p-1)/3} \equiv w^r \pmod{\pi},\ 0 \leq r \leq 2, \\ 0, & \text{if } \alpha \equiv 0 \pmod{\pi}. \end{cases} \tag{2.2}$$

If g is a primitive root (mod p), so that $\chi_\pi(g) = w$ or w^2, for any integers h and k $(0 \leq h, k \leq 2)$ the cyclotomic number $(h, k)_3$ of order 3 is defined to be the number of solutions (r, s) of $g^{3r+h} + 1 \equiv g^{3s+k} \pmod{p}$ with $0 \leq r, s < (p-1)/3$. Our first lemma, which is well known, gives expressions for these cyclotomic numbers in terms of g, L, M and π.

Lemma 1.

$$9(0, 0)_3 = p - 8 + L,$$

$$18(0, 1)_3 = 18(1, 0)_3 = 18(2, 2)_3 = \begin{cases} 2p - 4 - L + 9M, & \text{if } \chi_\pi(g) = w, \\ 2p - 4 - L - 9M, & \text{if } \chi_\pi(g) = w^2, \end{cases}$$

$$18(0, 2)_3 = 18(2, 0)_3 = 18(1, 1)_3 = \begin{cases} 2p - 4 - L - 9M, & \text{if } \chi_\pi(g) = w, \\ 2p - 4 - L + 9M, & \text{if } \chi_\pi(g) = w^2, \end{cases}$$

$$9(1, 2)_3 = 9(2, 1)_3 = p + 1 + L.$$

For any integer $a \not\equiv 0 \pmod{p}$ *we define the index of* a *with respect to* g, *written* $\operatorname{ind}_g(a)$, *to be the unique integer* b *such that* $a \equiv g^b \pmod{p}$, $0 \leq b \leq p-2$.

The next lemma consists of well-known results from the theory of cyclotomy (see for example [7, Lemma 4, p. 26], and [6, Theorem 1 ($e = 3$), p. 257]).

Lemma 2. (a) *Let* $h = 0, 1, 2$. *Then* $\operatorname{ind}_g(2) \equiv h \pmod{3}$ *if and only if* $(0, h)_g \equiv 1 \pmod{2}$.

(b) $\operatorname{ind}_3(3) \equiv (0, 2)_3 - (0, 1)_3 \pmod{3}$.

As D is a cubic residue (mod p) if and only if $\operatorname{ind}_g(D) \equiv 0 \pmod{3}$, we obtain immediately from Lemmas 1, 2 and (1.1) that, for $D = 2, 3$, D is a cubic residue (mod p) if and only if $M \equiv 0 \pmod{D}$. Thus if D (= 2 or 3) is not a cubic residue (mod p) we can distinguish between the two solutions $(L, \pm M)$ of (1.1) as follows: (a) if 2 is not a cubic residue (mod p) then (1.1) has a unique solution (L, M) satisfying $L \equiv M \pmod{4}$, and (b) if 3 is not a cubic residue (mod p) then (1.1) has a unique solution (L, M) satisfying $M \equiv -1 \pmod{3}$.

We can now prove Theorem 1.

Theorem 1. (a) *If* 2 *is not a cubic residue* (mod p) *and* (L, M) *is the unique solution of* (1.1) *satisfying* $L \equiv M \pmod{4}$ *then*

$$2^{(p-1)/3} \equiv (L + 9M)/(L - 9M) \pmod{p}.$$

(b) *If* 3 *is not a cubic residue* (mod p) *and* (L, M) *is the unique solution of* (1.1) *satisfying* $M \equiv -1 \pmod{3}$ *then*

$$3^{(p-1)/3} \equiv (L + 9M)/(L - 9M) \pmod{p}.$$

Proof. (a) Let (L, M) be the unique solution of (1.1) satisfying $L \equiv M \pmod{4}$ and define π by (2.1). Let g be a primitive root (mod p), such that $\chi_\pi(g) = w$. Thus for this primitive root g we have, by Lemma 1, $18(0, 1)_3 = 2p - 4 - L + 9M$, so that, as $L \equiv M \pmod{4}$, we have $(0, 1)_3 \equiv 1 \pmod{2}$. Thus by Lemma 2(a) we have $\operatorname{ind}_g(2) \equiv 1 \pmod{3}$, which gives

$$2^{(p-1)/3} \equiv w \pmod{\pi}. \tag{2.3}$$

It follows from (2.1) that

$$(L + 9M)/(L - 9M) \equiv w \pmod{\pi}. \tag{2.4}$$

Putting (2.3) and (2.4) together we obtain

$$2^{(p-1)/3} \equiv (L + 9M)/(L - 9M) \pmod{\pi},$$

and the required result follows as both sides are real.

(b) Let (L, M) be the unique solution of (1.1) satisfying $M \equiv -1 \pmod 3$ and define π by (2.1). Again we choose g to be a primitive root (mod p) such that $\chi_\pi(g) = w$, and for this primitive root we have by Lemma 1, $(0, 2)_3 - (0, 1)_3 = -M$, so that, as $M = -1 \pmod 3$, we have by Lemma 2(b), $\operatorname{ind}_g(3) \equiv (0, 2)_3 - (0, 1)_3 \equiv 1 \pmod 3$, which gives $3^{(p-1)/3} \equiv w \pmod{\pi}$. The rest of the proof is now the same as in (a).

Example 1. Let $p = 37$ so that $4p = 148 = 11^2 + 27.1^2$. The unique solution given by Lemma 3(a) is $L = -11$, $M = 1$, and the one given by Lemma 3(b) is $L = -11$, $M = -1$. Thus by Theorem 1 we have

$$2^{12} \equiv \frac{(-11) + 9(1)}{(-11) - 9(1)} = \frac{1}{10} \equiv 26 \pmod{37}$$

and

$$3^{12} \equiv \frac{(-11) + 9(-1)}{(-11) - 9(-1)} = 10 \pmod{37}.$$

3. $D \geq 5$. Let D be a prime ≥ 5. The Gauss sum $G(\chi_\pi)$ is defined by

$$G(\chi_\pi) = \sum_{n=1}^{p-1} \chi_\pi(n) \exp(2\pi i n/p),$$

and Ankeny [1] has shown that, if $D \neq p$, $G(\chi_\pi)$ satisfies the congruence

$$G(\chi_\pi)^{D^f - 1} \equiv \chi_\pi(D)^{-f} \pmod{D}, \tag{3.1}$$

where f is the least positive integer such that $D^f \equiv 1 \pmod 3$. Using (3.1) and the well-known result $G(\chi_\pi)^3 = p\pi$ (see for example [3, p. 116]) we obtain modulo D

$$\chi_\pi(D) \equiv \begin{cases} p^{2(D-1)/3}\pi^{2(D-1)/3}, & \text{if } D \equiv 1 \pmod 3, \\ p^{(D-2)/3}\pi^{(D+1)/3}, & \text{if } D \equiv 2 \pmod 3. \end{cases} \tag{3.2}$$

We next define for any integer k

$$F_D(k) = \begin{cases} (k^2 + 27)^{2(D-1)/3}(k + 3 + 6w)^{2(D-1)/3}, & \text{if } D \equiv 1 \pmod 3, \\ (k^2 + 27)^{(D-2)/3}(k + 3 + 6w)^{(D+1)/3}, & \text{if } D \equiv 2 \pmod 3. \end{cases} \tag{3.3}$$

Now Emma Lehmer [4] has shown that for *any* prime $p \equiv 1 \pmod 3$ with $LM \not\equiv$

0 (mod D), there is a set $\mathfrak{L}(D)$ *depending only on* D, such that D is a cubic nonresidue (mod p) if and only if $L^2 \equiv k^2M^2 \pmod{D}$ for some $k \in \mathfrak{L}(D)$. Clearly $\mathfrak{L}(D)$ may be taken as a subset of $\{\pm 1, \pm 2, \cdots, \pm\frac{1}{2}(D-1)\}$ and to have the property that if $k \in \mathfrak{L}(D)$ then $-k \in \mathfrak{L}(D)$. Further we may assume that for each $k \in \mathfrak{L}(D)$ there is some prime $p \equiv 1 \pmod 3$ with $LM \not\equiv 0 \pmod{D}$ for which $L^2 \equiv k^2M^2 \pmod{D}$. We also remark that $\pm h \notin \mathfrak{L}(D)$, where $h^2 + 27 \equiv 0 \pmod{D}$ when $D \equiv 1 \pmod 3$.

We prove

Lemma 3. *If* $k \in \mathfrak{L}(D)$ *then*

$$F_D(k) \equiv w \pmod{D}, \qquad F_D(-k) \equiv w^2 \pmod{D},$$

or

$$F_D(k) \equiv w^2 \pmod{D}, \qquad F_D(-k) \equiv w \pmod{D}.$$

Proof. As $(k + 3 + 6w)(-k + 3 + 6w) = -(k^2 + 27)$ we have

$$F_D(k)F_D(-k) = \begin{cases} (k^2 + 27)^{2(D-1)}, & \text{if } D \equiv 1 \pmod 3, \\ (-1)^{(D+1)/3}(k^2 + 27)^{D-1}, & \text{if } D \equiv 2 \pmod 3. \end{cases}$$

Since D is prime, we have $(D + 1)/3 \equiv 0 \pmod 2$ when $D \equiv 2 \pmod 3$. Also as $k^2 + 27 \not\equiv 0 \pmod{D}$ for $k \in \mathfrak{L}(D)$, we have $(k^2 + 27)^{D-1} \equiv 1 \pmod{D}$. Hence we have

$$F_D(k)F_D(-k) \equiv 1 \pmod{D}. \tag{3.4}$$

Further since $k \in \mathfrak{L}(D)$ there exists a prime p for which D is a cubic nonresidue (mod p) and such that $LM \not\equiv 0 \pmod{D}$ and $L \equiv kM \pmod{D}$. Hence we have

$$4p \equiv (k^2 + 27)M^2, \qquad 2\pi \equiv (k + 3 + 6w)M \pmod{D},$$

and so

$$F_D(k) \equiv \begin{cases} (4p/M^2)^{2(D-1)/3}(2\pi/M)^{2(D-1)/3}, & \text{if } D \equiv 1 \pmod 3, \\ (4p/M^2)^{(D-2)/3}(2\pi/M)^{(D+1)/3}, & \text{if } D \equiv 2 \pmod 3, \end{cases}$$

$$\equiv \begin{cases} p^{2(D-1)/3}\pi^{2(D-1)/3}, & \text{if } D \equiv 1 \pmod 3, \\ p^{(D-2)/3}\pi^{(D+1)/3}, & \text{if } D \equiv 2 \pmod 3, \end{cases}$$

that is

$$F_D(k) \equiv \chi_\pi(D) \pmod{D}. \tag{3.5}$$

The result now follows as $\chi_\pi(D) = w$ or w^2 since D is a cubic nonresidue (mod p).

Lemma 3 enables us to define for $i = 1, 2$

$$\mathfrak{L}_i(D) = \{k \in \mathfrak{L}(D): F_D(k) \equiv w^i \pmod{D}\}, \tag{3.6}$$

so that

$$\mathfrak{L}_1(D) \cup \mathfrak{L}_2(D) = \mathfrak{L}(D), \qquad \mathfrak{L}_1(D) \cap \mathfrak{L}_2(D) = \emptyset.$$

Lemma 4. *Let D be a prime ≥ 5. If p is a prime $\equiv 1 \pmod 3$, for which D is a cubic nonresidue* (mod p), *then we can define M uniquely by requiring it to satisfy $L \equiv kM \pmod D$ for some $k \in \mathfrak{L}_1(D)$.*

Proof. As D is a cubic nonresidue (mod p) by Lehmer's criterion, $L^2 \equiv k^2M^2 \pmod D$ for some $k \in \mathfrak{L}(D)$ and some solution (L, M) of (1.1). By replacing k by $-k$ if necessary we may assume that $k \in \mathfrak{L}_1(D)$. Now $L \equiv \pm kM \pmod D$ with $k \in \mathfrak{L}_1(D)$, and as L cannot satisfy both these congruences we may choose M uniquely so that $L \equiv kM \pmod D$.

We can now prove Theorem 2.

Theorem 2. *Let D be a prime ≥ 5. If p is a prime $\equiv 1 \pmod 3$ for which D is a cubic nonresidue* (mod p) *and M is defined uniquely by $L \equiv kM \pmod D$ for some $k \in \mathfrak{L}_1(D)$ then* (1.2) *holds.*

Proof. It follows from (2.1) that (2.4) holds. Further as $L \equiv kM \pmod D$ with $k \in \mathfrak{L}_1(D)$, we have $F_D(k) \equiv w \pmod D$ and so $\chi_\pi(D) \equiv w \pmod D$, that is, $\chi_\pi(D) = w$. Hence we have

$$D^{(p-1)/3} \equiv (L + 9M)/(L - 9M) \pmod \pi,$$

and the result follows as both sides are real.

Example 2. From Lehmer's criterion for cubic residuacity [4] we deduce that

$$\mathfrak{L}(5) = \{\pm 1, \pm 2\}, \qquad \mathfrak{L}(7) = \{\pm 2, \pm 3\},$$
$$\mathfrak{L}(11) = \{\pm 1, \pm 2, \pm 3, \pm 5\}, \qquad \mathfrak{L}(13) = \{\pm 2, \pm 3, \pm 4, \pm 6\},$$
$$\mathfrak{L}(17) = \{\pm 1, \pm 2, \pm 4, \pm 5, \pm 6, \pm 7\},$$
$$\mathfrak{L}(19) = \{\pm 1, \pm 2, \pm 4, \pm 5, \pm 6, \pm 8\}.$$

Using (3.3) and (3.6) we obtain

$$\mathfrak{L}_1(5) = \{+1, -2\}, \qquad \mathfrak{L}_1(7) = \{+2, -3\},$$
$$\mathfrak{L}_1(11) = \{-1, -2, -3, +5\}, \qquad \mathfrak{L}_1(13) = \{-2, +3, +4, -6\},$$
$$\mathfrak{L}_1(17) = \{-1, +2, +4, -5, +6, -7\},$$
$$\mathfrak{L}_1(19) = \{+1, +2, -4, -5, -6, -8\}.$$

Thus Theorem 2 gives as a particular case: let D denote one of 5, 7, 11, 13, 17, 19. If p is a prime $\equiv 1 \pmod 3$ for which D is a cubic nonresidue (mod p) and M is defined uniquely by $L \equiv kM \pmod D$ where

$$k = \begin{cases} 1 \text{ or } -2, & \text{if } D = 5, \\ 2 \text{ or } -3, & \text{if } D = 7, \\ -1, -2, -3 \text{ or } 5, & \text{if } D = 11, \\ -2, 3, 4 \text{ or } -6, & \text{if } D = 13, \\ -1, 2, 4, -5, 6 \text{ or } -7, & \text{if } D = 17, \\ 1, 2, -4, -5, -6 \text{ or } -8, & \text{if } D = 19, \end{cases}$$

then (1.2) holds.

Thus if $p = 61$ and $D = 19$ the required unique solution is $L = 1$, $M = 3$, so that

$$19^{20} \equiv \frac{1 + 9 \cdot 3}{1 - 9 \cdot 3} \equiv \frac{-14}{13} \equiv 13 \pmod{61}.$$

It is interesting to note that the sum of the elements in each of the sets $\mathfrak{L}_1(D)$ ($D = 5, 7, \cdots, 19$) is congruent to $-1 \pmod D$!

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