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ON URQUHART'S ELEMENTARY THEOREM OF EUCLIDEAN GEOMETRY

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In a recent article Pedoe [1] mentions that he has attempted to find a proof of Urquhart's Theorem (stated below) which does not involve circles (see also [2]). Here is a simple proof which only involves the sine formula for triangles and a few simple trigonometric identities.

LEMMA 1. In ΔABC we have

$$\frac{BC + CA}{AB} = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} .$$

Proof. By the sine formula we have (as $C = \pi - (A+B)$)

$$\frac{BC + CA}{\sin A + \sin B} = \frac{AB}{\sin C} = \frac{AB}{\sin (A+B)}$$

and so

$$\frac{BC + CA}{AB} = \frac{\sin A + \sin B}{\sin (A+B)} = \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A+B)} = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} .$$

LEMMA 2. If

$$\cos \tfrac{1}{2}(L-M) \cos \tfrac{1}{2}(N+P) = \cos \tfrac{1}{2}(L+M) \cos \tfrac{1}{2}(N-P) \qquad (1)$$

then we have

$$\sin \tfrac{1}{2}(L-P) \sin \tfrac{1}{2}(N+M) = \sin \tfrac{1}{2}(L+P) \sin \tfrac{1}{2}(N-M). \qquad (2)$$

Proof. Applying the identity $2 \cos R \cos S = \cos (R+S) + \cos (R-S)$ to each side of (1) we obtain

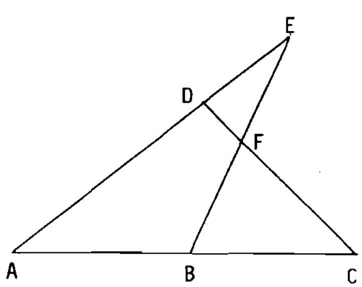
$$\cos \tfrac{1}{2}(L-M+N+P) + \cos \tfrac{1}{2}(L-M-N-P) = \cos \tfrac{1}{2}(L+M+N-P) + \cos \tfrac{1}{2}(L+M-N+P),$$

which gives

$$\cos \tfrac{1}{2}(L-M-N-P) - \cos \tfrac{1}{2}(L+M+N-P) = \cos \tfrac{1}{2}(L+M-N+P) - \cos \tfrac{1}{2}(L-M+N+P). \qquad (3)$$

Then applying the identity $\cos R - \cos S = 2 \sin \frac{1}{2}(R+S) \sin \frac{1}{2}(S-R)$ to each side of (3) we obtain (2).

URQUHART'S THEOREM. In the figure, if $AB + BF = AD + DF$ then we have $AC + CF = AE + EF$.



Proof. We join AF and set

$$\angle BAF = \phi, \quad \angle DAF = \theta, \quad \angle AFB = \mu, \quad \angle AFD = \lambda.$$

Applying Lemma 1 to Δs ABF and ADF we obtain

$$\frac{AB + BF}{AF} = \frac{\cos \frac{1}{2}(\mu-\phi)}{\cos \frac{1}{2}(\mu+\phi)} , \qquad \frac{AD + DF}{AF} = \frac{\cos \frac{1}{2}(\lambda-\theta)}{\cos \frac{1}{2}(\lambda+\theta)} ,$$

and as $AB + BF = AD + DF$ we have

$$\frac{\cos\frac{1}{2}(\mu-\phi)}{\cos\frac{1}{2}(\mu+\phi)} = \frac{\cos\frac{1}{2}(\lambda-\theta)}{\cos\frac{1}{2}(\lambda+\theta)} .$$

Thus by Lemma 2 (as $0 < \lambda-\phi < \pi$, $0 < \mu-\theta < \pi$) we have

$$\frac{\sin\frac{1}{2}(\lambda+\phi)}{\sin\frac{1}{2}(\lambda-\phi)} = \frac{\sin\frac{1}{2}(\mu+\theta)}{\sin\frac{1}{2}(\mu-\theta)} . \qquad (4)$$

Finally applying Lemma 1 to Δs ACF and AEF we obtain

$$\frac{AC+CF}{AF} = \frac{\cos\frac{1}{2}(\pi-\lambda-\phi)}{\cos\frac{1}{2}(\pi-\lambda+\phi)} = \frac{\sin\frac{1}{2}(\lambda+\phi)}{\sin\frac{1}{2}(\lambda-\phi)} ,$$

$$\frac{AE+EF}{AF} = \frac{\cos\frac{1}{2}(\pi-\mu-\theta)}{\cos\frac{1}{2}(\pi-\mu+\theta)} = \frac{\sin\frac{1}{2}(\mu+\theta)}{\sin\frac{1}{2}(\mu-\theta)} ,$$

and the required result $AC + CF = AE + EF$ now follows from (4).

REFERENCES

1. Dan Pedoe, The most elementary theorem of Euclidean geometry, *Mathematics Magazine*, 49 (1976), 40-42.

2. Léo Sauvé, On Circumscribable Quadrilaterals, EUREKA, Vol. 2 (1976), 63-67.

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