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Pascal's triangle (mod 9)

by

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1. Introduction. Let n denote a nonnegative integer. The nth row of Pascal's triangle consists of the $n+1$ binomial coefficients

$$\binom{n}{0}\ \binom{n}{1}\ \binom{n}{2}\ \cdots\ \binom{n}{n}.$$

We denote by $N_n(t,m)$ the number of binomial coefficients in the nth row of Pascal's triangle which are congruent to t modulo m, where t and m are integers with $m \ge 2$. Explicit formulae for $N_n(t,m)$ for certain values of t and m have been given by a number of authors, for example $m=2$ (Glaisher [3]), $m=3$ (Hexel and Sachs [5]), $m=4$ (Davis and Webb [2], Granville [4]), $m=5$ (Hexel and Sachs [5]), $m=8$ (Granville [4], Huard, Spearman and Williams [6]), and $m=p$ (prime) (Hexel and Sachs [5], Webb [10]).

In this paper we treat the case $m=9$. We determine explicit formulae for $N_n(t,9)$ for $t=0,1,2,\ldots,8$; see the Theorem in Section 2.

We use throughout the 3-ary representation of n, namely,

$$n = a_0 + a_1 3 + a_2 3^2 + \ldots + a_l 3^l = a_0 a_1 a_2 \ldots a_l, \tag{1.1}$$

where $l \ge 0$, each $a_i = 0, 1$ or 2, and $a_l = 1$ or 2 unless $n = 0$ in which case $l = 0$ and $a_0 = 0$. We denote by r an arbitrary integer between 0 and n inclusive, and we suppose that the 3-ary representation of r is (with additional zeros at the right hand end if necessary) $r = b_0 b_1 \ldots b_l$. From a theorem of Kummer [8, Lehrsatz, pp. 115–116] (proved in 1852), we can

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deduce the exact power of 3 dividing $\binom{n}{r}$, namely,

$$3^{c(n,r)} \,\|\, \binom{n}{r}, \tag{1.2}$$

where $c(n,r)$ is the number of carries when adding the 3-ary representations of r and $n-r$ in base 3. A special case of a theorem of Lucas [9, p. 52] (proved in 1878) gives the residue of $\binom{n}{r}$ modulo 3, namely,

$$\binom{n}{r} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots\binom{a_l}{b_l} \pmod 3, \tag{1.3}$$

with the usual interpretation that $\binom{a_i}{b_i} = 0$ if $b_i > a_i$. If $3 \nmid \binom{n}{r}$ (equivalently $c(n,r) = 0$) the residue of $\binom{n}{r}$ modulo 9 follows from a theorem of Granville [4, Proposition 2, p. 326] (proved in 1992), namely, if $3 \nmid \binom{n}{r}$ and $l \geq 1$ then

$$\binom{n}{r} \equiv \frac{\binom{a_0+3a_1}{b_0+3b_1}\binom{a_1+3a_2}{b_1+3b_2}\cdots\binom{a_{l-1}+3a_l}{b_{l-1}+3b_l}}{\binom{a_1}{b_1}\cdots\binom{a_{l-1}}{b_{l-1}}} \pmod 9, \tag{1.4}$$

with the convention that when $l = 1$ the denominator is the empty product $=$ 1. Further, if $3 \| \binom{n}{r}$ (equivalently $c(n,r) = 1$), then a theorem of Kazandzidis [7] gives the residue of $\binom{n}{r}$ modulo 9, namely,

$$\binom{n}{r} \equiv -3\frac{a_0!a_1!\ldots a_l!}{b_0!b_1!\ldots b_l!c_0!c_1!\ldots c_l!} \pmod 9, \tag{1.5}$$

where $c_0c_1\ldots c_l$ is the 3-ary representation of $n-r$. Both (1.4) and (1.5) also follow from an extension of Lucas' theorem given by Davis and Webb [1].

We conclude this introduction by giving the following formulae of Hexel and Sachs [5]: if n_1 denotes the number of 1's and n_2 the number of 2's in the string $a_0a_1\ldots a_l$ then

$$\begin{aligned} N_n(0,3) &= n+1-2^{n_1}3^{n_2}, \\ N_n(1,3) &= \tfrac{1}{2}(2^{n_1}3^{n_2}+2^{n_1}), \\ N_n(2,3) &= \tfrac{1}{2}(2^{n_1}3^{n_2}-2^{n_1}). \end{aligned}$$

2. Statement of results. If S is a string of 0's, 1's and 2's, we denote by $n_S = n_S(\mathbf{a})$ the number of occurrences of S in the string $\mathbf{a} = a_0a_1\ldots a_l$. Thus, for example, if $a_0a_1\ldots a_l = 01112010012$ then $n_{11} = 2$, $n_{12} = 2$, $n_{001} = 1$, and $n_{121} = 0$. Making use of the results (1.2)–(1.5), we prove the following theorem in Sections 4 and 5. We note that for a nonnegative integer m,

$$0^m = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}$$

THEOREM.

$$N_n(0,9) = n + 1 - 2^{n_1}3^{n_2} - n_{01}2^{n_1}3^{n_2} - n_{02}2^{n_1+2}3^{n_2-1} - n_{11}2^{n_1-2}3^{n_2} - n_{12}2^{n_1}3^{n_2-1}.$$

For $t = 3, 6$,

$$N_n(t,9) = n_{01}2^{n_1-1}(3^{n_2} - (-1)^t) + n_{02}2^{n_1+1}(3^{n_2-1} + (-1)^t) + n_{11}2^{n_1-3}(3^{n_2} + (-1)^t) + n_{12}2^{n_1-1}(3^{n_2-1} - (-1)^t).$$

For $t = 1, 2, 4, 5, 7, 8$,

$$N_n(t,9) = \tfrac{1}{6}\{2^{n_1}3^{n_2} + (-1)^{\mathrm{ind}_2 t}2^{n_1} + (-1)^{\mathrm{ind}_2 t + n_{11} + n_{122}}2^{n_1 - n_{11} + n_{122} + 1}\,\mathrm{Re}(X) + 0^{n_{122}}(-1)^{n_{11}}2^{n_1 - n_{11} + 1}3^{n_{22} - n_{122}}\,\mathrm{Re}(Y)\},$$

where $\mathrm{ind}_2\, t$ *denotes the unique integer* j *such that* $t \equiv 2^j \pmod 9$, $0 \le j \le 5$,

$$X = \beta^{\mathrm{ind}_2 t - n_{11} - n_{12} + n_{121} - n_{122}}(2 - \beta)^{n_{21} - n_{121}} \times (3 + \beta)^{n_2 - n_{12} - n_{21} - n_{22} + n_{121} + n_{122}},$$

$$Y = \beta^{-\mathrm{ind}_2 t + n_{11}}(1 - \beta)^{n_{21} - n_{121}}(2 + \beta)^{n_2 - n_{21} - n_{22}}(1 + 2\beta)^{n_{121}},$$

and $\beta = \exp(2\pi i/3)$.

For $n = 0, 1, \ldots, 8$ Table 1 gives the values of the expressions involving $n, n_1, n_2, n_{01}, n_{02}, n_{11}, n_{12}, n_{21}, n_{22}, n_{121}, n_{122}$ occurring on the right hand sides of the formulae for $N_n(t,9)$ given in the Theorem. Clearly $n_{121} = n_{122} = 0$ for $n = 0, 1, \ldots, 8$.

Table 1

										Right hand sides of formulae in Theorem for $t = 0, 1, \ldots, 8$								
n	n in base 3	n_1	n_2	n_{01}	n_{02}	n_{11}	n_{12}	n_{21}	n_{22}	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0
2	2	0	1	0	0	0	0	0	0	0	2	1	0	0	0	0	0	0
3	01	1	0	1	0	0	0	0	0	0	2	0	2	0	0	0	0	0
4	11	2	0	0	0	1	0	0	0	0	2	0	0	2	0	1	0	0
5	21	1	1	0	0	0	0	1	0	0	4	0	0	0	2	0	0	0
6	02	0	1	0	1	0	0	0	0	0	2	1	0	0	0	4	0	0
7	12	1	1	0	0	0	1	0	0	0	2	0	2	0	0	0	2	2
8	22	0	2	0	0	0	0	0	1	0	4	2	0	0	0	0	1	2

The first nine rows of Pascal's triangle (mod 9) are

0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	1	1	5	1			
6	1	6	6	2	6	6	1		
7	1	7	3	8	8	3	7	1	
8	1	8	1	2	7	2	1	8	1

From this triangle we deduce easily the values of $N_n(t,9)$ for $n = 0, 1, \ldots, 8$ and $t = 0, 1, \ldots, 8$. These values are in agreement with those in Table 1 so the Theorem holds for $n = 0, 1, \ldots, 8$. Thus in the proof of the Theorem in Sections 4 and 5, we may suppose that $n \geq 9$, so that $l \geq 2$.

In the next section we evaluate a sum which will be used in the determination of $N_n(t,9)$ $(3 \nmid t)$ in Section 4.

3. Evaluation of the sum $S(\mathbf{c};\alpha)$. Let k be a positive integer. Let $\mathbf{c} = c_0c_1\ldots c_k$ be a string of length $k+1$ (≥ 2) with each $c_i = 0, 1, 2$. Let $\mathbf{d} = d_0d_1\ldots d_k$ be a string of length $k+1$ with each $d_i = 0, 1, 2$ and $d_i \leq c_i$. As $0 \leq d_i \leq c_i \leq 2$ $(i = 0, 1, \ldots, k)$ we have

$$\binom{c_i}{d_i} \not\equiv 0 \pmod 3 \qquad (i = 0, \ldots, k)$$

and by Lucas' theorem (see (1.3))

$$\binom{c_{i-1} + 3c_i}{d_{i-1} + 3d_i} \equiv \binom{c_{i-1}}{d_{i-1}} \binom{c_i}{d_i} \not\equiv 0 \pmod 3 \qquad (i = 1, \ldots, k)$$

so that

$$\frac{\binom{c_0 + 3c_1}{d_0 + 3d_1} \binom{c_1 + 3c_2}{d_1 + 3d_2} \cdots \binom{c_{k-1} + 3c_k}{d_{k-1} + 3d_k}}{\binom{c_1}{d_1} \cdots \binom{c_{k-1}}{d_{k-1}}} \not\equiv 0 \pmod 3,$$

where the denominator is understood to be the empty product $(= 1)$ when $k = 1$.

Thus we can define $e(\mathbf{c}, \mathbf{d}) = 1, 2, 4, 5, 7, 8$ by

$$\text{(3.1)} \qquad e(\mathbf{c}, \mathbf{d}) \equiv \frac{\binom{c_0 + 3c_1}{d_0 + 3d_1} \binom{c_1 + 3c_2}{d_1 + 3d_2} \cdots \binom{c_{k-1} + 3c_k}{d_{k-1} + 3d_k}}{\binom{c_1}{d_1} \cdots \binom{c_{k-1}}{d_{k-1}}} \pmod 9.$$

We set

$$i(\mathbf{c},\mathbf{d}) = 0,1,2,3,4,5 \quad \text{according as} \quad e(\mathbf{c},\mathbf{d}) = 1,2,4,8,7,5,$$

so that

$$i(\mathbf{c},\mathbf{d}) = \operatorname{ind}_2(e(\mathbf{c},\mathbf{d})). \tag{3.2}$$

Then, for any sixth root of unity α, we define the sum $S(\mathbf{c};\alpha)$ by

$$S(\mathbf{c};\alpha) = \sum_{d_0=0}^{c_0} \cdots \sum_{d_k=0}^{c_k} \alpha^{i(\mathbf{c},\mathbf{d})}. \tag{3.3}$$

The objective of this section is to evaluate the sum $S(\mathbf{c};\alpha)$ explicitly. This evaluation will be used in Section 4 to determine $N_n(t,9)$ for $3 \nmid t$.

We denote by $\mathbf{c}'$ the substring of $\mathbf{c} = c_0c_1\ldots c_k$ formed by removing the first term, that is, $\mathbf{c}' = c_1\ldots c_k$. Our first lemma relates $i(\mathbf{c},\mathbf{d})$ and $i(\mathbf{c}', \mathbf{d}')$ modulo 6.

LEMMA 1. *For $k \geq 2$ we have*

$$i(\mathbf{c},\mathbf{d}) \equiv \operatorname{ind}_2 \left\{ \frac{\binom{c_0+3c_1}{d_0+3d_1}}{\binom{c_1}{d_1}} \right\} + i(\mathbf{c}',\mathbf{d}') \pmod 6.$$

Proof. From (3.1) we have

$$e(\mathbf{c},\mathbf{d}) \equiv \frac{\binom{c_0+3c_1}{d_0+3d_1}}{\binom{c_1}{d_1}} e(\mathbf{c}',\mathbf{d}') \pmod 9.$$

Thus

$$\begin{aligned} i(\mathbf{c},\mathbf{d}) &= \operatorname{ind}_2(e(\mathbf{c},\mathbf{d})) \\ &= \operatorname{ind}_2 \left(\frac{\binom{c_0+3c_1}{d_0+3d_1}}{\binom{c_1}{d_1}} e(\mathbf{c}',\mathbf{d}') \right) \\ &\equiv \operatorname{ind}_2 \left\{ \frac{\binom{c_0+3c_1}{d_0+3d_1}}{\binom{c_1}{d_1}} \right\} + \operatorname{ind}_2(e(\mathbf{c}',\mathbf{d}')) \pmod 6 \end{aligned}$$

$$\equiv \operatorname{ind}_2 \left\{ \frac{\binom{c_0+3c_1}{d_0+3d_1}}{\binom{c_1}{d_1}} \right\} + i(\mathbf{c}', \mathbf{d}') \pmod 6. \blacksquare$$

Our second lemma gives a relationship between $S(\mathbf{c};\alpha)$ and $S(\mathbf{c}';\alpha)$ if $c_0c_1 \neq 12$ and between $S(\mathbf{c};\alpha)$ and $S(\mathbf{c}'';\alpha)$ if $c_0c_1 = 12$, where $\mathbf{c}'' = (\mathbf{c}')' = c_2 \ldots c_k$.

LEMMA 2. *For $k \geq 2$, we have $S(\mathbf{c};\alpha) = f(c_0c_1;\alpha)S(\mathbf{c}';\alpha)$, where*

$$f(c_0c_1;\alpha) = \begin{cases} 1 & \text{if } c_0 = 0, \\ 2 & \text{if } c_0c_1 = 10, \\ 1+\alpha^2 & \text{if } c_0c_1 = 11, \\ 2+\alpha & \text{if } c_0c_1 = 20, \\ 2+\alpha^5 & \text{if } c_0c_1 = 21, \\ 2+\alpha^3 & \text{if } c_0c_1 = 22. \end{cases} \tag{3.4}$$

For $k \geq 3$, we have $S(\mathbf{c};\alpha) = g(c_0c_1c_2;\alpha)S(\mathbf{c}'';\alpha)$, where

$$g(c_0c_1c_2;\alpha) = \begin{cases} 2(1+\alpha^3+\alpha^4) & \text{if } c_0c_1c_2 = 120, \\ 2(1+\alpha+\alpha^4) & \text{if } c_0c_1c_2 = 121, \\ 2(1+\alpha^4+\alpha^5) & \text{if } c_0c_1c_2 = 122. \end{cases} \tag{3.5}$$

Proof. For $k \geq 2$ and any integer d_0 satisfying $0 \leq d_0 \leq c_0$, we define

$$F(d_0,\mathbf{c};\alpha) = \sum_{d_1=0}^{c_1} \cdots \sum_{d_k=0}^{c_k} \alpha^{i(\mathbf{c},\mathbf{d})}. \tag{3.6}$$

Then

$$\sum_{d_0=0}^{c_0} F(d_0,\mathbf{c};\alpha) = \sum_{d_0=0}^{c_0}\sum_{d_1=0}^{c_1} \cdots \sum_{d_k=0}^{c_k} \alpha^{i(\mathbf{c},\mathbf{d})},$$

so that

$$S(\mathbf{c};\alpha) = \sum_{d_0=0}^{c_0} F(d_0,\mathbf{c};\alpha). \tag{3.7}$$

Also for $k \geq 2$ we have, by (3.6) and Lemma 1,

$$\begin{aligned} F(d_0,\mathbf{c};\alpha) &= \sum_{d_1=0}^{c_1} \cdots \sum_{d_k=0}^{c_k} \alpha^{\operatorname{ind}_2\{\binom{c_0+3c_1}{d_0+3d_1}/\binom{c_1}{d_1}\}+i(\mathbf{c}',\mathbf{d}')} \\ &= \sum_{d_1=0}^{c_1} \alpha^{\operatorname{ind}_2\{\binom{c_0+3c_1}{d_0+3d_1}/\binom{c_1}{d_1}\}} \sum_{d_2=0}^{c_2} \cdots \sum_{d_k=0}^{c_k} \alpha^{i(\mathbf{c}',\mathbf{d}')}, \end{aligned}$$

that is,

$$F(d_0,\mathbf{c};\alpha)=\sum_{d_1=0}^{c_1}\alpha^{\mathrm{ind}_2\{\binom{c_0+3c_1}{d_0+3d_1}/\binom{c_1}{d_1}\}}F(d_1,\mathbf{c}';\alpha). \tag{3.8}$$

Next we define the $(c_0+1)\times 1$ matrix $A(\mathbf{c};\alpha)$ by

$$A(\mathbf{c};\alpha)=\begin{bmatrix}F(0,\mathbf{c};\alpha)\\ \vdots\\ F(c_0,\mathbf{c};\alpha)\end{bmatrix}. \tag{3.9}$$

Then, from (3.8), we deduce that for $k\geq 2$,

$$A(\mathbf{c};\alpha)=M(c_0c_1;\alpha)A(\mathbf{c}';\alpha), \tag{3.10}$$

where $M(c_0c_1;\alpha)$ is the $(c_0+1)\times(c_1+1)$ matrix whose entry in the (i,j) place $(i=0,1,\dots,c_0;\ j=0,1,\dots,c_1)$ is

$$\alpha^{\mathrm{ind}_2\{\binom{c_0+3c_1}{i+3j}/\binom{c_1}{j}\}}.$$

Thus

$$M(00;\alpha)=[1],\qquad M(01;\alpha)=[1\quad 1],$$

$$M(10;\alpha)=\begin{bmatrix}1\\1\end{bmatrix},\qquad M(11;\alpha)=\begin{bmatrix}1&\alpha^2\\ \alpha^2&1\end{bmatrix},$$

$$M(20;\alpha)=\begin{bmatrix}1\\ \alpha\\ 1\end{bmatrix},\qquad M(21;\alpha)=\begin{bmatrix}1&1\\ \alpha^5&\alpha^5\\ 1&1\end{bmatrix},$$

$$M(02;\alpha)=[1\quad 1\quad 1],$$

$$M(12;\alpha)=\begin{bmatrix}1&\alpha^2&\alpha^4\\ \alpha^4&\alpha^2&1\end{bmatrix},$$

$$M(22;\alpha)=\begin{bmatrix}1&1&1\\ \alpha^3&\alpha^3&\alpha^3\\ 1&1&1\end{bmatrix}.$$

If $c_0c_1\neq 12$ then the c_1+1 columns of the matrix $M(c_0c_1;\alpha)$ all have the same sum. Hence summing the rows in (3.9), and appealing to (3.7) and (3.10), we obtain

$$S(\mathbf{c};\alpha)=f(c_0c_1;\alpha)S(\mathbf{c}';\alpha),$$

where

$$f(c_0c_1;\alpha)=\begin{cases}1 & \text{if } c_0c_1=00,\\ 2 & \text{if } c_0c_1=10,\\ 2+\alpha & \text{if } c_0c_1=20,\\ 1 & \text{if } c_0c_1=01,\\ 1+\alpha^2 & \text{if } c_0c_1=11,\\ 2+\alpha^5 & \text{if } c_0c_1=21,\\ 1 & \text{if } c_0c_1=02,\\ 2+\alpha^3 & \text{if } c_0c_1=22.\end{cases}$$

Now suppose that $c_0c_1=12$ and $k\geq 3$. By (3.10) we have

$$A(\mathbf{c};\alpha)=M(c_0c_1;\alpha)M(c_1c_2;\alpha)A(\mathbf{c}'',\alpha), \tag{3.11}$$

where $\mathbf{c}''=(\mathbf{c}')'=c_2\ldots c_l$. Now

$$M(12;\alpha)M(20;\alpha)=\begin{bmatrix}1 & \alpha^2 & \alpha^4\\ \alpha^4 & \alpha^2 & 1\end{bmatrix}\begin{bmatrix}1\\ \alpha\\ 1\end{bmatrix}=\begin{bmatrix}1+\alpha^3+\alpha^4\\ 1+\alpha^3+\alpha^4\end{bmatrix},$$

$$\begin{aligned}M(12;\alpha)M(21;\alpha)&=\begin{bmatrix}1 & \alpha^2 & \alpha^4\\ \alpha^4 & \alpha^2 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ \alpha^5 & \alpha^5\\ 1 & 1\end{bmatrix}\\ &=\begin{bmatrix}1+\alpha+\alpha^4 & 1+\alpha+\alpha^4\\ 1+\alpha+\alpha^4 & 1+\alpha+\alpha^4\end{bmatrix},\end{aligned}$$

and

$$\begin{aligned}M(12;\alpha)M(22;\alpha)&=\begin{bmatrix}1 & \alpha^2 & \alpha^4\\ \alpha^4 & \alpha^2 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ \alpha^3 & \alpha^3 & \alpha^3\\ 1 & 1 & 1\end{bmatrix}\\ &=\begin{bmatrix}1+\alpha^4+\alpha^5 & 1+\alpha^4+\alpha^5 & 1+\alpha^4+\alpha^5\\ 1+\alpha^4+\alpha^5 & 1+\alpha^4+\alpha^5 & 1+\alpha^4+\alpha^5\end{bmatrix}.\end{aligned}$$

For each of these products, the column sums are the same. Hence summing the rows in (3.11), we obtain

$$S(\mathbf{c};\alpha)=g(c_0c_1c_2;\alpha)S(\mathbf{c}'';\alpha),$$

where

$$g(c_0c_1c_2;\alpha)=\begin{cases}2(1+\alpha^3+\alpha^4) & \text{if } c_0c_1c_2=120,\\ 2(1+\alpha+\alpha^4) & \text{if } c_0c_1c_2=121,\\ 2(1+\alpha^4+\alpha^5) & \text{if } c_0c_1c_2=122. \blacksquare\end{cases}$$

We are now ready to use Lemma 2 to evaluate $S(\mathbf{c};\alpha)$.

PROPOSITION. *Let $\mathbf{c} = c_0c_1\dots c_k$ be a string of length $k+1 \geq 2$ with each $c_i = 0, 1$ or 2. Denote by n_S the number of occurrences of the string S in $\mathbf{c}$. Let α be a sixth root of unity. Then*

$$\begin{aligned}(3.12)\quad S(\mathbf{c};\alpha) = {} & 2^{n_1-n_{11}}(1+\alpha^2)^{n_{11}}(2+\alpha)^{n_2-n_{12}-n_{21}-n_{22}+n_{121}+n_{122}} \\ & \times (2+\alpha^5)^{n_{21}-n_{121}}(2+\alpha^3)^{n_{22}-n_{122}} \\ & \times (1+\alpha^3+\alpha^4)^{n_{12}-n_{121}-n_{122}}(1+\alpha+\alpha^4)^{n_{121}} \\ & \times (1+\alpha^4+\alpha^5)^{n_{122}}.\end{aligned}$$

Proof. The proof of (3.12) is by induction on $k \geq 1$. When $k = 1$ we have

$$S(\mathbf{c};\alpha) = S(c_0c_1;\alpha) = \sum_{d_0=0}^{c_0}\sum_{d_1=0}^{c_1} \alpha^{\operatorname{ind}_2\binom{c_0+3c_1}{d_0+3d_1}}.$$

The values of this sum for $c_0c_1 = 00, 01, \dots, 22$ are given in Table 2. The values of the expression on the right hand side of (3.12) when $k = 1$ are given in Table 3. These two tables show that the Proposition is true for $k = 1$.

Table 2

c_0c_1	$S(c_0c_1;\alpha)$
00	1
01	2
02	$2+\alpha$
10	2
11	$2(1+\alpha^2)$
12	$2(1+\alpha^3+\alpha^4)$
20	$2+\alpha$
21	$2(2+\alpha^5)$
22	$(2+\alpha)(2+\alpha^3)$

Table 3

c_0c_1	n_1	n_2	n_{11}	n_{12}	n_{21}	n_{22}	n_{121}	n_{122}	Right side of (3.12)
00	0	0	0	0	0	0	0	0	1
01	1	0	0	0	0	0	0	0	2
02	0	1	0	0	0	0	0	0	$2+\alpha$
10	1	0	0	0	0	0	0	0	2
11	2	0	1	0	0	0	0	0	$2(1+\alpha^2)$
12	1	1	0	1	0	0	0	0	$2(1+\alpha^3+\alpha^4)$
20	0	1	0	0	0	0	0	0	$2+\alpha$
21	1	1	0	0	1	0	0	0	$2(2+\alpha^5)$
22	0	2	0	0	0	1	0	0	$(2+\alpha)(2+\alpha^3)$

When $k = 2$ we have

$$S(\mathbf{c};\alpha) = S(c_0c_1c_2;\alpha) = \sum_{d_0=0}^{c_0}\sum_{d_1=0}^{c_1}\sum_{d_2=0}^{c_2} \alpha^{\mathrm{ind}_2\left\{\frac{\binom{c_0+3c_1}{d_0+3d_1}\binom{c_1+3c_2}{d_1+3d_2}}{\binom{c_1}{d_1}}\right\}}.$$

Taking $c_0c_1c_2 = 000, 001, \ldots, 222$, and working out the sum in each case, we obtain the values of $S(c_0c_1c_2;\alpha)$ given in Table 4. The values of the expression on the right side of (3.12) when $k = 2$ are given in Table 5. Thus the Proposition is true for $k = 2$.

Table 4

$c_0c_1c_2$	$S(c_0c_1c_2;\alpha)$	$c_0c_1c_2$	$S(c_0c_1c_2;\alpha)$
000	1	112	$2(1+\alpha^2)(1+\alpha^3+\alpha^4)$
001	2	120	$2(1+\alpha^3+\alpha^4)$
002	$2+\alpha$	121	$2^2(1+\alpha+\alpha^4)$
010	2	122	$2(2+\alpha)(1+\alpha^4+\alpha^5)$
011	$2(1+\alpha^2)$	200	$2+\alpha$
012	$2(1+\alpha^3+\alpha^4)$	201	$2(2+\alpha)$
020	$2+\alpha$	202	$(2+\alpha)^2$
021	$2(2+\alpha^5)$	210	$2(2+\alpha^5)$
022	$(2+\alpha)(2+\alpha^3)$	211	$2(2+\alpha^5)(1+\alpha^2)$
100	2	212	$2(2+\alpha^5)(1+\alpha^3+\alpha^4)$
101	2^2	220	$(2+\alpha)(2+\alpha^3)$
102	$2(2+\alpha)$	221	$2(2+\alpha^3)(2+\alpha^5)$
110	$2(1+\alpha^2)$	222	$(2+\alpha)(2+\alpha^3)^2$
111	$2(1+\alpha^2)^2$		

Table 5

$c_0c_1c_2$	n_1	n_2	n_{11}	n_{12}	n_{21}	n_{22}	n_{121}	n_{122}	Right side of (3.12)
000	0	0	0	0	0	0	0	0	1
001	1	0	0	0	0	0	0	0	2
002	0	1	0	0	0	0	0	0	$2+\alpha$
010	1	0	0	0	0	0	0	0	2
011	2	0	1	0	0	0	0	0	$2(1+\alpha^2)$
012	1	1	0	1	0	0	0	0	$2(1+\alpha^3+\alpha^4)$
020	0	1	0	0	0	0	0	0	$2+\alpha$
021	1	1	0	0	1	0	0	0	$2(2+\alpha^5)$
022	0	2	0	0	0	1	0	0	$(2+\alpha)(2+\alpha^3)$
100	1	0	0	0	0	0	0	0	2
101	2	0	0	0	0	0	0	0	2^2
102	1	1	0	0	0	0	0	0	$2(2+\alpha)$

Table 5 (cont.)

$c_0c_1c_2$	n_1	n_2	n_{11}	n_{12}	n_{21}	n_{22}	n_{121}	n_{122}	Right side of (3.12)
110	2	0	1	0	0	0	0	0	$2(1+\alpha^2)$
111	3	0	2	0	0	0	0	0	$2(1+\alpha^2)^2$
112	2	1	1	1	0	0	0	0	$2(1+\alpha^2)(1+\alpha^3+\alpha^4)$
120	1	1	0	1	0	0	0	0	$2(1+\alpha^3+\alpha^4)$
121	2	1	0	1	1	0	1	0	$2^2(1+\alpha+\alpha^4)$
122	1	2	0	1	0	1	0	1	$2(2+\alpha)(1+\alpha^4+\alpha^5)$
200	0	1	0	0	0	0	0	0	$2+\alpha$
201	1	1	0	0	0	0	0	0	$2(2+\alpha)$
202	0	2	0	0	0	0	0	0	$(2+\alpha)^2$
210	1	1	0	0	1	0	0	0	$2(2+\alpha^5)$
211	2	1	1	0	1	0	0	0	$2(1+\alpha^2)(2+\alpha^5)$
212	1	2	0	1	1	0	0	0	$2(2+\alpha^5)(1+\alpha^3+\alpha^4)$
220	0	2	0	0	0	1	0	0	$(2+\alpha)(2+\alpha^3)$
221	1	2	0	0	1	1	0	0	$2(2+\alpha^3)(2+\alpha^5)$
222	0	3	0	0	0	2	0	0	$(2+\alpha)(2+\alpha^3)^2$

We now make the inductive hypothesis (IH) that the Proposition is true for all strings of lengths $2, 3, \ldots, k$, where $k \geq 3$. We consider the string $\mathbf{c} = c_0c_1\ldots c_k$ of length $k+1$. We set

$$\mathcal{B} = \{1, 2, 11, 12, 21, 22, 121, 122\},$$

and, for $B \in \mathcal{B}$, $n_B = n_B(\mathbf{c})$, $n'_B = n_B(\mathbf{c}')$, $n''_B = n_B(\mathbf{c}'')$. Recall that if $\mathbf{c} = c_0c_1\ldots c_k$ then $\mathbf{c}' = c_1\ldots c_k$ and $\mathbf{c}'' = (\mathbf{c}')'$. The information needed for the inductive step is provided in Table 6.

Table 6

c_0c_1	$n_B = n'_B$	$n_B = n'_B + 1$	$S(\mathbf{c};\alpha)/S(\mathbf{c}';\alpha)$
00	all B		1
01	all B		1
02	all B		1
10	all $B \neq 1$	1	2
11	all $B \neq 1, 11$	1, 11	$1+\alpha^2$
20	all $B \neq 2$	2	$2+\alpha$
21	all $B \neq 2, 21$	2, 21	$2+\alpha^5$
22	all $B \neq 2, 22$	2, 22	$2+\alpha^3$

$c_0c_1c_2$	$n_B = n''_B$	$n_B = n''_B + 1$	$S(\mathbf{c};\alpha)/S(\mathbf{c}'';\alpha)$
120	11, 21, 22, 121, 122	1, 2, 12	$2(1+\alpha^3+\alpha^4)$
121	11, 22, 122	1, 2, 12, 21, 121	$2(1+\alpha+\alpha^4)$
122	11, 21, 121	1, 2, 12, 22, 122	$2(1+\alpha^4+\alpha^5)$

We just do the case $c_0c_1c_2 = 120$ in detail. We have

$$\begin{aligned}
S(\mathbf{c};\alpha) &= 2(1+\alpha^3+\alpha^4)S(\mathbf{c}'';\alpha) && \text{(by Lemma 2)}\\
&= 2(1+\alpha^3+\alpha^4)2^{n_1''-n_{11}''}(1+\alpha^2)^{n_{11}''}\\
&\quad\times(2+\alpha)^{n_2''-n_{12}''-n_{21}''-n_{22}''+n_{121}''+n_{122}''}\\
&\quad\times(2+\alpha^5)^{n_{21}''-n_{121}''}(2+\alpha^3)^{n_{22}''-n_{122}''}\\
&\quad\times(1+\alpha^3+\alpha^4)^{n_{12}''-n_{121}''-n_{122}''}(1+\alpha+\alpha^4)^{n_{121}''}\\
&\quad\times(1+\alpha^4+\alpha^5)^{n_{122}''} && \text{(by (IH))}\\
&= 2^{n_1-n_{11}}(1+\alpha^2)^{n_{11}}(2+\alpha)^{n_2-n_{12}-n_{21}-n_{22}+n_{121}+n_{122}}\\
&\quad\times(2+\alpha^5)^{n_{21}-n_{121}}(2+\alpha^3)^{n_{22}-n_{122}}\\
&\quad\times(1+\alpha^3+\alpha^4)^{n_{12}-n_{121}-n_{122}}(1+\alpha+\alpha^4)^{n_{121}}(1+\alpha^4+\alpha^5)^{n_{122}},
\end{aligned}$$

from Table 6. This completes the inductive step and the Proposition follows by the principle of mathematical induction. ■

4. Evaluation of $N_n(t,9)$ $(3\nmid t)$. Let n be an integer with $n \geq 9$. Let $a_0a_1\ldots a_l$ be the 3-ary representation of n so that $l \geq 2$. In this section $n_1, n_2, n_{11}, \ldots$ refer to the string $a_0a_1\ldots a_l$. Set $\omega = e^{2\pi i/6}$ and $\beta = \omega^2 = e^{2\pi i/3}$. We note that $\omega = -\beta^2$. For $t = 1,2,4,5,7,8$ we have

$$\begin{aligned}
N_n(t,9) &= \sum_{\substack{r=0\\ \binom{n}{r}\equiv t \,(\mathrm{mod}\, 9)}}^{n} 1 = \sum_{\substack{r=0\\ \binom{n}{r}\equiv t\,(\mathrm{mod}\,9)\\ 3\nmid\binom{n}{r}}}^{n} 1 && \text{(as } 3\nmid t)\\
&= \sum_{\substack{r=0\\ \mathrm{ind}_2\binom{n}{r}\equiv \mathrm{ind}_2 t\,(\mathrm{mod}\,6)\\ 3\nmid\binom{n}{r}}}^{n} 1 = \frac{1}{6}\sum_{\substack{r=0\\ 3\nmid\binom{n}{r}}}^{n}\sum_{s=0}^{5}\omega^{s(\mathrm{ind}_2\binom{n}{r}-\mathrm{ind}_2 t)}\\
&= \frac{1}{6}\sum_{s=0}^{5}\omega^{-s\,\mathrm{ind}_2 t}\sum_{\substack{r=0\\ 3\nmid\binom{n}{r}}}^{n}\omega^{s\,\mathrm{ind}_2\binom{n}{r}}\\
&= \frac{1}{6}\sum_{s=0}^{5}\omega^{-s\,\mathrm{ind}_2 t}\sum_{\substack{b_0=0\\ b_0+b_13+\cdots+b_l3^l\leq a_0+a_13+\cdots+a_l3^l\\ 3\nmid\binom{a_0}{b_0}\binom{a_1}{b_1}\cdots\binom{a_l}{b_l}}}^{2}\cdots\sum_{b_l=0}^{2}\omega^{si(\mathbf{a},\mathbf{b})}\\
&&& \text{(by (1.3), (1.4), (3.1), (3.2))}
\end{aligned}$$

$$= \frac{1}{6}\sum_{s=0}^{5}\omega^{-s\,\mathrm{ind}_2 t}\sum_{b_0=0}^{a_0}\cdots\sum_{b_l=0}^{a_l}\omega^{si(\mathbf{a},\mathbf{b})}$$

$$= \frac{1}{6}\sum_{s=0}^{5}\omega^{-s\,\mathrm{ind}_2 t}S(\mathbf{a};\omega^s) \qquad \text{(by (3.3))}$$

$$\begin{aligned}= \frac{1}{6}\sum_{s=0}^{5}&\omega^{-s\,\mathrm{ind}_2 t}2^{n_1-n_{11}}(1+\omega^{2s})^{n_{11}}(2+\omega^s)^{n_2-n_{12}-n_{21}-n_{22}+n_{121}+n_{122}}\\ &\times(2+\omega^{5s})^{n_{21}-n_{121}}(2+\omega^{3s})^{n_{22}-n_{122}}(1+\omega^{3s}+\omega^{4s})^{n_{12}-n_{121}-n_{122}}\\ &\times(1+\omega^s+\omega^{4s})^{n_{121}}(1+\omega^{4s}+\omega^{5s})^{n_{122}},\end{aligned}$$

by the Proposition. The term in the sum with $s=0$ is

$$\begin{aligned}&2^{n_1-n_{11}}2^{n_{11}}\\ &\quad\times 3^{n_2-n_{12}-n_{21}-n_{22}+n_{121}+n_{122}}3^{n_{21}-n_{121}}3^{n_{22}-n_{122}}3^{n_{12}-n_{121}-n_{122}}3^{n_{121}}3^{n_{122}}\\ &=2^{n_1}3^{n_2}.\end{aligned}$$

The term with $s=1$ is

$$\begin{aligned}&\omega^{-\,\mathrm{ind}_2 t}2^{n_1-n_{11}}(1+\omega^2)^{n_{11}}(2+\omega)^{n_2-n_{12}-n_{21}-n_{22}+n_{121}+n_{122}}\\ &\qquad\times(2+\omega^5)^{n_{21}-n_{121}}(2+\omega^3)^{n_{22}-n_{122}}\\ &\qquad\times(1+\omega^3+\omega^4)^{n_{12}-n_{121}-n_{122}}(1+\omega+\omega^4)^{n_{121}}(1+\omega^4+\omega^5)^{n_{122}}\\ &\quad=(-1)^{\mathrm{ind}_2 t}\beta^{\mathrm{ind}_2 t}2^{n_1-n_{11}}(1+\beta)^{n_{11}}(2-\beta^2)^{n_2-n_{12}-n_{21}-n_{22}+n_{121}+n_{122}}\\ &\qquad\times(2-\beta)^{n_{21}-n_{121}}1^{n_{22}-n_{122}}\beta^{2n_{12}-2n_{121}-2n_{122}}1^{n_{121}}(1-\beta+\beta^2)^{n_{122}}\\ &\quad=(-1)^{\mathrm{ind}_2 t}\beta^{\mathrm{ind}_2 t}2^{n_1-n_{11}}(-\beta^2)^{n_{11}}(3+\beta)^{n_2-n_{12}-n_{21}-n_{22}+n_{121}+n_{122}}\\ &\qquad\times(2-\beta)^{n_{21}-n_{121}}\beta^{2n_{12}-2n_{121}-2n_{122}}(-2\beta)^{n_{122}}\\ &\quad=(-1)^{\mathrm{ind}_2 t+n_{11}+n_{122}}2^{n_1-n_{11}+n_{122}}\beta^{\mathrm{ind}_2 t+2n_{11}+2n_{12}-2n_{121}-n_{122}}\\ &\qquad\times(3+\beta)^{n_2-n_{12}-n_{21}-n_{22}+n_{121}+n_{122}}(2-\beta)^{n_{21}-n_{121}}\\ &\quad=(-1)^{\mathrm{ind}_2 t+n_{11}+n_{122}}2^{n_1-n_{11}+n_{122}}X.\end{aligned}$$

The term with $s=2$ is

$$\begin{aligned}&\omega^{-2\,\mathrm{ind}_2 t}2^{n_1-n_{11}}(1+\omega^4)^{n_{11}}(2+\omega^2)^{n_2-n_{12}-n_{21}-n_{22}+n_{121}+n_{122}}\\ &\quad\times(2+\omega^4)^{n_{21}-n_{121}}3^{n_{22}-n_{122}}\\ &\quad\times(2+\omega^2)^{n_{12}-n_{121}-n_{122}}(1+2\omega^2)^{n_{121}}(1+\omega^2+\omega^4)^{n_{122}}\\ &=\beta^{-\,\mathrm{ind}_2 t}2^{n_1-n_{11}}(1+\beta^2)^{n_{11}}(2+\beta)^{n_2-n_{12}-n_{21}-n_{22}+n_{121}+n_{122}}\\ &\quad\times(2+\beta^2)^{n_{12}-n_{121}}3^{n_{22}-n_{122}}\\ &\quad\times(2+\beta)^{n_{12}-n_{121}-n_{122}}(1+2\beta)^{n_{121}}(1+\beta+\beta^2)^{n_{122}}\end{aligned}$$

$$
\begin{aligned}
&= \beta^{-\operatorname{ind}_2 t} 2^{n_1-n_{11}} 3^{n_{22}-n_{122}} (-\beta)^{n_{11}} (2+\beta)^{n_2-n_{12}-n_{21}-n_{22}+n_{121}+n_{122}} \\
&\quad \times (1-\beta)^{n_{21}-n_{121}} (2+\beta)^{n_{12}-n_{121}-n_{122}} (1+2\beta)^{n_{121}} 0^{n_{122}} \\
&= 0^{n_{122}} (-1)^{n_{11}} \beta^{-\operatorname{ind}_2 t+n_{11}} 2^{n_1-n_{11}} 3^{n_{22}-n_{122}} \\
&\quad \times (2+\beta)^{n_2-n_{21}-n_{22}} (1-\beta)^{n_{21}-n_{121}} (1+2\beta)^{n_{121}} \\
&= 0^{n_{122}} (-1)^{n_{11}} 2^{n_1-n_{11}} 3^{n_{22}-n_{122}} Y.
\end{aligned}
$$

The term with $s = 3$ is

$$(-1)^{\operatorname{ind}_2 t} 2^{n_1-n_{11}} 2^{n_{11}} = (-1)^{\operatorname{ind}_2 t} 2^{n_1}.$$

The term with $s = 4$ is the complex conjugate of the term with $s = 2$, and the term with $s = 5$ is the complex conjugate of the term with $s = 1$. Hence

$$
\begin{aligned}
N_n(t,9) &= \frac{1}{6}\{2^{n_1}3^{n_2} + (-1)^{\operatorname{ind}_2 t+n_{11}+n_{122}} 2^{n_1-n_{11}+n_{122}} X \\
&\quad + 0^{n_{122}} (-1)^{n_{11}} 2^{n_1-n_{11}} 3^{n_{22}-n_{122}} Y \\
&\quad + (-1)^{\operatorname{ind}_2 t} 2^{n_1} \\
&\quad + 0^{n_{122}} (-1)^{n_{11}} 2^{n_1-n_{11}} 3^{n_{22}-n_{122}} \overline{Y} \\
&\quad + (-1)^{\operatorname{ind}_2 t+n_{11}+n_{122}} 2^{n_1-n_{11}+n_{122}} \overline{X}\} \\
&= \frac{1}{6}\{2^{n_1}3^{n_2} + (-1)^{\operatorname{ind}_2 t} 2^{n_1} \\
&\quad + (-1)^{\operatorname{ind}_2 t+n_{11}+n_{122}} 2^{n_1-n_{11}+n_{122}+1} \operatorname{Re}(X) \\
&\quad + 0^{n_{122}} (-1)^{n_{11}} 2^{n_1-n_{11}+1} 3^{n_{22}-n_{122}} \operatorname{Re}(Y)\},
\end{aligned}
$$

as asserted. ■

5. Evaluation of $N_n(t,9)$ $(3 \mid t)$. Let n be an integer with $n \geq 9$. We recall that the 3-ary representations of n, r and $n-r$ $(0 \leq r \leq n)$ are

$$
\begin{aligned}
n &= a_0 + a_1 3 + \ldots + a_l 3^l, && \text{each } a_i = 0,1,2, \\
r &= b_0 + b_1 3 + \ldots + b_l 3^l, && \text{each } b_i = 0,1,2, \\
n-r &= c_0 + c_1 3 + \ldots + c_l 3^l, && \text{each } c_i = 0,1,2.
\end{aligned}
$$

As $n \geq 9$ we have $l \geq 2$. We first consider $t = 3$ and $t = 6$. By Kummer's theorem (see (1.2)), we have

$$3 \,\|\, \binom{n}{r} \Leftrightarrow \text{there is a single carry when adding } r \text{ and } n-r \text{ in base 3.}$$

If this carry occurs in the jth place $(0 \leq j \leq l-1)$ then

$$
\begin{aligned}
b_j + c_j &= a_j + 3, \\
b_{j+1} + c_{j+1} &= a_{j+1} - 1, \\
b_i + c_i &= a_i \quad (i \neq j, j+1).
\end{aligned}
$$

Clearly

$$a_j \neq 2, \quad a_{j+1} \neq 0, \quad a_j < b_j \leq 2, \quad 0 \leq b_{j+1} < a_{j+1}.$$

Moreover, by Kazandzidis' theorem (see (1.5)), we have

$$\binom{n}{r} \equiv -3 \prod_{i=0}^{l} \frac{a_i!}{b_i! c_i!} \pmod 9,$$

that is,

$$\binom{n}{r} \equiv -3 \frac{a_j!}{b_j!(a_j+3-b_j)!} \cdot \frac{a_{j+1}!}{b_{j+1}!(a_{j+1}-1-b_{j+1})!} \prod_{\substack{i=0 \\ i \neq j,j+1}}^{l} \binom{a_i}{b_i} \pmod 9.$$

Set

$$f(a_j, b_j, a_{j+1}, b_{j+1}) = \frac{b_j!(a_j+3-b_j)!}{a_j!} \cdot \frac{b_{j+1}!(a_{j+1}-1-b_{j+1})!}{a_{j+1}!},$$

so that

$$\binom{n}{r} \equiv t \pmod 9 \Leftrightarrow \prod_{\substack{i=0 \\ i \neq j,j+1}}^{l} \binom{a_i}{b_i} \equiv -\frac{t}{3} f(a_j, b_j, a_{j+1}, b_{j+1}) \pmod 3.$$

Hence we have

$$N_n(t,9) = \sum_{\substack{r=0 \\ \binom{n}{r} \equiv t \pmod 9}}^{n} 1 = \sum_{\substack{r=0 \\ 3 \| \binom{n}{r} \\ \binom{n}{r} \equiv t \pmod 9}}^{n} 1 \qquad \text{(as } t = 3, 6\text{)}$$

$$= \sum_{\substack{r=0 \\ c(n,r)=1 \\ \binom{n}{r} \equiv t \pmod 9}}^{n} 1 = \sum_{j=0}^{l-1} \sum_{\substack{r=0 \\ c(n,r)=1 \\ \text{carry in } j\text{th place} \\ \binom{n}{r} \equiv t \bmod 9}}^{n} 1$$

$$= \sum_{j=0}^{l-1} \sum_{b_0=0}^{a_0} \cdots \sum_{b_{j-1}=0}^{a_{j-1}} \sum_{b_j=a_j+1}^{2} \sum_{b_{j+1}=0}^{a_{j+1}-1} \sum_{b_{j+2}=0}^{a_{j+2}} \cdots \sum_{\substack{b_l=0 \\ \prod_{\substack{i=0 \\ i \neq j,j+1}}^{l} \binom{a_i}{b_i} \equiv -\frac{t}{3} f(a_j,b_j,a_{j+1},b_{j+1}) \pmod 3}}^{a_l} 1.$$

Now

$$\sum_{\substack{b_0=0 \\ \prod_{\substack{i=0\\ i\neq j,j+1}}^{l} \binom{a_i}{b_i} \equiv -\frac{t}{3}f(a_j,b_j,a_{j+1},b_{j+1}) \pmod 3}}^{a_0} \cdots \sum_{b_{j-1}=0}^{a_{j-1}} \sum_{b_{j+1}=0}^{a_{j+1}} \cdots \sum_{b_l=0}^{a_l} 1$$

$$= \sum_{\substack{s_0=0 \\ s_0+s_1 3+\dots+s_l 3^l \le a_0+a_1 3+\dots+a_{j-1}3^{j-1}+a_{j+2}3^{j+2}+\dots+a_l 3^l \\ =n-a_j 3^j-a_{j+1}3^{j+1} \\ \binom{a_0}{s_0}\cdots\binom{a_{j-1}}{s_{j-1}}\binom{0}{s_j}\binom{0}{s_{j+1}}\binom{a_{j+2}}{s_{j+2}}\cdots\binom{a_l}{s_l} \equiv -\frac{t}{3}f(a_j,b_j,a_{j+1},b_{j+1}) \pmod 3}}^{2} \cdots \sum_{s_l=0}^{2} 1$$

$$= \sum_{\substack{s=0 \\ \binom{n-a_j 3^j-a_{j+1}3^{j+1}}{s} \equiv -\frac{t}{3}f(a_j,b_j,a_{j+1},b_{j+1}) \pmod 3}}^{n-a_j 3^j-a_{j+1}3^{j+1}} 1$$

$$= N_{n-a_j 3^j-a_{j+1}3^{j+1}}\left(-\frac{t}{3}f(a_j,b_j,a_{j+1},b_{j+1}),3\right).$$

Hence

$$(5.1)\qquad N_n(t,9) = \sum_{j=0}^{l-1} \sum_{b_j=a_j+1}^{2} \sum_{b_{j+1}=0}^{a_{j+1}-1} N_{n-a_j 3^j-a_{j+1}3^{j+1}}\left(-\frac{t}{3}f(a_j,b_j,a_{j+1},b_{j+1}),3\right).$$

We recall from the introduction that for $k=1$ and 2,

$$N_n(k,3) = \frac{1}{2}2^{n_1(n)}(3^{n_2(n)}-(-1)^k).$$

In order to use this formula in (5.1) we consider four cases according as $a_j a_{j+1} = 01, 02, 11$ or 12.

Case (i): $a_j a_{j+1} = 01$ (so $b_j = 1$ or 2, $b_{j+1}=0$). Here

$$\begin{aligned} n_1(n-a_j 3^j - a_{j+1}3^{j+1}) &= n_1 - 1,\\ n_2(n-a_j 3^j - a_{j+1}3^{j+1}) &= n_2,\\ f(a_j,b_j,a_{j+1},b_{j+1}) &= 2, \end{aligned}$$

so that

$$N_{n-a_j 3^j-a_{j+1}3^{j+1}}\left(-\frac{t}{3}f(a_j,b_j,a_{j+1},b_{j+1}),3\right) = \frac{1}{2}2^{n_1-1}(3^{n_2}-(-1)^{t/3}).$$

Case (ii): $a_j a_{j+1} = 02$ (so $b_j = 1$ or 2, $b_{j+1} = 0$ or 1). Here

$$n_1(n - a_j 3^j - a_{j+1}3^{j+1}) = n_1,$$
$$n_2(n - a_j 3^j - a_{j+1}3^{j+1}) = n_2 - 1,$$
$$f(a_j, b_j, a_{j+1}, b_{j+1}) = 1,$$

so that

$$N_{n-a_j3^j-a_{j+1}3^{j+1}}\left(-\frac{t}{3}f(a_j, b_j, a_{j+1}, b_{j+1}), 3\right) = \frac{1}{2}2^{n_1}(3^{n_2-1} - (-1)^{3-t/3}) = \frac{1}{2}2^{n_1}(3^{n_2-1} + (-1)^{t/3}).$$

Case (iii): $a_j a_{j+1} = 11$ (so $b_j = 2, b_{j+1} = 0$). Here

$$n_1(n - a_j 3^j - a_{j+1}3^{j+1}) = n_1 - 2,$$
$$n_2(n - a_j 3^j - a_{j+1}3^{j+1}) = n_2,$$
$$f(a_j, b_j, a_{j+1}, b_{j+1}) = 4,$$

so that

$$N_{n-a_j3^j-a_{j+1}3^{j+1}}\left(-\frac{t}{3}f(a_j, b_j, a_{j+1}, b_{j+1}), 3\right) = \frac{1}{2}2^{n_1-2}(3^{n_2} - (-1)^{3-t/3}) = \frac{1}{2}2^{n_1-2}(3^{n_2} + (-1)^{t/3}).$$

Case (iv): $a_j a_{j+1} = 12$ (so $b_j = 2$, $b_{j+1} = 0$ or 1). Here

$$n_1(n - a_j 3^j - a_{j+1}3^{j+1}) = n_1 - 1,$$
$$n_2(n - a_j 3^j - a_{j+1}3^{j+1}) = n_2 - 1,$$
$$f(a_j, b_j, a_{j+1}, b_{j+1}) = 2,$$

so that

$$N_{n-a_j3^j-a_{j+1}3^{j+1}}\left(-\frac{t}{3}f(a_j, b_j, a_{j+1}, b_{j+1}), 3\right) = \frac{1}{2}2^{n_1-1}(3^{n_2-1} - (-1)^{t/3}).$$

Hence, using these evaluations in (5.1), we obtain

$$\begin{aligned} N_n(t, 9) = &\sum_{\substack{j=0 \\ a_ja_{j+1}=01}}^{l-1} 2 \cdot \frac{1}{2}2^{n_1-1}(3^{n_2} - (-1)^{t/3}) \\ &+ \sum_{\substack{j=0 \\ a_ja_{j+1}=02}}^{l-1} 2^2 \cdot \frac{1}{2}2^{n_1}(3^{n_2-1} + (-1)^{t/3}) \\ &+ \sum_{\substack{j=0 \\ a_ja_{j+1}=11}}^{l-1} \frac{1}{2}2^{n_1-2}(3^{n_2} + (-1)^{t/3}) \end{aligned}$$

$$+ \sum_{\substack{j=0 \\ a_j a_{j+1}=12}}^{l-1} 2 \cdot \frac{1}{2} 2^{n_1-1}(3^{n_2-1} - (-1)^{t/3})$$

$$= n_{01} 2^{n_1-1}(3^{n_2} - (-1)^t) + n_{02} 2^{n_1+1}(3^{n_2-1} + (-1)^t)$$
$$+ n_{11} 2^{n_1-3}(3^{n_2} + (-1)^t) + n_{12} 2^{n_1-1}(3^{n_2-1} - (-1)^t),$$

which is the asserted formula.

Finally, we treat the case $t = 0$. We have

$$N_n(3,9) + N_n(6,9)$$
$$= n_{01} 2^{n_1} 3^{n_2} + n_{02} 2^{n_1+2} 3^{n_2-1} + n_{11} 2^{n_1-2} 3^{n_2} + n_{12} 2^{n_1} 3^{n_2-1},$$

so that

$$\begin{aligned} N_n(0,9) &= N_n(0,3) - (N_n(3,9) + N_n(6,9)) \\ &= n + 1 - 2^{n_1} 3^{n_2} - n_{01} 2^{n_1} 3^{n_2} - n_{02} 2^{n_1+2} 3^{n_2-1} \\ &\quad - n_{11} 2^{n_1-2} 3^{n_2} - n_{12} 2^{n_1} 3^{n_2-1}. \ \blacksquare \end{aligned}$$

6. Concluding comments. We remark that our formulae for $N_n(t,9)$ when $3 \nmid t$ are consistent with the following result of Webb [10, Theorem 3].

If p is a prime and $p \nmid t$ then $N_n(t,p^2)$ depends only on t and the number of occurrences of each block of nonzero digits in the base p expansion of n and not on where they occur nor on the number of zeros in the expansion.

The formulae for $N_n(t,p^2)$ $(p \nmid t)$ for $p = 2$ and $p = 3$ suggest that perhaps only blocks of length at most p are needed.

When p is a prime and $p \,\|\, t$ we have shown (in a paper submitted for publication) that $N_n(t,p^2)$ depends only on t, $n_1, \ldots, n_{p-1}$ and n_{ij} ($i = 0, 1, \ldots, p-2$; $j = 1, \ldots, p-1$). Our formulae for $N_n(3,9)$ and $N_n(6,9)$, as well as that of Davis and Webb [2] for $N_n(2,4)$, are in conformity with this result. Compare this result with Webb's comment [10, sentence preceding first example on p. 278].

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