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SOME SERIES REPRESENTATIONS OF $\zeta(2n+1)$

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1. Introduction. For $\mathrm{Re}\,(s) > 1$ the Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It is well known that $\zeta(s)$ can be continued analytically to the whole complex plane except for a simple pole at $s = 1$ with residue 1. Moreover, $\zeta(0) = -1/2$.

In [**2**] Boo Rim Choe gives an elementary proof of the classical result

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{1.1}$$

by making use of the power series expansion of $\arcsin x$. In [**4**] Ewell modifies Boo Rim Choe's method to give a new series representation of $\zeta(3)$, namely,

$$\zeta(3) = -\frac{4\pi^2}{7} \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}}. \tag{1.2}$$

Then in [**5**] Ewell further modifies the method of Boo Rim Choe to obtain the following representation of $\zeta(r)$ (valid for an integer $r > 2$):

$$\zeta(r) = \frac{2^{r-2}}{2^r - 1}\pi^2 \sum_{m=0}^{\infty} (-1)^m A_{2m}(r-2)\pi^{2m}/(2m+2)!. \tag{1.3}$$

The coefficients $A_{2m}(r)$ are given by

$$A_{2m}(r) = \sum \frac{\binom{2m}{2i_1, 2i_2, \dots, 2i_r}}{(2i_1+1)(2(i_1+i_2)+1)\cdots(2(i_1+i_2+\cdots+i_{r-1})+1)} \cdot B_{2i_1}B_{2i_2}\cdots B_{2i_r},$$

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where the sum is taken over all r-tuples $(i_1, \ldots, i_r)$ of nonnegative integers whose sum is m,

$$\binom{2m}{2i_1, 2i_2, \ldots, 2i_r}$$

is a multinomial coefficient, and B_{2i} is a Bernoulli number as defined in [**5**]. When $r = 3$, the formula (1.3) reduces to (1.2) recalling Euler's result

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}, \qquad k = 1, 2, \ldots .$$

The aim of this paper is two-fold. First in Section 2 we replace the use of the power series expansion of $\arcsin x$ in [**2, 4**] by that of $(\arcsin x)^2$ in order to give a new short proof of (1.1) as well as a new series representation of $\zeta(3)$ analogous to (1.2), namely,

$$\zeta(3) = -2\pi^2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+2)(2n+3)2^{2n}}. \tag{1.4}$$

Then, in Section 3, we use an idea of Moiseyev [**7**] and a result of Elizalde about $\zeta'(-2n)$, $n = 1, 2, \ldots$, [**3**] to obtain a series representation of $\zeta(2n+1)$, $n = 1, 2, \ldots$, which is simpler than that given by (1.3).

2. Some series representations of $\zeta(3)$. Recall that

$$(\arcsin x)^2 = \sum_{n=1}^{\infty} \frac{2^{2n-1}(n!)^2 x^{2n}}{n^2 (2n)!}, \qquad |x| \leq 1, \tag{2.1}$$

(see, for example, [**1**, p. 262]). Taking $x = \sin t$ in (2.1), we have

$$t^2 = \sum_{n=1}^{\infty} \frac{2^{2n-1}(n!)^2}{n^2 (2n)!} \sin^{2n} t, \qquad |t| \leq \pi/2. \tag{2.2}$$

Then, integrating both sides of (2.2) from 0 to $\pi/2$, we obtain

$$\frac{\pi^3}{24} = \sum_{n=1}^{\infty} \frac{2^{2n-1}(n!)^2}{n^2 (2n)!} \int_0^{\pi/2} \sin^{2n} t \, dt,$$

and, appealing to the well-known formula of Wallis

$$\int_0^{\pi/2} \sin^{2n} t\, dt = \frac{(2n)!\pi}{2^{2n+1}(n!)^2}, \tag{2.3}$$

we deduce $\zeta(2) = \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

Further, dividing (2.1) by x and integrating from 0 to $\sin t$, we have

$$\int_0^{\sin t} \frac{(\arcsin x)^2}{x}\, dx = \sum_{n=1}^{\infty} \frac{2^{2n-2}(n!)^2}{n^3(2n)!} \sin^{2n} t.$$

Making the substitution $x = \sin u$ in the integral, we obtain

$$\int_0^{t} u^2 \cot u\, du = \sum_{n=1}^{\infty} \frac{2^{2n-2}(n!)^2}{n^3(2n)!} \sin^{2n} t.$$

Recalling the power series expansion of $u \cot u$ (see, for example, [**6**, p. 35])

$$u \cot u = -2 \sum_{n=0}^{\infty} \frac{\zeta(2n) u^{2n}}{\pi^{2n}}, \qquad |u| \le \pi, \tag{2.4}$$

we have

$$-2 \int_0^{t} \sum_{n=0}^{\infty} \frac{\zeta(2n) u^{2n+1}}{\pi^{2n}}\, du = \sum_{n=1}^{\infty} \frac{2^{2n-2}(n!)^2}{n^3(2n)!} \sin^{2n} t,$$

that is,

$$-2 \sum_{n=1}^{\infty} \frac{\zeta(2n) t^{2n+2}}{(2n+2)\pi^{2n}} = \sum_{n=1}^{\infty} \frac{2^{2n-2}(n!)^2}{n^3(2n)!} \sin^{2n} t.$$

Integrating both sides from 0 to $\pi/2$, and appealing to (2.3), gives

$$-\frac{\pi^3}{4} \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+2)(2n+3)2^{2n}} = \frac{\pi}{8} \sum_{n=1}^{\infty} \frac{1}{n^3},$$

that is,

$$\zeta(3) = -2\pi^2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+2)(2n+3)2^{2n}}. \tag{2.5}$$

In addition, from (2.4), we have

$$2\sum_{n=1}^{\infty}\frac{\zeta(2n)u^{2n-1}}{\pi^{2n}}=\frac{1}{u}-\cot u,\qquad |u|<\pi. \tag{2.6}$$

Integrating (2.6) from 0 to x gives

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)x^{2n}}{n\pi^{2n}}=\int_0^x\left(\frac{1}{u}-\cot u\right)du=\log(x/\sin x). \tag{2.7}$$

Taking $x=\pi/2$ in (2.7) gives

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n2^{2n}}=\log\frac{\pi}{2}, \tag{2.8}$$

which is formula (6) of [**8**]. Next, integrating (2.7) from 0 to $\pi/2$ gives

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)\pi}{n(2n+1)2^{2n+1}}=\int_0^{\pi/2}(\log x-\log\sin x)\,dx=\frac{\pi}{2}(\log\pi-1),$$

that is,

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(2n+1)2^{2n}}=\log\pi-1, \tag{2.9}$$

which is formula (7) of [**8**]. Further, we obtain successively from (2.8), (2.9) and (2.10) using

$$\frac{1}{2n+1}=\frac{1}{2n}-\frac{1}{2n(2n+1)};$$

(2.11) and (1.2) using

$$\frac{1}{2n+2}=\frac{1}{2n+1}-\frac{1}{(2n+1)(2n+2)};$$

(2.12) and (2.5) using

$$\frac{1}{2n+3}=\frac{1}{2n+2}-\frac{1}{(2n+2)(2n+3)};$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n)2^{2n}} = \frac{1}{2}\log \pi - \frac{1}{2}\log 2, \tag{2.10}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)2^{2n}} = \frac{1}{2} - \frac{1}{2}\log 2, \tag{2.11}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+2)2^{2n}} = \frac{7}{4\pi^2}\zeta(3) - \frac{1}{2}\log 2 + \frac{1}{4}, \tag{2.12}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+3)2^{2n}} = \frac{9}{4\pi^2}\zeta(3) - \frac{1}{2}\log 2 + \frac{1}{6}. \tag{2.13}$$

Then, setting

$$F(a,b) = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+a)(2n+b)2^{2n}}, \quad 0 \le a < b \le 3$$

we deduce from (2.10)–(2.13) and the identity

$$\frac{1}{(2n+a)(2n+b)} = \frac{1}{b-a}\left(\frac{1}{2n+a} - \frac{1}{2n+b}\right)$$

the following table of values

a	b	$F(a,b)$	a	b	$F(a,b)$
0	1	$\frac{1}{2}\log\pi - \frac{1}{2}$	1	2	$-\frac{7}{4\pi^2}\zeta(3) + \frac{1}{4}$
0	2	$-\frac{7}{8\pi^2}\zeta(3) + \frac{1}{4}\log\pi - \frac{1}{8}$	1	3	$-\frac{9}{8\pi^2}\zeta(3) + \frac{1}{6}$
0	3	$-\frac{3}{4\pi^2}\zeta(3) + \frac{1}{6}\log\pi - \frac{1}{18}$	2	3	$-\frac{1}{2\pi^2}\zeta(3) + \frac{1}{12}$

Thus we have the following five different series representations of $\zeta(3)$:

$$\begin{aligned}\zeta(3) &= -\frac{8\pi^2}{7}\sum_{n=1}^{\infty}\frac{\zeta(2n)}{(2n)(2n+2)2^{2n}} + \frac{2\pi^2}{7}\log\pi - \frac{\pi^2}{7}\\ &= -\frac{4\pi^2}{3}\sum_{n=1}^{\infty}\frac{\zeta(2n)}{(2n)(2n+3)2^{2n}} + \frac{2\pi^2}{9}\log\pi - \frac{2}{27}\pi^2\\ &= -\frac{4\pi^2}{7}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}}\\ &= -\frac{8\pi^2}{9}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+3)2^{2n}}\\ &= -2\pi^2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+2)(2n+3)2^{2n}}.\end{aligned}$$

3. Series representation of $\zeta(2n+1)$. For $0 < a \le 1$ and $\mathrm{Re}\,(s) > 1$ the Hurwitz zeta function $\zeta(s,a)$ is defined by

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}.$$

We set (see [**9**])

$$\mu(s,a) = \zeta(s,a) - \zeta(s,1-a), \qquad 0 < a < 1, \mathrm{Re}\,(s) > 1.$$

Then, following an approach of Moiseyev [**7**], we have

$$\begin{aligned}\mu(s,a) &= \sum_{n=0}^{\infty}\frac{1}{(n+a)^s} - \sum_{n=0}^{\infty}\frac{1}{(n+1-a)^s}\\ &= \frac{1}{a^s} + \sum_{n=1}^{\infty}\frac{1}{n^2(1+a/n)^s} - \sum_{n=1}^{\infty}\frac{1}{n^s(1-a/n)^s}.\end{aligned}$$

Since

$$\begin{aligned}\left(1+\frac{a}{n}\right)^{-s} &= \sum_{m=0}^{\infty}\frac{(-1)^m(s)_m}{m!}\left(\frac{a}{n}\right)^m,\\ \left(1-\frac{a}{n}\right)^{-s} &= \sum_{m=0}^{\infty}\frac{(s)_m}{m!}\left(\frac{a}{n}\right)^m,\end{aligned}$$

where

$$(s)_m = s(s+1)\cdots(s+m-1), \qquad (s)_0 = 1,$$

we have

$$\mu(s,a) = \frac{1}{a^s} - 2\sum_{m=1}^{\infty} \frac{(s)_{2m-1}a^{2m-1}}{(2m-1)!} \sum_{n=1}^{\infty} \frac{1}{n^{s+2m-1}},$$

that is,

$$\mu(s,a) = \frac{1}{a^s} - 2\sum_{m=1}^{\infty} \frac{(s)_{2m-1}\zeta(s+2m-1)}{(2m-1)!} a^{2m-1}. \tag{3.1}$$

Similarly, with

$$\lambda(s,a) = \zeta(s,a) + \zeta(s,1-a), \qquad 0 < a < 1, \mathrm{Re}\,(s) > 1,$$

we have

$$\lambda(s,a) = \frac{1}{a^s} + 2\sum_{m=0}^{\infty} \frac{(s)_{2m}\zeta(s+2m)}{(2m)!} a^{2m}. \tag{3.2}$$

Letting $a = 1/2$ and changing s into $s+1$ in (3.1), we have

$$2^{s-1} = \sum_{m=1}^{\infty} \frac{(s+1)_{2m-1}\zeta(s+2m)}{(2m-1)!2^{2m}}, \qquad \mathrm{Re}\,(s) > 0. \tag{3.3}$$

Letting $a = 1/2$ in (3.2) and recalling $\lambda(s,1/2) = 2\zeta(s,1/2) = 2(2^s-1)\zeta(s)$, we obtain

$$(2^s - 2)\zeta(s) - 2^{s-1} = \sum_{m=1}^{\infty} \frac{(s)_{2m}\zeta(s+2m)}{(2m)!2^{2m}}. \tag{3.4}$$

Adding (3.3) to (3.4) and noticing

$$(s)_{2m} + 2m(s+1)_{2m-1} = (s+1)_{2m},$$

we obtain

$$(2^s - 2)\zeta(s) = \sum_{m=1}^{\infty} \frac{(s+1)_{2m}\zeta(s+2m)}{(2m)!2^{2m}}. \tag{3.5}$$

Recalling the functional equation for $\zeta(s)$, namely,

$$2^s\Gamma(1-s)\zeta(1-s)\sin\frac{\pi s}{2} = \pi^{1-s}\zeta(s),$$

we obtain from (3.5)

$$(3.6)\quad (2^s-2)2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\frac{\pi s}{2} = \sum_{m=1}^{\infty}\frac{(s+1)_{2m}\zeta(s+2m)}{(2m)!2^{2m}},$$

or equivalently,

$$(3.6)'\quad (2^{2s}-2^{s+1})\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\frac{\pi s}{2} = \sum_{m=1}^{\infty}\frac{(s+1)(s+2)\cdots(s+2m)\zeta(s+2m)}{(2m)!2^{2m}}.$$

Dividing (3.3) by $(s+1)/4$ and letting $s\to -1$, we have

$$\lim_{s\to -1}\left(\frac{2^{s+1}}{s+1}-\zeta(s+2)\right) = \sum_{m=1}^{\infty}\frac{\zeta(2m+1)}{(2m+1)2^{2m}}.$$

However, as

$$\begin{aligned}\frac{2^{s+1}}{s+1}-\zeta(s+2) &= \frac{e^{(s+1)\log 2}}{s+1} - \left(\frac{1}{s+1}+\gamma+O(|s+1|)\right)\\ &= \frac{1}{s+1}\{1+\log 2(s+1)+O(|s+1|^2)\} - \left\{\frac{1}{s+1}+\gamma+O(|s+1|)\right\}\\ &= (\log 2-\gamma)+O(|s+1|),\end{aligned}$$

we deduce

$$(3.7)\qquad \sum_{m=1}^{\infty}\frac{\zeta(2m+1)}{(2m+1)2^{2m}} = \log 2-\gamma.$$

Similarly, taking $s=1$ in (3.3), we obtain

$$(3.8)\qquad \sum_{m=1}^{\infty}\frac{2m\zeta(2m+1)}{2^{2m}} = 1.$$

Letting $s \to 1$ in (3.6), and recalling that $\zeta(0) = -1/2$, we have

$$\sum_{m=1}^{\infty} \frac{(2m+1)\zeta(2m+1)}{2^{2m+1}} = \log 2. \tag{3.9}$$

We remark that (3.7)–(3.9) can also be obtained by using the Euler-Maclaurin summation formula (see [**8**]).

Dividing $(3.6)'$ by $(s+1)(s+2)$ and letting $s \to -2$ gives

$$\begin{aligned}\frac{7}{16\pi^3}\Gamma(3)\zeta(3) \lim_{s\to -2} \frac{\sin(\pi s/2)}{(s+2)} &= \sum_{m=1}^{\infty} \frac{(s+3)\cdots(s+2m)}{(2m)!2^{2m}}\zeta(s+2m)|_{s=-2} \\ &= \sum_{m=1}^{\infty} \frac{(2m-2)!}{(2m)!2^{2m}}\zeta(2m-2) \\ &= \sum_{m=0}^{\infty} \frac{\zeta(2m)}{(2m+1)(2m+2)2^{2m+2}}.\end{aligned}$$

Since

$$\lim_{s\to -2} \frac{\sin(\pi s/2)}{s+2} = -\pi/2,$$

we have

$$\zeta(3) = -\frac{4\pi^2}{7} \sum_{m=0}^{\infty} \frac{\zeta(2m)}{(2m+1)(2m+2)2^{2m}}, \tag{3.10}$$

which is (1.2).

We now carry out the above argument in general. First, we separate the right side of $(3.6)'$ into two parts as follows:

$$\begin{aligned}&(2^{2s} - 2^{s+1})\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\frac{\pi s}{2} \\ &\quad = \sum_{m=1}^{n-1} \frac{(s+1)\cdots(s+2m)}{(2m)!2^{2m}}\zeta(s+2m) \\ &\qquad + \sum_{m=n}^{\infty} \frac{(s+1)\cdots(s+2m)}{(2m)!2^{2m}}\zeta(s+2m), \qquad n \geq 2.\end{aligned} \tag{3.11}$$

Next we divide (3.11) by $(s+1)(s+2)\cdots(s+2n)$ to obtain

$$(3.12)\qquad \begin{aligned}(2^{2s} &- 2^{s+1})\pi^{s-1}\Gamma(1-s)\zeta(1-s) \\ &\cdot \frac{1}{(s+1)\cdots(s+2n-1)}\cdot\frac{\sin(\pi s/2)}{s+2n} \\ &= \sum_{m=1}^{n-1}\frac{1}{(2m)!2^{2m}(s+2m+1)\cdots(s+2n-1)}\cdot\frac{\zeta(s+2m)}{s+2n} \\ &\quad + \sum_{m=n}^{\infty}\frac{(s+2n+1)\cdots(s+2m)}{(2m)!2^{2m}}\zeta(s+2m).\end{aligned}$$

Then, letting $s\to -2n$ in (3.12), we have as

$$\left(\lim_{s\to -2n}\frac{\sin(\pi s/2)}{s+2n} = \frac{(-1)^n\pi}{2}\right):$$

$$\begin{aligned}\frac{n(2^{2n+1}-1)(-1)^n}{2^{4n}\pi^{2n}}\zeta(2n+1) &= -\sum_{m=1}^{n-1}\frac{\zeta'(-(2n-2m))}{(2m)!2^{2m}(2n-2m-1)!} \\ &\quad + \frac{1}{2^{2n}}\sum_{m=0}^{\infty}\frac{\zeta(2m)}{(2m+1)\cdots(2m+2n)2^{2m}}.\end{aligned}$$

Hence, we obtain for $n\geq 2$:

$$(3.13)\qquad \begin{aligned}\zeta(2n+1) = \frac{(-1)^n(2\pi)^{2n}}{(2^{2n+1}-1)n}\bigg\{ &-\sum_{m=1}^{n-1}\frac{2^{2m}\zeta'(-2m)}{(2m-1)!(2n-2m)!} \\ &+\sum_{m=0}^{\infty}\frac{\zeta(2m)}{(2m+1)\cdots(2m+2n)2^{2m}}\bigg\}.\end{aligned}$$

In particular, if $n=2$, we have

$$(3.14)\qquad \zeta(5) = \frac{8\pi^4}{31}\bigg\{-2\zeta'(-2)+\sum_{m=0}^{\infty}\frac{\zeta(2m)}{(2m+1)\cdots(2m+4)2^{2m}}\bigg\}.$$

Elizalde [**3**] has shown for $k\geq 1$ that

$$(3.15)\qquad \begin{aligned}\zeta'(-k) &= -\frac{1}{(k+1)^2} \\ &\quad -\sum_{l=1}^{\infty}\frac{(-1)^l(2l-1)!}{2^{2l-1}\pi^{2l}}\left(\sum_{h=0}^{\min(2l-2,k)}\binom{k}{h}\frac{(-1)^h}{2l-h-1}\right)\zeta(2l).\end{aligned}$$

Using (3.15) in (3.13), we obtain

Theorem. *For $n \geq 2$,*

$$(3.16)\quad \zeta(2n+1) = \frac{(-1)^n(2\pi)^{2n}}{(2^{2n+1}-1)n}\left\{\sum_{m=1}^{n-1}\frac{2^{2m}}{(2m-1)!(2n-2m)!(2m+1)^2}\right.$$
$$+\sum_{m=1}^{n-1}\frac{2^{2m}}{(2m-1)!(2n-2m)!}\sum_{l=1}^{\infty}\frac{(-1)^l(2l-1)!}{2^{2l-1}\pi^{2l}}$$
$$\left.\cdot\sum_{h=0}^{\min(2l-2,2m)}\binom{2m}{h}\frac{(-1)^h\zeta(2l)}{2l-h-1}+\sum_{m=0}^{\infty}\frac{(2m)!\zeta(2m)}{(2m+4)!2^{2m}}\right\}.$$

Although complicated, this expansion for $\zeta(2n+1)$, $n \geq 2$, is simpler than the formula given by Ewell [**5**]. In particular, we have

$$(3.17)\quad \zeta(5) = \frac{8\pi^4}{31}\left(\frac{1}{18}+\sum_{l=2}^{\infty}\frac{(-1)^l(2l-4)!}{2^{2l-3}\pi^{2l}}\zeta(2l)\right.$$
$$\left.+\sum_{l=0}^{\infty}\frac{\zeta(2l)}{(2l+1)(2l+2)(2l+3)(2l+4)2^{2l}}\right).$$

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