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ON A FORMULA OF HOWARD

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Abstract: Let n be a positive integer. The nth row of Pascal's triangle consists of the integers $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$. In 1971 F.T. Howard [6] showed that the number $f(n,m)$ of entries in the nth row of Pascal's triangle which are exactly divisible by 2^m, where m is a positive integer, is given by

$$f(n,m) = \sum_{w=1}^{m} a_{mw} 2^{n_1+m-2w},$$

where n_1 denotes the number of 1's in the binary representation of n and the a_{mw} are nonnegative integers given by a complicated combinatorial prescription [6, p. 237]. In this paper we show that

$$f(n,m) = \sum_{w=1}^{m} b_{mw} 2^{n_1+m-2w},$$

where the nonnegative integers b_{mw} are given by a simple formula (Theorem 1). It is conjectured that $a_{mw} = b_{mw}$ for all positive integer m and w with $1 \leq w \leq m$. This is verified for $m = 1, 2, 3, 4$. Finally it is shown (Theorem 3) that for fixed m

$$f(n,m)/2^{n_1} = n_{01}^m/2^m m! + O_m\left(n_{01}^{m-1}\right),$$

as $n_{01} \to +\infty$.

Key Words: Binomial coefficients

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0 NOTATION

We denote the binary representation of the positive integer n by $a_0a_1\ldots a_\ell$, that is

$$n = a_0 + a_1 2 + \cdots + a_\ell 2^\ell, \quad a_i = 0 \text{ or } 1 \quad (i = 0,1,\ldots,\ell-1), \quad a_\ell = 1.$$

If S and A are finite nonempty strings of 0's and 1's the number of occurrences of the string S in the string A is denoted by $n_S(A)$. Thus, for example, $n_{010}(00110) = 0$, $n_{11}(11101011) = 3$, $n_{001}(01) = 0$. In particular if A is the string $a_0a_1\ldots a_\ell$ we set

$$n_S := n_S(A) = n_S(a_0a_1\ldots a_\ell).$$

Thus if $n = 670$ then $a_0a_1\ldots a_\ell = 0111100101$ and $n_0 = 4$, $n_1 = 6$, $n_{11} = 3$, $n_{01} = 3$, $n_{101} = 1$. We denote the empty string by $\emptyset$ and define

$$n_S(\emptyset) = 0, \quad \text{if } S \neq \emptyset.$$

We do not need to define $n_\emptyset(A)$. The length of the string S is denoted by $|S|$, so that $|0011| = 4$, $|0| = 1$, $|\emptyset| = 0$.

1 INTRODUCTION

The nth row of Pascal's triangle is

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}. \tag{1.1}$$

Let m be a positive integer. We denote by $f(n,m)$ the number of entries in (1.1) which are exactly divisible by 2^m. Howard [6: eqn. (2.2)] has shown that

$$f(n,m) = \sum_{w=1}^{m} a_{mw} 2^{n_1+m-2w} \tag{1.2}$$

for certain integers a_{mw}. He also gave a complicated combinatorial interpretation of the integers a_{mw} [6: p. 237]. Using this viewpoint he evaluated $f(n,m)$ explicitly for $m = 1,2,3$ and 4 [6: pp. 237-238]. In this paper we show that $f(n,m)$ can be expressed in the form

$$f(n,m) = \sum_{w=1}^{m} b_{mw} 2^{n_1+m-2w},$$

where the integers b_{mw} are given by a compact formula (see (1.4) below). In order to do this we introduce a certain sum (Definition 1) which counts the number of occurrences of certain patterns of 0's and 1's in the string $a_0a_1\ldots a_\ell$.

DEFINITION 1 Let n be a positive integer. Let $a_0a_1\dots a_\ell$ be the binary representation of n. Let u be a positive integer. For $i=1,\dots,u$ let B_i be a string of 0's and 1's of length k_i-1. (If $k_i=1$ then B_i is the empty string $\emptyset$.) We define

$$S_n(B_1,\dots,B_u) := \sum_{\substack{i_1,\dots,i_u \\ 0\le i_1<i_1+k_1<i_2<i_2+k_2<\dots<i_u<i_u+k_u\le \ell \\ a_{i_1}a_{i_1+1}\cdots a_{i_1+k_1}=0B_11 \\ \cdots \\ a_{i_u}a_{i_u+1}\cdots a_{i_u+k_u}=0B_u1}} 1 .$$

We remark that if $k_1+\cdots+k_u>\ell+1-u$ then the above sum is empty and we have $S_n(B_1,\dots,B_u)=0$. Properties of $S_n(B_1,\dots,B_u)$ are given in Section 2. In Section 3 two lemmas are proved and these are used in Section 4 to prove the following theorem.

THEOREM 1 Let n and m be positive integers. Let $a_0a_1\dots a_\ell$ be the binary representation of n. Then the number $f(n,m)$ of binomial coefficeints $\binom{n}{r}$ $(r=0,1,\dots,n)$, which are exactly divisible by 2^m, is given by

$$f(n,m)=\sum_{w=1}^{m} b_{mw}2^{n_1+m-2w}, \tag{1.3}$$

where

$$b_{mw}=\sum_{u=1}^{w}\ \sum_{\substack{B_1,\dots,B_u \\ n_0(B_1)+\cdots+n_0(B_u)=m-w \\ n_1(B_1)+\cdots+n_1(B_u)=w-u}} S_n(B_1,\dots,B_u), \tag{1.4}$$

and

$B_i(i=1,2,\dots,u)$ is a finite string of 0's and 1's,
$n_0(B_i)(i=1,\dots,u)$ denotes the number of 0's in the string B_i,
$n_1(B_i)(i=1,\dots,u)$ denotes the number of 1's in the string B_i,
the inner sum in (1.4) is taken over all possible strings
(including the empty string) $B_1,\dots,B_u$ such that
$n_0(B_1)+\cdots+n_0(B_u)=m-w$,
$n_1(B_1)+\cdots+n_1(B_u)=w-u$,
n_1 denotes the number of 1's in $a_0a_1\dots a_\ell$.

In Section 5 we determine the integers $b_{m1},\dots,b_{mm}$ for $m=1,2,3$ and 4.

THEOREM 2

$$\begin{aligned} b_{11} &= n_{01}, \\ b_{21} &= n_{001}, \end{aligned}$$

$$
\begin{aligned}
b_{22} &= n_{011} + \binom{n_{01}}{2}, \\
b_{31} &= n_{0001}, \\
b_{32} &= n_{0011} + n_{0101} + n_{001}(n_{01} - 1), \\
b_{33} &= n_{0111} + n_{011}(n_{01} - 1) + \binom{n_{01}}{3}, \\
b_{41} &= n_{00001}, \\
b_{42} &= n_{01001} + n_{00101} + n_{00011} + n_{0001}(n_{01} - 1) + \binom{n_{001}}{2}, \\
b_{43} &= n_{00111} + n_{01011} + n_{01101} + n_{0011}(n_{01} - 1) + n_{0101}(n_{01} - 2) \\
&\quad + n_{001}n_{011} - n_{0011} + n_{001}\binom{n_{01} - 1}{2}, \\
b_{44} &= n_{01111} + n_{0111}(n_{01} - 1) + \binom{n_{011}}{2} + n_{011}\binom{n_{01} - 1}{2} + \binom{n_{01}}{4}.
\end{aligned}
$$

Using the above formulae for $b_{11}, b_{21}, b_{31}, \ldots, b_{44}$ in Theorem 1, we obtain the following formulae for $f(n,1), f(n,2), f(n,3)$ and $f(n,4)$.

$$f(n,1) = b_{11}2^{n_1-1} = n_{01}2^{n_1-1}.$$

This result is a special case of a result due to Carlitz [1: eqn. (2.5)], see also Howard [5], [6: eqn. (2.4)], Davis and Webb [2], Huard, Spearman and Williams [7].

$$f(n,2) = b_{21}2^{n_1} + b_{22}2^{n_1-2} = n_{001}2^{n_1} + \left(n_{011} + \binom{n_{01}}{2}\right)2^{n_1-2}.$$

This result is due to Howard [6: eqn. (2.5)], see also Huard, Spearman and Williams [7].

$$
\begin{aligned}
f(n,3) &= b_{31}2^{n_1+1} + b_{32}2^{n_1-1} + b_{33}2^{n_1-3} \\
&= n_{0001}2^{n_1+1} + (n_{0011} + n_{0101} + n_{001}(n_{01} - 1))\,2^{n_1-1} \\
&\quad + \left(n_{0111} + n_{011}(n_{01} - 1) + \binom{n_{01}}{3}\right)2^{n_1-3}.
\end{aligned}
$$

This result is due to Howard [6: eqn. (2.6)]. It was also discovered independently by K. Hardy and K.S. Williams (unpublished).

$$
\begin{aligned}
f(n,4) &= b_{41}2^{n_1+2} + b_{42}2^{n_1} + b_{43}2^{n_1-2} + b_{44}2^{n_1-4} \\
&= n_{00001}2^{n_1+2} \\
&\quad + \left(n_{01001} + n_{00101} + n_{00011} + n_{0001}(n_{01} - 1) + \binom{n_{001}}{2}\right)2^{n_1}
\end{aligned}
$$

$$+\Bigg(n_{00111}+n_{01011}+n_{01101}+n_{0011}(n_{01}-1)$$

$$+n_{0101}(n_{01}-2)+n_{001}n_{011}-n_{0011}+n_{001}\left(\begin{array}{c} n_{01}-1 \\ 2 \end{array}\right)\Bigg)2^{n_1-2}$$

$$+\Bigg(n_{01111}+n_{0111}(n_{01}-1)+\left(\begin{array}{c} n_{011} \\ 2 \end{array}\right)$$

$$+n_{011}\left(\begin{array}{c} n_{01}-1 \\ 2 \end{array}\right)+\left(\begin{array}{c} n_{01} \\ 4 \end{array}\right)\Bigg)2^{n_1-4}.$$

This result is due to Howard [6: eqn. (2.7)]. For the convenience of the reader we note the correspondence between Howard's nonstandard notation and the standard notation used in this paper: $q_1 = n_{01}$, $q_2 = n_{001}$, $q_3 = n_{0001}$, $q_4 = n_{00001}$, $q_5 = n_{011}$, $q_6 = n_{011} - n_{0011}$, $q_7 = n_{0011}$, $q_8 = n_{00011}$, $q_9 = n_{0101}$, $q_{10} = n_{00101}$, $q_{11} = n_{0111}$, $q_{12} = n_{00111}$, $q_{13} = n_{01011}$, $q_{14} = n_{01101}$, $q_{15} = n_{01111}$, $q_{16} = n_{01001}$. Theorem 2 shows that Howard's a_{mw} and our b_{mw} agree for $m = 1, 2, 3, 4$ and $1 \le w \le m$. It is conjectured that $a_{mw} = b_{mw}$ for all m and w satisfying $1 \le w \le m$. Finally in Section 6 we obtain the following asymptotic result as a consequence of Theorem 1.

THEOREM 3 Let m be a fixed positive integer. Then

$$f(n,m)\o 2^{n_1} = n_{01}^m \o 2^m m! + O_m\left(n_{01}^{m-1}\right), \text{ as } n_{01} \to +\infty.$$

2 PROPERTIES OF $S_n(B_1, \cdots, B_u)$

If $B_i = \emptyset$ for $i = 1, \ldots, u$, then $S_n(B_1, \ldots, B_u)$ is the sum

$$\sum_{\substack{i_1,\ldots,i_u \\ 0\le i_1<i_1+1<i_2<i_2+1<\ldots<i_u<i_u+1\le \ell \\ a_{i_1}a_{i_1+1}=a_{i_2}a_{i_2+1}=\ldots=a_{i_u}a_{i_u+1}=01}} 1 \quad .$$

This sum counts the number of ways of choosing u disjoint strings 01 from the string $a_0a_1 \ldots a_\ell$ without regard to order. This number is $\left(\begin{array}{c} n_{01} \\ u \end{array}\right)$ so we have

PROPERTY 1 $\quad S_n(\emptyset, \ldots, \emptyset) = \left(\begin{array}{c} n_{01} \\ u \end{array}\right).$

If each string $B_i = 0$ $(i = 1, \ldots, u)$ then the sum $S_n(B_1, \ldots, B_u)$ is

$$\sum_{\substack{i_1,\ldots,i_u \\ 0\le i_1<i_1+2<i_2<i_2+2<\ldots<i_u<i_u+2\le \ell \\ a_{i_1}a_{i_1+1}a_{i_1+2}=a_{i_2}a_{i_2+1}a_{i_2+2}=\ldots=a_{i_u}a_{i_u+1}a_{i_u+2}=001}} 1 \quad .$$

This sum counts the number of ways of choosing u disjoint strings 001 from $a_0 a_1 \dots a_\ell$ without regard to order. This number is $\binom{n_{001}}{u}$ so we have

PROPERTY 2 $S_n(0,\dots,0) = \binom{n_{001}}{u}$. In exactly the same way we have

PROPERTY 3 $S_n(1,\dots,1) = \binom{n_{011}}{u}$. Another case when $S_n(B_1,\dots,B_u)$ is easily evaluated occurs when $u = 1$. In this case $S_n(B_1,\dots,B_u)$ is

$$\sum_{\substack{i_1 \\ 0 \le i_1 < i_1 + k_1 \le \ell \\ a_{i_1} a_{i_1+1} \cdots a_{i_1+k_1} = 0B_1 1}} 1 \qquad = n_{0B_1 1}.$$

PROPERTY 4 $S_n(B) = n_{0B1}$. The evaluation of $S_n(B_1,\dots,B_u)$ becomes more difficult when some of the B_i are different. In fact it seems quite difficult to evaluate a sum such as

$$S_n(\emptyset, 0) = \sum_{\substack{i_1, i_2 \\ 0 \le i_1 < i_1+1 < i_2 < i_2+2 \le \ell \\ a_{i_1} a_{i_1+1} = 01 \\ a_{i_2} a_{i_2+1} a_{i_2+2} = 001}} 1$$

explicitly. Fortunately for our purposes in applying Theorem 1 it suffices to evaluate

$$\text{(2.1)} \qquad \sum_{\substack{\text{distinct permutations} \\ \text{of } B_1,\dots,B_u}} S_n(B_1,\dots,B_u).$$

However when evaluating (2.1) the problem of the strings $0B_i1$ ($i = 1,\dots,u$) overlapping arises and must be taken into consideration. This is illustrated in the remaining properties. We consider the case $u = 2$, $B_1 = \emptyset$, $B_2 = 0$. Here we have

$$\begin{aligned} &S_n(B_1,B_2) + S_n(B_2,B_1) \\ = &\sum_{\substack{0 \le i_1 < i_1+1 < i_2 < i_2+2 \le \ell \\ a_{i_1} a_{i_1+1} = 01 \\ a_{i_2} a_{i_2+1} a_{i_2+2} = 001}} 1 + \sum_{\substack{0 \le i_1 < i_1+2 < i_2 < i_2+1 \le \ell \\ a_{i_1} a_{i_1+1} a_{i_1+2} = 001 \\ a_{i_2} a_{i_2+1} = 01}} 1 . \end{aligned}$$

The combined sum counts the number of ways of choosing nonoverlapping strings 01 and 001 from $a_0 a_1 \dots a_\ell$. Since there are exactly n_{001} overlapping pairs of strings 01 and 001, the combined sum is $n_{01} n_{001} - n_{001}$.

PROPERTY 5 $S_n(\emptyset, 0) + S_n(0, \emptyset) = n_{01} n_{001} - n_{001}$.

Similarly we obtain

PROPERTY 6 $S_n(\emptyset,1)+S_n(1,\emptyset)=n_{01}n_{011}-n_{011}$.

More generally when 0 and 1 in Properties 5 and 6 are replaced by an arbitrary string B, we have

PROPERTY 7 $S_n(\emptyset,B)+S_n(B,\emptyset)=n_{01}n_{0B1}-n_{01}(0B1)n_{0B1}$.

We remark that when $B=\emptyset$, Property 7 gives $2S_n(\emptyset,\emptyset)=n_{01}^2-n_{01}$, that is, $S_n(\emptyset,\emptyset)=\binom{n_{01}}{2}$ in agreement with Property 1.

Next we determine $S_n(0,1)+S_n(1,0)$. This sum is clearly $n_{001}n_{011}-k$, where k is the number of overlapping pairs 001, 011 in the string $a_0a_1\dots a_\ell$. Since these pairs can only overlap in the string 0011, and each such string contributes exactly one overlap, we have $k=n_{0011}$.

PROPERTY 8 $S_n(0,1)+S_n(0,1)=n_{001}n_{011}-n_{0011}$.

Finally we evaluate $S_n(i,\emptyset,\emptyset)+S_n(\emptyset,i,\emptyset)+S_n(\emptyset,\emptyset,i)$ for $i=0$ and 1.

PROPERTY 9 For $i=0$ and 1

$$S_n(i,\emptyset,\emptyset)+S_n(\emptyset,i,\emptyset)+S_n(\emptyset,\emptyset,i)=n_{0i1}\binom{n_{01}-1}{2}.$$

Proof We just treat the case $i=1$ as the case $i=0$ can be handled in a similar manner. From Definition 1 we have

$$\begin{aligned}
&S_n(1,\emptyset,\emptyset)+S_n(\emptyset,1,\emptyset)+S_n(\emptyset,\emptyset,1)\\
&=\sum_{\substack{i_1,i_2,i_3\\ 0\le i_1<i_1+2<i_2<i_2+1<i_3<i_3+1\le\ell\\ a_{i_1}a_{i_1+1}a_{i_1+2}=011\\ a_{i_2}a_{i_2+1}=01\\ a_{i_3}a_{i_3+1}=01}}1
\quad+\sum_{\substack{i_1,i_2,i_3\\ 0\le i_1<i_1+1<i_2<i_2+2<i_3<i_3+1\le\ell\\ a_{i_1}a_{i_1+1}=01\\ a_{i_2}a_{i_2+1}a_{i_2+2}=011\\ a_{i_3}a_{i_3+1}=01}}1
\quad+\sum_{\substack{i_1,i_2,i_3\\ 0\le i_1<i_1+1<i_2<i_2+1<i_3<i_3+2\le\ell\\ a_{i_1}a_{i_1+1}=01\\ a_{i_2}a_{i_2+1}=01\\ a_{i_3}a_{i_3+1}a_{i_3+2}=011}}1\\
&=\frac{1}{2}\sum_{\substack{i,j,k=0\\ i\ne j-2,j-1,j,j+1\\ i\ne k-2,k-1,k,k+1\\ j\ne k-1,k,k+1\\ a_ia_{i+1}a_{i+2}=011\\ a_ja_{j+1}=01\\ a_ka_{k+1}=01}}^{\ell-2,\ell-1,\ell-1}1
=\frac{1}{2}\sum_{\substack{i,j,k=0\\ i\ne j,\,i\ne k,\,j\ne k\\ a_ia_{i+1}a_{i+2}=011\\ a_ja_{j+1}=01\\ a_ka_{k+1}=01}}^{\ell-2,\ell-1,\ell-1}1\,,
\end{aligned}$$

since

$$\begin{array}{rll} a_{i+2}=1\neq 0=a_j=a_k & \text{implies} & i\neq j-2, i\neq k-2,\\ a_{i+1}=1\neq 0=a_j=a_k & \text{implies} & i\neq j-1, i\neq k-1,\\ a_i=0\neq 1=a_{j+1}=a_{k+1} & \text{implies} & i\neq j+1, i\neq k+1,\\ a_{j+1}=1\neq 0=a_k & \text{implies} & j\neq k-1,\\ a_j=0\neq 1=a_{k+1} & \text{implies} & j\neq k+1. \end{array}$$

Let $(*)$ indicate the conditions

$$a_ia_{i+1}a_{i+2}=011,\ a_ja_{j+1}=01,\ a_ka_{k+1}=01.$$

By the inclusion-exclusion principle we have

$$\begin{aligned} &\sum_{\substack{i,j,k=0\\ i\neq j\\ i\neq k\\ j\neq k}}^{\ell-2,\ell-1,\ell-1}{}^{*}\,1\\ =\;& \sum_{i,j,k=0}^{\ell-2,\ell-1,\ell-1}{}^{*}\,1 - \sum_{\substack{i,j,k=0\\ i=j}}^{\ell-2,\ell-1,\ell-1}{}^{*}\,1 - \sum_{\substack{i,j,k=0\\ i=k}}^{\ell-2,\ell-1,\ell-1}{}^{*}\,1 - \sum_{\substack{i,j,k=0\\ j=k}}^{\ell-2,\ell-1,\ell-1}{}^{*}\,1\\ &+ \sum_{\substack{i,j,k=0\\ i=j,i=k}}^{\ell-2,\ell-1,\ell-1}{}^{*}\,1 + \sum_{\substack{i,j,k=0\\ i=j,j=k}}^{\ell-2,\ell-1,\ell-1}{}^{*}\,1 + \sum_{\substack{i,j,k=0\\ i=k,j=k}}^{\ell-2,\ell-1,\ell-1}{}^{*}\,1 - \sum_{\substack{i,j,k=0\\ i=j=k}}^{\ell-2,\ell-1,\ell-1}{}^{*}\,1\\ =\;& n_{011}n_{01}^2 - n_{011}n_{01} - n_{011}n_{01} - n_{011}n_{01} + n_{011} + n_{011} + n_{011} - n_{011}\\ =\;& n_{011}(n_{01}-1)(n_{01}-2). \end{aligned}$$

3 TWO LEMMAS

In this section we prove two lemmas which will be used in the proof of Theorem 1.

Lemma 1 Let n be a positive integer. Let $a_0a_1\ldots a_\ell$ be the binary representation of n. Let r be an integer with $0\le r\le n$. Let $b_0b_1\ldots b_\ell$ be the binary representation of r. Let $c_0c_1\ldots c_\ell$ be the binary representation of $n-r$. When adding $b_0b_1\ldots b_\ell$ to $c_0c_1\ldots c_\ell$ in base 2 to obtain $a_0a_1\ldots a_\ell$ suppose that a carry occurs when adding b_j and c_j for $j=i, i+1,\ldots,i+k-1$, where i and k are integers with $0\le i<i+k\le \ell$, no carry occurs when adding b_{i+k} and c_{i+k}, and if $i\neq 0$ no carry occurs when adding b_{i-1} and c_{i-1}. Then

(i) $a_i=0$,

(ii) $a_{i+k}=1$,

(iii) *the total number of possibilities for* $b_ib_{i+1}\ldots b_{i+k-1}$ is $2^{k-1-n_1(a_i\ldots a_{i+k-1})}$.

Proof (i) When adding b_i and c_i in base 2, as there is no carry coming from the addition of b_{i-1} and c_{i-1} if $i \geq 1$, the only way we can obtain a carry is with $b_i = c_i = 1$, so that $a_i = 0$. (ii) When adding b_{i+k} and c_{i+k} in base 2, as there is a carry coming from the addition of b_{i+k-1} and c_{i+k-1}, the only way that no carry can occur is with $b_{i+k} = c_{i+k} = 0$, so that $a_{i+k} = 1$. (iii) First we treat the case $k \geq 2$. We consider the addition of b_j and c_j for $j = i+1, \ldots, i+k-1$ in the addition of $b_0b_1\ldots b_\ell$ and $c_0c_1\ldots c_\ell$ in base 2 to obtain $a_0a_1\ldots a_\ell$. As there is a carry coming in and a carry going out, we must have $1 + b_j + c_j = a_j + 2$, so that $b_j + c_j = a_j + 1$. If $a_j = 0$ then $(b_j, c_j) = (0,1)$ or $(1,0)$. If $a_j = 1$ then $(b_j, c_j) = (1,1)$. Hence the number of possibilities for b_j is $2^{1-n_1(a_j)}$. Thus the number of possibilities for $b_ib_{i+1}\ldots b_{i+k-1}$ is

$$1 \times 2^{1-n_1(a_{i+1})} \times \cdots \times 2^{1-n_1(a_{i+k-1})} = 2^{k-1-n_1(a_{i+1}\ldots a_{i+k-1})} = 2^{k-1-n_1(a_ia_{i+1}\ldots a_{i+k-1})},$$

as $b_i = 1$ and $a_i = 0$ by part (i). Secondly we treat the case $k = 1$. Here the addition

$$\begin{array}{c} b_ib_{i+1} \\ \underline{c_ic_{i+1}} \\ a_ia_{i+1} \end{array} \quad \text{is} \quad \begin{array}{c} 1\,0 \\ \underline{1\,0} \\ 0\,1 \end{array}$$

by parts (i) and (ii), so there is exactly one choice for b_i, namely 1. This agrees with the asserted formulas as $2^{k-1-n_1(a_i\ldots a_{i+k-1})} = 2^{1-1-n_1(a_i)} = 2^{-n_1(0)} = 2^0 = 1$.

Lemma 2 Let n be a positive integer. Let $a_0a_1\ldots a_\ell$ be the binary representation of n. Let r be an integer with $0 \leq r \leq n$. Let $b_0b_1\ldots b_\ell$ be the binary representation of r and $c_0c_1\ldots c_\ell$ the binary representation of $n - r$. When adding $b_0b_1\ldots b_\ell$ to $c_0c_1\ldots c_\ell$ in base 2 to obtain $a_0a_1\ldots a_\ell$ suppose that no carry occurs when adding b_j and c_j for $j = i$, $i+1, \ldots, i+k$, where i and k are integers with $0 \leq i \leq i+k \leq \ell$, and, if $i \neq 0$ a carry occurs when adding b_{i-j} and c_{i-1}, and if $i+k \neq \ell$ a carry occurs when adding b_{i+k+1} and c_{i+k+1}. Then the total number of possibilities for $b_ib_{i+1}\ldots b_{i+k}$ is

$$\begin{cases} 2^{n_1(a_0a_1\ldots a_k)}, & \text{if } i = 0, \\ 2^{n_1(a_i\ldots a_{i+k})-1}, & \text{if } i \geq 1. \end{cases}$$

Proof Suppose first that $i \geq 1$. Consider the addition of b_i and c_i in the addition of $b_0b_1\ldots b_\ell$ to $c_0c_1\ldots c_\ell$ to obtain $a_0a_1\ldots a_\ell$. There is a carry coming from the addition in the $i-1$th place and no carry going to the addition in the $i+1$th place. Hence we must have $b_i = c_i = 0$ and $a_i = 1$. For the addition in the jth place ($j = i+1, \ldots, i+k$) there is no carry coming in or going out so we must have

$$(b_j, c_j) = (0,0), \quad \text{if } a_j = 0, \text{ and } (b_j, c_j) = (0,1) \text{ or } (1,1), \text{ if } a_j = 1.$$

Hence the number of possibilities for $b_j (j = i+1, \ldots, i+k)$ is $2^{n_1(a_j)}$, and so the number of possibilities for $b_i b_{i+1} \ldots b_{i+k}$ is

$$1 \times 2^{n_1(a_{i+1})} \times \cdots \times 2^{n_1(a_{i+k})} = 2^{n_1(a_{i+1} \ldots a_{i+k})} = 2^{n_1(a_i \ldots a_{i+k})-1}.$$

Now suppose that $i = 0$. By the above reasoning we see that the number of possiblities for $b_0 b_1 \ldots b_k$ is

$$2^{n_1(a_0)} \times 2^{n_1(a_1)} \times \cdots \times 2^{n_1(a_k)} = 2^{n_1(a_0 a_1 \ldots a_k)}.$$

4 PROOF OF THEOREM 1

Before giving the proof we give a definition.

Definition 2 Let n be a positive integer. Let $a_0 a_1 \ldots a_\ell$ be the binary representation of n. Let r be an integer with $0 \le r \le n$. Let $b_0 b_1 \ldots b_\ell$ be the binary representation of r and $c_0 c_1 \ldots c_\ell$ the binary representation of $n - r$. In the base 2 addition

$$\begin{array}{rl} & b_0 b_1 \ldots b_i \ldots b_\ell \\ + & \underline{c_0 c_1 \ldots c_i \ldots c_\ell} \\ & a_0 a_1 \ldots a_i \ldots a_\ell \end{array} \tag{4.1}$$

we suppose that a carry occurs in the ith position for

$$i = i_1, i_1+1, \ldots, i_1+k_1-1, i_2, i_2+1, \ldots, i_2+k_2-1, \ldots, i_u, i_u+1, \ldots, i_u+k_u-1,$$

and there is no carry in the ith position for

$$i = 0, \ldots, i_1-1, i_1+k_1, \ldots, i_2-1, i_2+k_2, \ldots, i_3-1, \ldots, i_u+k_u, \ldots, \ell,$$

where i_j, k_j $(j = 1, 2, \ldots, u)$ are integers such that

$$0 \le i_1 < i_1+k_1 < i_2 < i_2+k_2 < \ldots < i_u+k_u \le \ell.$$

We define strings N_j $(j = 1, 2, \ldots, u+1)$ and C_j $(j = 1, 2, \ldots, u)$ by

$$\begin{array}{rcl} N_1 & = & b_0 \ldots b_{i_1-1} (= \emptyset \text{ if } i_1 = 0), \\ C_1 & = & b_{i_1} \ldots b_{i_1+k_1-1}, \\ N_2 & = & b_{i_1+k_1} \ldots b_{i_2-1}, \\ C_2 & = & b_{i_2} \ldots b_{i_2+k_2-1}, \\ & \ldots & \\ N_u & = & b_{i_{u-1}+k_{u-1}} \ldots b_{i_u-1}, \\ C_u & = & b_{i_u} \ldots b_{i_u+k_u-1}, \\ N_{u+1} & = & b_{i_u+k_u} \ldots b_\ell (\ne \emptyset), \end{array}$$

so that

$$b_0b_1\dots b_\ell = N_1C_1N_2C_2\dots N_uC_uN_{u+1}. \tag{4.2}$$

We call (5.2) the carry partition of r. We also define

$$\begin{aligned}
&\text{carry number of } r = c(r) = k_1 + \cdots + k_u,\\
&\text{carry block number of } r = cbn(r) = u,\\
&\text{carry block vector of } r = \mathbf{cbv}(r) = (k_1,\dots,k_u),\\
&\text{index vector of } r = \mathbf{i}(r) = (i_1,\dots,i_u).
\end{aligned}$$

Clearly the C_j $(j = 1,\dots,u)$ are the blocks of consecutive positions where carries occur in the addition (5.1), the carry number is the total number of carries, the carry block number is the number of blocks of consecutive carries, the carry block vector is the vector of lengths of the blocks of consecutive carries, and the index vector is the vector of initial positions of the blocks of consecutive carries. We give an example to illustrate Definition 2.

Example We choose $n = 803031$ and $r = 33630$ so that $n - r = 769401$. The binary representations of $n, r, n-r$ are 11101011000000100011, 0111101011000001, 11001111010111101110 respectively. The addition of r and $n - r$ in base 2 is

$$\begin{array}{cccccccccccccccccccccc}
 & & & \star & \star & \star & \star & & \star & \star & \star & \star & \star & \star & & \star & \star & \star & & & \\
 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
+ & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1\\
\hline
 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1
\end{array}$$

where the asterisk indicates a position in which a carry is generated. Hence

$$\begin{aligned}
i_1 &= 3, i_2 = 8, i_3 = 15, u = 3,\\
k_1 &= 4, k_2 = 6, k_3 = 3,\\
N_1 &= 011, N_2 = 0, N_3 = 0, N_4 = 00,\\
C_1 &= 1101, C_2 = 110000, C_3 = 100,\\
c(33630) &= 13,\\
cbn(33630) &= 3,\\
\mathbf{cbv}(33630) &= (4,6,3)\\
\mathbf{i}(33630) &= (3,8,15).
\end{aligned}$$

Proof of theorem 1

$$f(n,m) = \sum^{n} 1$$

By Kummer's theorem [8] we have $2^m \| \binom{n}{r} \Leftrightarrow c(r) = m$, so that

$$f(n,m) = \sum_{\substack{r=0 \\ c(r)=m}}^{n} 1.$$

First we split up this sum according to the value of the carry block number of r. As $m \geq 1$ we have

$$f(n,m) = \sum_{u=1}^{m} \sum_{\substack{r=0 \\ c(r)=m \\ cbn(r)=u}}^{n} 1 .$$

Next we split up the inner sum according to the carry block vector of r. We have

$$f(n,m) = \sum_{u=1}^{m} \sum_{k_1,\ldots,k_u \geq 1} \sum_{\substack{r=0 \\ c(r)=m \\ cbn(r)=u \\ cbv(r)=(k_1,\ldots,k_u)}}^{n} 1 .$$

Since $c(r) = k_1 + \cdots + k_u$ we can rewrite the expression for $f(n,m)$ as

$$f(n,m) = \sum_{u=1}^{m} \sum_{\substack{k_1,\ldots,k_u \geq 1 \\ k_1+\cdots+k_u=m}} \sum_{\substack{r=0 \\ cbn(r)=u \\ cbv(r)=(k_1,\ldots,k_u)}}^{n} 1 .$$

Now we split up the inner sum according to the initial vector $\mathrm{i}(r)$. We obtain

$$f(n,m) = \sum_{u=1}^{m} \sum_{\substack{k_1,\ldots,k_u \geq 1 \\ k_1+\cdots+k_u=m}} \sum_{\substack{i_1,\ldots,i_u \\ 0 \leq i_1 < i_1+k_1 < \cdots < i_u < i_u+k_u \leq \ell}} \sum_{\substack{r=0 \\ cbn(r)=u \\ cbv(r)=(k_1,\ldots,k_u) \\ \mathrm{i}(r)=(i_1,\ldots,i_u)}}^{n} 1 .$$

By Lemmas 1 and 2 we have $a_{i_1} = \ldots = a_{i_u} = 0$, $a_{i_1+k_1} = \ldots = a_{i_u+k_u} = 1$, and

$$\begin{aligned} & \sum_{\substack{r=0 \\ cbn(r)=u \\ cbv(r)=(k_1,\ldots,k_u) \\ \mathrm{i}(r)=(i_1,\ldots,i_u)}}^{n} 1 \\ = \; & 2^{n_1(a_0 \ldots a_{i_1-1})} \times 2^{k_1-1-n_1(a_{i_1} \ldots a_{i_1+k_1-1})} \times 2^{n_1(a_{i_1+k_1} \ldots a_{i_2-1})-1} \\ & \times 2^{k_1-1-n_1(a_{i_2} \ldots a_{i_2+k_2-1})} \times \cdots \\ & \times 2^{k_u-1-n_1(a_{i_u} \ldots a_{i_u+k_u-1})} \times 2^{n_1(a_{i_u+k_u} \ldots a_\ell)-1} . \end{aligned}$$

The exponent of 2 on the right hand side is

$$\begin{aligned} & (k_1 + \cdots + k_u) - 2u + n_1(a_0 \ldots a_\ell) - 2 \sum_{j=1}^{u} n_1(a_{i_j} \ldots a_{i_j+k_j-1}) \\ = \; & m - 2u + n_1 - 2 \sum_{j=1}^{u} n_1(a_{i_j+1} \ldots a_{i_j+k_j-1}). \end{aligned}$$

Thus

$$
\begin{aligned}
&f(n,m)\\
&= \sum_{u=1}^{m} \sum_{\substack{k_1,\ldots,k_u \geq 1\\ k_1+\cdots+k_u=m}} \sum_{\substack{i_1,\ldots,i_u\\ 0\leq i_1<i_1+k_1<\ldots<i_u<i_u+k_u\leq \ell\\ a_{i_1}a_{i_1+k_1}=\cdots=a_{i_u}a_{i_u+k_u}=01}} 2^{m-2u+n_1-2\sum_{j=1}^{u} n_1(a_{i_j+1}\cdots a_{i_j+k_j-1})}\\
&= \sum_{u=1}^{m} 2^{n_1+m-2u} \sum_{\substack{k_1,\ldots,k_u \geq 1\\ k_1+\cdots+k_u=m}} \sum_{\substack{B_1,\ldots,B_u\\ |B_1|=k_1-1\\ \cdots\\ |B_u|=k_u-1}} \sum_{\substack{i_1,\ldots,i_u\\ 0\leq i_1<i_1+k_1<\ldots<i_u<i_u+k_u\leq \ell\\ a_{i_1}\cdots a_{i_1+k_1}=0B_11\\ \cdots\\ a_{i_u}\cdots a_{i_u+k_u}=0B_u1}} 2^{-2(n_1(B_1)+\cdots+n_1(B_u))}\\
&= \sum_{u=1}^{m} 2^{n_1+m-2u} \sum_{\substack{B_1,\ldots,B_u\\ |B_1|+\cdots+|B_u|=m-u}} 2^{-2(n_1(B_1)+\cdots+n_1(B_u))} \sum_{\substack{i_1,\ldots,i_u\\ 0\leq i_1<i_1+k_1<\ldots<i_u<i_u+k_u\leq \ell\\ a_{i_1}\cdots a_{i_1+k_1}=0B_11\\ \cdots\\ a_{i_u}\cdots a_{i_u+k_u}=0B_u1}} 1\\
&= \sum_{u=1}^{m} 2^{n_1+m-2u} \sum_{\substack{B_1,\ldots,B_u\\ |B_1|+\cdots+|B_u|=m-u}} 2^{-2(n_1(B_1)+\cdots+n_1(B_u))} S_n(B_1,\ldots,B_u)\\
&= \sum_{u=1}^{m} 2^{n_1+m-2u} \sum_{v=0}^{m-u} 2^{-2v} \sum_{\substack{B_1,\ldots,B_u\\ |B_1|+\cdots+|B_u|=m-u\\ n_1(B_1)+\cdots+n_1(B_u)=v}} S_n(B_1,\ldots,B_u)\\
&= \sum_{u=1}^{m} \sum_{v=0}^{m-u} 2^{n_1+m-2(u+v)} \sum_{\substack{B_1,\ldots,B_u\\ |B_1|+\cdots+|B_u|=m-u\\ n_1(B_1)+\cdots+n_1(B_u)=v}} S_n(B_1,\ldots,B_u)\\
&= \sum_{w=1}^{m} \sum_{u=1}^{w} 2^{n_1+m-2w} \sum_{\substack{B_1,\ldots,B_u\\ |B_1|+\cdots|B_u|=m-u\\ n_1(B_1)+\cdots+n_1(B_u)=w-u}} S_n(B_1,\ldots,B_u)\\
&= \sum_{w=1}^{m} 2^{n_1+m-2w} \sum_{u=1}^{w} \sum_{\substack{B_1,\ldots,B_u\\ n_0(B_1)+\cdots+n_0(B_u)=m-w\\ n_1(B_1)+\cdots+n_1(B_u)=w-u}} S_n(B_1,\ldots,B_u)\\
&= \sum_{w=1}^{m} 2^{n_1+m-2w} b_{mw},
\end{aligned}
$$

where

$$b_{mw} = \sum_{u=1}^{w} \sum_{B_1,\ldots,B_u} S_n(B_1,\ldots,B_u).$$

5 PROOF OF THEOREM 2

b_{11}, b_{21}, b_{22}, b_{31}, b_{32}, b_{33}, b_{41}, b_{42}, b_{43} and b_{44} are easily evaluated using Properties 1–7. We just give the details for b_{44}.

$$\begin{aligned}
b_{44} &= \sum_{\substack{B_1 \\ n_1(B_1)=0 \\ n_1(B_1)=3}} S_n(B_1) + \sum_{\substack{B_1,B_2 \\ n_1(B_1)+n_0(B_2)=0 \\ n_1(B_1)+n_1(B_2)=2}} S_n(B_1,B_2) \\
&\quad + \sum_{\substack{B_1,B_2,B_3 \\ n_0(B_1)+n_0(B_2)+n_0(B_3)=0 \\ n_1(B_1)+n_1(B_2)+n_1(B_3)=1}} S_n(B_1,B_2,B_3) \\
&\quad + \sum_{\substack{B_1,B_2,B_3,B_4 \\ n_0(B_1)+n_0(B_2)+n_0(B_3)+n_0(B_4)=0 \\ n_1(B_1)+n_1(B_2)+n_1(B_3)+n_1(B_4)=0}} S_n(B_1,B_2,B_3,B_4) \\
&= S_n(111) + \{(S_n(\emptyset,11) + S_n(11,\emptyset)) + S_n(1,1)\} \\
&\quad + \{S_n(\emptyset,\emptyset,1) + S_n(\emptyset,1,\emptyset) + S_n(1,\emptyset,\emptyset)\} + S_n(\emptyset,\emptyset,\emptyset,\emptyset) \\
&= n_{01111} + \left\{(n_{01}n_{0111} - n_{0111}) + \binom{n_{011}}{2}\right\} + n_{011}\binom{n_{01}-1}{2} + \binom{n_{01}}{4}.
\end{aligned}$$

6 PROOF OF THEOREM 3

From the proof of Theorem 1 we have

$$f(n,m) = \sum_{u=1}^{m} 2^{n_1+m-2u} \sum_{\substack{B_1,\ldots,B_u \\ |B_1|+\cdots+|B_u|=m-u}} 2^{-2(n_1(B_1)+\cdots+n_1(B_u))} S_n(B_1,\ldots,B_u). \tag{6.1}$$

The term with $u = m$ is by Property 1

$$2^{n_1-m} S_n(\emptyset,\ldots,\emptyset) = 2^{n_1-m}\binom{n_{01}}{m} = 2^{n_1-m}\left\{\frac{n_{01}^m}{m!} + O_m(n_{01}^{m-1})\right\},$$

as $n_{01} \to +\infty$. For $1 \le u \le m-1$ we have

$$S_n(B_1,\ldots,B_u) \le n_{0B_11}\cdots n_{0B_u1} \le n_{01}^u \le n_{01}^{m-1},$$

so that

$$\sum_{u=1}^{m-1} 2^{n_1+m-2u} \sum_{\substack{B_1,\ldots,B_u \\ |B_1|+\cdots+|B_u|=m-u}} 2^{-2(n_1(B_1)+\cdots+n_1(B_u))} S_n(B_1,\ldots,B_u)$$

$$
\begin{aligned}
&\le \sum_{u=1}^{m-1} 2^{n_1+m-2u} \sum_{\substack{B_1,\ldots,B_u \\ |B_1|+\cdots+|B_u|=m-u}} n_{01}^{m-1} \\
&= 2^{n_1+m-2} n_{01}^{m-1} \sum_{u=1}^{m-1} \sum_{\substack{B_1,\ldots,B_u \\ |B_1|+\cdots+|B_u|=m-u}} 1 \\
&\le 2^{n_1+m-2} n_{01}^{m-1} \sum_{u=1}^{m-1} \Big(\sum_{\substack{B \\ |B|\le m-u}} 1 \Big)^u \\
&\le 2^{n_1+m-2} n_{01}^{m-1} \sum_{u=1}^{m-1} (2^{m-u+1})^u \\
&\le 2^{n_1+m-2} n_{01}^{m-1} (m-1) 2^{m(m-1)}.
\end{aligned}
$$

Hence

$$f(n,m) = 2^{n_1-m} \frac{n_{01}^m}{m!} + O_m(2^{n_1} n_{01}^{m-1}),$$

as $n_{01} \to +\infty$, which completes the proof of Theorem 3.

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