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COMPARISON OF THE LENGTHS OF THE CONTINUED FRACTIONS OF $\sqrt{D}$ AND $\frac{1}{2}(1+\sqrt{D})$

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ABSTRACT. Let D denote a positive nonsquare integer such that $D \equiv 1 \pmod 4$. Let $l(\sqrt{D})$ (resp. $l(\frac{1}{2}(1+\sqrt{D}))$) denote the length of the period of the continued fraction expansion of $\sqrt{D}$ (resp. $\frac{1}{2}(1+\sqrt{D})$). Recently Ishii, Kaplan, and Williams (*On Eisenstein's problem*, Acta Arith. **54** (1990), 323–345) established inequalities between $l(\sqrt{D})$ and $l(\frac{1}{2}(1+\sqrt{D}))$. In this note it is shown that these inequalities are best possible in a strong sense.

Throughout this note D denotes a positive nonsquare integer > 16 such that $D \equiv 1 \pmod 4$. Let $l = l(\sqrt{D})$ (resp. $l' = l(\frac{1}{2}(1+\sqrt{D}))$) denote the length of the period of the continued fraction expansion of $\sqrt{D}$ (resp. $\frac{1}{2}(1+\sqrt{D})$). In their work on Eisenstein's problem, Ishii, Kaplan, and Williams [3] established a number of inequalities relating $l(\sqrt{D})$ and $l(\frac{1}{2}(1+\sqrt{D}))$. These are summarized as Theorems A, B, and C.

Theorem A. *If there exist odd integers T and U such that $T^2 - DU^2 = 4$ (so that $D \equiv 5 \pmod 8$) then*

$$l(\tfrac{1}{2}(1+\sqrt{D})) + 4 \le l(\sqrt{D}) \le 5l(\tfrac{1}{2}(1+\sqrt{D})).$$

Theorem B. *If there do not exist odd integers T and U such that $T^2 - DU^2 = 4$ but there do exist integers V and W such that $V^2 - DW^2 = -1$ then*

$$\tfrac{1}{3}l(\tfrac{1}{2}(1+\sqrt{D})) \le l(\sqrt{D}) \le 3l(\tfrac{1}{2}(1+\sqrt{D})) - 4.$$

Theorem C. *If there do not exist odd integers T and U such that $T^2 - DU^2 = 4$ and there do not exist integers V and W such that $V^2 - DW^2 = -1$ then*

$$\tfrac{1}{3}l(\tfrac{1}{2}(1+\sqrt{D})) \le l(\sqrt{D}) \le 3l(\tfrac{1}{2}(1+\sqrt{D})) - 8.$$

We show that all of the six inequalities in Theorems A, B, and C are best possible in a strong sense (see Theorems 1–6).

We begin by giving a few basic facts about the continued fractions of $\sqrt{D}$ and $\frac{1}{2}(1+\sqrt{D})$, all of which can be found, for example, in [6].

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The continued fraction of $\sqrt{D}$ is

(1) $$\sqrt{D} = [a_0; \overline{a_1, a_2, \ldots, a_l}], \qquad l = l(\sqrt{D}),$$

where the positive integers a_i are given by

(2) $$a_i = [x_i], \qquad x_{i+1} = \frac{1}{x_i - a_i} \quad (i = 0, 1, 2, \ldots), \qquad x_0 = \sqrt{D}.$$

Moreover, we have

(3) $$x_i = (P_i + \sqrt{D})/Q_i \qquad (i = 0, 1, 2, \ldots),$$

where the integers P_i and Q_i are given by

(4) $$P_{i+1} = a_i Q_i - P_i, \quad Q_{i+1} = (D - P_{i+1}^2)/Q_i \qquad (i = 0, 1, 2, \ldots),$$

(5) $$P_0 = 0, \qquad Q_0 = 1.$$

From (4) we deduce

(6) $$Q_{i+1} = a_i(P_i - P_{i+1}) + Q_{i-1} \qquad (i = 1, 2, \ldots).$$

The integers a_i, P_i, Q_i have the properties

(7) $$a_0 = [\sqrt{D}], \qquad a_{l-i} = a_i \quad (i = 1, 2, \ldots, l-1), \qquad a_l = 2a_0,$$

(8) $$P_i = P_{l+1-i} \qquad (i = 1, 2, \ldots, l),$$

(9) $$Q_i = Q_{l-i} \qquad (i = 0, 1, 2, \ldots, l).$$

The equation $V^2 - DW^2 = -1$ is solvable in integers V and W if and only if $l = l(\sqrt{D})$ is odd, and the equation $T^2 - DU^2 = 4$ is solvable in *odd* integers T and U if and only if $Q_i = 4$ for some *even* integer i satisfying $0 < i < l$.

The continued fraction of $\frac{1}{2}(1+\sqrt{D})$ is

(10) $$\tfrac{1}{2}(1+\sqrt{D}) = [b_0; \overline{b_1, b_2, \ldots, b_{l'}}], \qquad l' = l(\tfrac{1}{2}(1+\sqrt{D})),$$

where the positive integers b_i are given by

(11) $$b_i = [x_i'], \qquad x_{i+1}' = \frac{1}{x_i' - b_i} \quad (i = 0, 1, 2, \ldots), \qquad x_0' = \frac{1}{2}(1+\sqrt{D}).$$

Moreover, we have

(12) $$x_i' = (P_i' + \sqrt{D})/Q_i', \qquad (i = 0, 1, 2, \ldots),$$

where the integers P_i' and Q_i' are given by

(13) $$P_{i+1}' = b_i Q_i' - P_i', \quad Q_{i+1}' = (D - P_{i+1}'^2)/Q_i' \qquad (i = 0, 1, 2, \ldots),$$

(14) $$P_0' = 1, \qquad Q_0' = 2.$$

From (13) we deduce

(15) $$Q_{i+1}' = b_i(P_i' - P_{i+1}') + Q_{i-1}' \qquad (i = 1, 2, \ldots).$$

The integers b_i, P_i', Q_i' have the properties

(16) $$b_0 = [\tfrac{1}{2}(1+\sqrt{D})], \qquad b_{l'-i} = b_i \quad (i = 1, 2, \ldots, l'-1), \qquad b_{l'} = 2b_0 - 1,$$

(17) $$P_i' = P_{l'+1-i}' \qquad (i = 1, 2, \ldots, l'),$$

(18) $$Q_i' = Q_{l'-i}' \qquad (i = 0, 1, \ldots, l').$$

We are now ready to prove Theorems 1–6. All of the families of D used in the proofs of Theorems 1–6 were suggested on the basis of numerical evidence found by the second author by means of an extensive computer search. Some (but not all) of the continued fractions expansions of $\sqrt{D}$ and $\frac{1}{2}(1+\sqrt{D})$ for these families of D were located in the literature.

Theorem 1. *There exist infinitely many nonsquare positive integers* $D \equiv 1 \pmod 4$ *such that:*

(i) $l(\frac{1}{2}(1+\sqrt{D})) + 4 = l(\sqrt{D})$,
(ii) $T^2 - DU^2 = 4$ *is solvable in odd integers* T *and* U,
(iii) $l(\sqrt{D})$ *is unbounded.*

Proof. We let F_n denote the nth Fibonacci number so that

$$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, \ldots$$

and generally

$$F_n = F_{n-1} + F_{n-2} \qquad (n = 2, 3, \ldots).$$

We choose

$$D = (2F_{6n+1} + 1)^2 + (8F_{6n} + 4) \qquad (n = 1, 2, \ldots).$$

As

$$F_n \equiv \begin{cases} 0 \pmod 2 & \text{if } n \equiv 0 \pmod 3, \\ 1 \pmod 2 & \text{if } n \not\equiv 0 \pmod 3, \end{cases}$$

we see that $D \equiv 5 \pmod 8$ so that D is not a square.

By a straightforward induction argument, we find, making use of the identity

$$F_{r+t}F_s - F_rF_{s+t} = (-1)^{s-1}F_tF_{r-s} \qquad (r \geq s \geq 0, t \geq 0),$$

the continued fraction expansions of $\sqrt{D}$ and $\frac{1}{2}(1+\sqrt{D})$ given in Tables 1 and 2 on the next page.

Thus $l(\sqrt{D}) = 6n+5$ and $l(\frac{1}{2}(1+\sqrt{D})) = 6n+1$ so that $l(\frac{1}{2}(1+\sqrt{D}))+4 = l(\sqrt{D})$ and $l(\sqrt{D})$ is unbounded. The equation $T^2 - DU^2 = 4$ is solvable in odd integers T and U as $Q_{4n+4} = 4$. As far as the authors are aware this is the first example of a continued fraction expansion of $\sqrt{D}$ or $\frac{1}{2}(1+\sqrt{D})$, where D involves the Fibonacci numbers F_n and their squares. Presumably the D's used here are a special case of an infinite family of D's involving Fibonacci numbers for which the continued fractions of $\sqrt{D}$ and $\frac{1}{2}(1+\sqrt{D})$ can be given explicitly.

Theorem 2. *There exist infinitely many nonsquare positive integers* $D \equiv 1 \pmod 4$ *such that:*

(i) $l(\sqrt{D}) = 5l(\frac{1}{2}(1+\sqrt{D}))$,
(ii) $T^2 - DU^2 = 4$ *is solvable in odd integers* T *and* U,
(iii) $l(\sqrt{D})$ *is unbounded.*

Proof. We take

$$D = 16.5^{2n} + 12.5^n + 1 \qquad (n = 1, 2, 3, \ldots).$$

Clearly $D \equiv 5 \pmod 8$ so that D is not a square. The continued fraction expansion of $\sqrt{D}$ was considered by Williams [7, Table 4(A) with $q = 4$,

TABLE 1. Continued fraction expansion of $\sqrt{D}$: $D = (2F_{6n+1}+1)^2 + (8F_{6n}+4)$ $(n \geq 1)$

t	P_t	Q_t	a_t
0	0	1	$2F_{6n+1}+2$
k $(1 \leq k \leq n)$	$4F_{3k}F_{6n-3k+1}$ $-2(-1)^k F_{6n-6k+1}+2$	$2F_{3k}F_{6n-3k+1}+1$	4
k $(n+1 \leq k \leq 2n-1)$	$4F_{3k}F_{6n-3k+1}$ $-2(-1)^k F_{6k-6n-1}+2$	$2F_{3k}F_{6n-3k+1}+1$	4
$2n$	$4F_{6n}-2F_{6n-1}+2$	$2F_{6n}+1$	3
$2n+1$	$2F_{6n+1}+1$	4	F_{6n+1}
$2n+2$	$2F_{6n+1}-1$	$2F_{6n+2}+1$	1
$2n+k+2$ $(1 \leq k \leq n)$	$4F_{3k-1}F_{6n-3k+2}$ $+2(-1)^k F_{6n-6k+3}+2$	$2F_{3k-1}F_{6n-3k+2}+1$	4
$2n+k+2$ $(n+1 \leq k \leq 2n)$	$4F_{3k-1}F_{6n-3k+2}$ $+2(-1)^k F_{6k-6n-3}+2$	$2F_{3k-1}F_{6n-3k+2}+1$	4
$4n+3$	$2F_{6n}+2$	$2F_{6n+2}+1$	1
$4n+4$	$2F_{6n+1}-1$	4	F_{6n+1}
$4n+5$	$2F_{6n+1}+1$	$2F_{6n}+1$	3
$4n+k+6$ $(0 \leq k \leq n-1)$	$4F_{3k+1}F_{6n-3k}-2(-1)^k$ $F_{6n-6k-1}+2$	$2F_{3k+4}F_{6n-3k-3}+1$	4
$4n+k+6$ $(n \leq k \leq 2n-2)$	$4F_{3k+1}F_{6n-3k}-2(-1)^k$ $F_{6k-6n+1}+2$	$2F_{3k+4}F_{6n-3k-3}+1$	4
$6n+5$	$2F_{6n+1}+2$	1	$4F_{6n+1}+4$

TABLE 2. Continued fraction expansion of $\frac{1}{2}(1+\sqrt{D})$: $D = (2F_{6n+1}+1)^2 + 8F_{6n}+4$ $(n \geq 1)$

t	P'_t	Q'_t	b_t
0	1	2	$F_{6n+1}+1$
k $(1 \leq k \leq 6n)$	$1+2F_{6n+1}$ $-4F_{k-1}F_{6n-k+1}$	$2+4F_kF_{6n-k+1}$	1
$6n+1$	$2F_{6n+1}+1$	2	$2F_{6n+1}+1$

$k=1$, $A=5]$ and that of $\frac{1}{2}(1+\sqrt{D})$ by deMille [1, p. 32, Table with $q=4$, $k=1$, $A=5]$. From these tables we have

$$l(\sqrt{D}) = 10n+5, \qquad l(\tfrac{1}{2}(1+\sqrt{D})) = 2n+1,$$

so (i) and (iii) hold. In addition, $Q_{8n+4}=4$, so (ii) holds.

Theorem 3. *There exist infinitely many nonsquare positive integers* $D \equiv 1 \pmod 4$ *such that:*

(i) $\frac{1}{3}l(\frac{1}{2}(1+\sqrt{D})) = l(\sqrt{D})$,
(ii) $T^2 - DU^2 = 4$ *has no solution in odd integers* T *and* U,
(iii) $V^2 - DW^2 = -1$ *is solvable in integers* V *and* W,
(iv) $l(\sqrt{D})$ *is unbounded.*

Proof. We take

$$D = 4.17^{2n} + 9.17^n + 4 \qquad (n = 1, 2, 3, \ldots).$$

Clearly $D \equiv 1 \pmod 8$. Further $D \equiv 2 \pmod 3$, so D is not a square. The continued fraction expansion of $\sqrt{D}$ was treated by Williams [7, Table 2 with

$q=k=2$, $A=17$] and that of $\frac{1}{2}(1+\sqrt{D})$ by deMille [1, p. 21, Table with $q=k=2$, $A=17$]. From these tables we have

$$l(\sqrt{D})=2n+1, \qquad l(\tfrac{1}{2}(1+\sqrt{D}))=6n+3,$$

so (i) and (iv) hold. Also

$$Q_{2k+1}=17^{n-k} \quad (0\le k\le n-1), \qquad Q_{2k+2}=17^{k+1} \quad (0\le k\le n-1),$$

so $Q_t\ne 4$ $(t=1,2,\dots,2n)$, which show that (ii) holds. As $l(\sqrt{D})$ is odd, the equation $V^2-DW^2=-1$ is solvable.

Theorem 4. *There exist infinitely many nonsquare integers* $D\equiv 1\,(\mathrm{mod}\,4)$ *for which*:

(i) $l(\sqrt{D})=3l(\frac{1}{2}(1+\sqrt{D}))-4$,
(ii) $T^2-DU^2=4$ *has no solutions in odd integers* T *and* U,
(iii) $V^2-DW^2=-1$ *is solvable in integers* V *and* W,
(iv) $l(\sqrt{D})$ *is unbounded.*

Proof. We take

$$D=9.4^{2n}+10.4^n+1 \qquad (n=1,2,\dots).$$

Clearly $D\equiv 1\,(\mathrm{mod}\,4)$. Also $D\equiv 2\,(\mathrm{mod}\,3)$, so D is not a square. The continued fraction expansion of $\sqrt{D}$ was treated by Williams [7, Table 4(A) with $q=3$, $k=1$, $A=4$] and that of $\frac{1}{2}(1+\sqrt{D})$ by deMille [1, p. 32, Table with $q=3$, $k=1$, $A=4$]. From these tables we see that

$$l(\sqrt{D})=6n-1, \qquad l(\tfrac{1}{2}(1+\sqrt{D}))=2n+1,$$

so (i) and (iv) hold. As $l(\sqrt{D})$ is odd, (iii) holds. Finally, as

$$Q_1=4^{n+1}, \qquad Q_{6k+2}=3.4^n-4^{n-k}+4^{k+1}-1 \quad (0\le k\le n-1),$$
$$Q_{6k+3}=3.4^n+4^{n-k}-4^{k+1}-1 \qquad (0\le k\le n-1),$$
$$Q_{6k+4}=4^{k+2} \qquad (0\le k\le n-2),$$
$$Q_{6k+5}=3.4^n+4^{n-k-1}-4^{k+1}+1 \qquad (0\le k\le n-2),$$
$$Q_{6k+6}=3.4^n-4^{n-k-1}+4^{k+1}+1 \qquad (0\le k\le n-2),$$
$$Q_{6k+7}=4^{n-k} \quad (0\le k\le n-2), \qquad Q_{6n-2}=4^{n+1},$$

we see that $Q_i\ne 4$ $(1\le i\le 6n-2)$, so that (ii) holds.

Theorem 5. *There exist infinitely many nonsquare integers* $D\equiv 1\,(\mathrm{mod}\,4)$ *for which*

(i) $\frac{1}{3}l(\frac{1}{2}(1+\sqrt{D}))=l(\sqrt{D})$,
(ii) $T^2-DU^2=4$ *has no solutions in odd integers* T *and* U,
(iii) $V^2-DW^2=-1$ *has no solutions in integers* V *and* W,
(iv) $l(\sqrt{D})$ *is unbounded.*

Proof. We take

$$D=225.61^{2n}+155.61^n+25 \qquad (n=1,2,\dots).$$

Clearly $D\equiv 1\,(\mathrm{mod}\,4)$. Also $D\equiv 0\,(\mathrm{mod}\,5)$, $D\not\equiv 0\,(\mathrm{mod}\,5^2)$, so D is a nonsquare integer. The continued fraction expansions of $\sqrt{D}$ and $\frac{1}{2}(1+\sqrt{D})$ are given in Tables 3 and 4.

TABLE 3. Continued fraction expansion of $\sqrt{D}$: $D = 225.61^{2n} + 155.61^n + 25$ $(n \geq 1)$

t	P_t	Q_t	a_t
0	0	1	$15.61^n + 5$
$2k+1$ $(0 \leq k \leq n-1)$	$15.61^n + 5$	5.61^{n-k}	6.61^k
$2k+2$ $(0 \leq k \leq n-1)$	$15.61^n - 5$	61^{k+1}	30.61^{n-k-1}
$2n+1$	$15.61^n + 5$	5	$6.61^n + 2$
$2n+2k+2$ $(0 \leq k \leq n-1)$	$15.61^n + 5$	61^{n-k}	30.61^k
$2n+2k+3$ $(0 \leq k \leq n-1)$	$15.61^n - 5$	5.61^{k+1}	6.61^{n-k-1}
$4n+2$	$15.61^n + 5$	1	$30.61^n + 10$

Note. This table is the special case $\lambda = \mu = \tau = 1$, $l = 3$, $q = c = 5$, $p = 61$ of [2, §4.2].

TABLE 4. Continued fraction expansion of $\frac{1}{2}(1 + \sqrt{D})$: $D = 225.61^{2n} + 155.61^n + 25$ $(n \geq 1)$

t	P'_t	Q'_t	b_t
0	1	2	$\frac{1}{2}(15.61^n + 5)$
$6k+1$ $(0 \leq k \leq n)$	$15.61^n - 61^k + 5$	$15.61^n + \frac{1}{2}(5.61^{n-k} - 61^k) + 5$	1
$6k+2$ $(0 \leq k \leq n)$	$\frac{1}{2}(61^k + 5.61^{n-k})$	$15.61^n - \frac{1}{2}(5.61^{n-k} - 61^k) + 5$	1
$6k+3$ $(0 \leq k \leq n-1)$	$15.61^n - 5.61^{n-k} + 5$	10.61^{n-k}	$3.61^k - 1$
$6k+4$ $(0 \leq k \leq n-1)$	$15.61^n - 5.61^{n-k} - 5$	$15.61^n - \frac{1}{2}(5.61^{n-k} - 61^{k+1}) - 5$	1
$6k+5$ $0 \leq k \leq n-1)$	$\frac{1}{2}(61^{k+1} + 5.61^{n-k})$	$15.61^n + \frac{1}{2}(5.61^{n-k} - 61^{k+1}) - 5$	1
$6k+6$ $(0 \leq k \leq n-1)$	$15.61^n - 61^{k+1} - 5$	2.61^{k+1}	$15.61^{n-k-1} - 1$
$6n+3$	15.61^n	10	3.61^n
$6n+6k+4$ $(0 \leq k \leq n)$	$15.61^n - 5.61^k + 5$	$15.61^n - \frac{1}{2}(5.61^k - 61^{n-k}) + 5$	1
$6n+6k+5$ $(0 \leq k \leq n)$	$\frac{1}{2}(61^{n-k} + 5.61^k)$	$15.61^n + \frac{1}{2}(5.61^k - 61^{n-k}) + 5$	1
$6n+6k+6$ $(0 \leq k \leq n-1)$	$15.61^n - 61^{n-k} + 5$	2.61^{n-k}	$15.61^k - 1$
$6n+6k+7$ $(0 \leq k \leq n-1)$	$15.61^n - 61^{n-k} - 5$	$15.61^n + \frac{1}{2}(5.61^{k+1} - 61^{n-k}) - 5$	1
$6n+6k+8$ $(0 \leq k \leq n-1)$	$\frac{1}{2}(61^{n-k} + 5.61^{k+1})$	$15.61^n - \frac{1}{2}(5.61^{k+1} - 61^{n-k}) - 5$	1
$6n+6k+9$ $(0 \leq k \leq n-1)$	$15.61^n - 5.61^{k+1} - 5$	10.61^{k+1}	$3.61^{n-k-1} - 1$
$12n+6$	$15.61^n + 4$	2	$15.61^n + 4$

Note. This table is not included in either [1] or [2]. Presumably there is an infinite family of D's for which the continued fraction expansion of $\frac{1}{2}(1 + \sqrt{D})$ has a similar structure.

TABLE 5. Continued fraction expansion of $\sqrt{D}$: $D = 81.10^{2n} + 66.10^n + 9$ $(n \geq 1)$

t	P_t	Q_t	a_t
0	0	1	$9.10^n + 3$
1	$9.10^n + 3$	12.10^n	1
$6k + 2$ $(0 \leq k \leq n-1)$	$9.10^n - 6.10^{n-k} - 3$	$9.10^n - 3.10^{n-k} + 10^{k+1} - 3$	1
$6k + 3$ $(0 \leq k \leq n-1)$	$3.10^{n-k} + 10^{k+1}$	$9.10^n + 3.10^{n-k} - 10^{k+1} - 3$	1
$6k + 4$ $(0 \leq k \leq n-2)$	$9.10^n - 2.10^{k+1} - 3$	4.10^{k+1}	$45.10^{n-k-2} - 1$
$6k + 5$ $(0 \leq k \leq n-2)$	$9.10^n - 2.10^{k+1} + 3$	$9.10^n + 3.10^{n-k-1} - 10^{k+1} + 3$	1
$6k + 6$ $(0 \leq k \leq n-2)$	$3.10^{n-k-1} + 10^{k+1}$	$9.10^n - 3.10^{n-k-1} + 10^{k+1} + 3$	1
$6k + 7$ $(0 \leq k \leq n-2)$	$9.10^n - 6.10^{n-k-1} + 3$	12.10^{n-k-1}	$15.10^k - 1$
$6n - 2$	$7.10^n - 3$	4.10^n	4
$6n - 1$	$9.10^n + 3$	3	$6.10^n + 2$
$6n$	$9.10^n + 3$	4.10^n	4
$6n + 6k + 1$ $(0 \leq k \leq n-1)$	$9.10^n - 2.10^{n-k} - 3$	$9.10^n + 3.10^{k+1} - 10^{n-k} - 3$	1
$6n + 6k + 2$ $(0 \leq k \leq n-1)$	$3.10^{k+1} + 10^{n-k}$	$9.10^n - 3.10^{k+1} + 10^{n-k} - 3$	1
$6n + 6k + 3$ $(0 \leq k \leq n-2)$	$9.10^n - 6.10^{k+1} - 3$	12.10^{k+1}	$15.10^{n-k-2} - 1$
$6n + 6k + 4$ $(0 \leq k \leq n-2)$	$9.10^n - 6.10^{k+1} + 3$	$9.10^n - 3.10^{k+1} + 10^{n-k-1} + 3$	1
$6n + 6k + 6$ $(0 \leq k \leq n-2)$	$3.10^{k+1} + 10^{n-k-1}$	$9.10^n + 3.10^{k+1} - 10^{n-k-1} + 3$	1
$6n + 6k + 6$ $(0 \leq k \leq n-2)$	$9.10^n - 2.10^{n-k-1} + 3$	4.10^{n-k-1}	$45.10^k - 1$
$12n - 3$	$3.10^n - 3$	12.10^n	1
$12n - 2$	$9.10^n + 3$	1	$18.10^n + 6$

Note. This table is not a special case of any of the continued fraction expansions considered in [2, 4, 5, 7].

TABLE 6. Continued fraction expansion of $\frac{1}{2}(1+\sqrt{D})$: $D = 81.10^{2n} + 66.10^n + 9$ $(n \geq 1)$

t	P'_t	Q'_t	b_t
0	1	2	$\frac{1}{2}(9.10^n + 4)$
$2k + 1$ $(0 \leq k \leq n-1)$	$9.10^n + 3$	6.10^{n-k}	3.10^k
$2k + 2$ $(0 \leq k \leq n-1)$	$9.10^n - 3$	2.10^{k+1}	9.10^{n-k-1}
$2n + 1$	$9.10^n + 3$	6	$3.10^n + 1$
$2n + 2k + 2$ $(0 \leq k \leq n-1)$	$9.10^n + 3$	2.10^{n-k}	9.10^k
$2n + 2k + 3$ $(0 \leq k \leq n-1)$	$9.10^n - 3$	6.10^{k+1}	3.10^{n-k-1}
$4n + 2$	$9.10^n + 3$	2	$9.10^n + 3$

Note. This table is the special case $\lambda = \mu = 1$, $p = 10$, $c = l = q = 3$ of [2, §4.1].

We have

$$l(\sqrt{D}) = 4n + 2, \qquad l(\tfrac{1}{2}(1+\sqrt{D})) = 12n + 6,$$

so (i) and (iv) hold. As $l(\sqrt{D})$ is even, (iii) holds. It is also clear that $Q_t \neq 4$ $(t = 1, 2, \ldots, 4n+1)$, so (ii) holds.

Theorem 6. *There exist infinitely many nonsquare positive integers* $D \equiv 1 \pmod 4$ *such that*:

(i) $l(\sqrt{D}) = 3l(\frac{1}{2}(1+\sqrt{D})) - 8$,

(ii) $T^2 - DU^2 = 4$ *is not solvable in odd integers* T *and* U,

(iii) $V^2 - DW^2 = -1$ *is not solvable in integers* V *and* W,

(iv) $l(\sqrt{D})$ *is unbounded.*

Proof. We take

$$D = 81.10^{2n} + 66.10^n + 9 \qquad (n = 1, 2, 3, \ldots).$$

Clearly $D \equiv 1 \pmod 8$. Further $D \equiv 0 \pmod 3$, $D \not\equiv 0 \pmod{3^2}$, so D is not a square. The continued fraction expansions of $\sqrt{D}$ and $\frac{1}{2}(1+\sqrt{D})$ are given in Tables 5 and 6. We have

$$l(\sqrt{D}) = 12n - 2, \qquad l(\tfrac{1}{2}(1+\sqrt{D})) = 4n + 2,$$

so (i) and (iv) hold. Further, as $l(\sqrt{D})$ is even, (iii) holds. Finally we note that $Q_t \neq 4$, $(1 \leq t \leq 12n - 3)$, so $T^2 - DU^2 = 4$ is not solvable in odd integers T and U.

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