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Small Solutions of $\phi_1 x_1^2 + \cdots + \phi_n x_n^2 = 0$

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Abstract. Let $\phi_1, \ldots, \phi_n$ $(n \geq 2)$ be nonzero integers such that the equation

$$\sum_{i=1}^{n} \phi_i x_i^2 = 0$$

is solvable in integers $x_1, \ldots, x_n$ not all zero. It is shown that there exists a solution satisfying

$$0 < \sum_{i=1}^{n} |\phi_i| x_i^2 \leq 2|\phi_1 \cdots \phi_n|,$$

and that the constant 2 is best possible.

1 Introduction

As a consequence of a more general result, Birch and Davenport [1] showed in 1958 that if $\phi_1, \ldots, \phi_n$ $(n \geq 2)$ are nonzero integers such that the equation

$$\sum_{i=1}^{n} \phi_i x_i^2 = 0 \tag{1.1}$$

is solvable in integers $x_1, \ldots, x_n$ not all zero then there exists a solution satisfying

$$0 < \sum_{i=1}^{n} |\phi_i| x_i^2 \leq (2n)^{\frac{1}{2}(n-1)} \gamma_{n-1}^{n-1} |\phi_1 \cdots \phi_n|, \tag{1.2}$$

where γ_{n-1} is Hermite's constant, defined as the upper bound of the minima of positive definite quadratic forms in $n-1$ variables of determinant 1. It is known that

$$\gamma_2 = 2/\sqrt{3}, \quad \gamma_3 = \sqrt[3]{2}, \quad \gamma_4 = \sqrt{2}, \quad \gamma_5 = \sqrt[5]{8}, \quad \gamma_6 = \sqrt[6]{\frac{64}{3}}, \tag{1.3}$$

see for example [3, p. 36].

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In this paper we prove the following improvement of Birch and Davenport's result.

Theorem *Let $\phi_1, \dots, \phi_n$ ($n \geq 2$) be nonzero integers such that the equation (1.1) is solvable in integers $x_1, \dots, x_n$ not all zero. Then there is a solution of (1.1) satisfying*

$$0 < \sum_{i=1}^{n} |\phi_i| x_i^2 \leq 2|\phi_1 \cdots \phi_n|. \tag{1.4}$$

Moreover the constant 2 on the right hand side of the inequality (1.4) is best possible in the sense that equality can hold.

To see that 2 is the best possible constant in (1.4) it suffices to consider the equation

$$x_1^2 + x_2^2 + \cdots + x_{n-1}^2 - x_n^2 = 0. \tag{1.5}$$

A solution of (1.5) having the least possible nonzero value of $x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x_n^2 = 2x_n^2$ is $(x_1, x_2, \dots, x_{n-1}, x_n) = (1, 0, \dots, 0, 1)$. Hence there is a solution of (1.5) with $x_1^2 + x_2^2 + \cdots + x_n^2 = 2$.

We remark that our theorem is easily seen to be true for $n = 2$. In this case we may suppose that $\phi_1 > 0$ and $\phi_2 < 0$. Set $g = (\phi_1, \phi_2)$. As the equation $\phi_1 x_1^2 + \phi_2 x_2^2 = 0$ is solvable nontrivially we see that $(\phi_1/g)(-\phi_2/g) = (\phi_2 x_2/g x_1)^2$ is a square. Hence there exist positive integers u and v such that $\phi_1/g = u^2$ and $-\phi_2/g = v^2$. A nontrivial solution of $\phi_1 x_1^2 + \phi_2 x_2^2 = 0$ is $(x_1, x_2) = (v, u)$ and this solution satisfies

$$0 < |\phi_1| x_1^2 + |\phi_2| x_2^2 = 2gu^2v^2 = \frac{2}{g}|\phi_1\phi_2| \leq 2|\phi_1|\,|\phi_2|.$$

When $n = 3$ it was shown by Mordell [4] that Legendre's equation

$$\phi_1 x_1^2 + \phi_2 x_2^2 + \phi_3 x_3^2 = 0,$$

when solvable nontrivially, has a solution in integers $(x_1, x_2, x_3) \neq (0, 0, 0)$ satisfying

$$|x_1| \leq \sqrt{|\phi_2\phi_3|}, \quad |x_2| \leq \sqrt{|\phi_1\phi_3|}, \quad |x_3| \leq \sqrt{|\phi_1\phi_2|}.$$

A small omission in Mordell's proof was provided by Williams [5]. Such a solution satisfies

$$0 < |\phi_1| x_1^2 + |\phi_2| x_2^2 + |\phi_3| x_3^2 \leq 2|\phi_1\phi_2\phi_3|,$$

which is the assertion of our theorem when $n = 3$.

For $n \geq 4$ our theorem is new. The theorem is proved in Section 4 after a lemma is proved in Section 2 and a preliminary form of the theorem is proved in Section 3. The calculation of the determinant of a particular quadratic form needed in Section 3 is carried out in Section 5.

We remark that as indefinite integral quadratic forms in 5 or more variables have nontrivial integral solutions, we have the following corollary to our theorem.

Corollary *Let $\phi_1, \dots, \phi_n$ ($n \geq 5$) be nonzero integers not all of the same sign. Then there is a solution of (1.1) satisfying (1.4).*

2 A Preliminary Lemma

Let $a_1, \dots, a_n$ $(n \geq 2)$ be nonzero integers such that

$$(a_1, \dots, a_n) = 1. \tag{2.1}$$

We set

$$d_i = (a_1, a_i), \quad i = 2, \dots, n, \tag{2.2}$$

$$\begin{cases} d_2' = 1, \\ d_i' = \frac{a_1}{(a_1,[a_2,\dots,a_i])}, \quad i = 3, \dots, n, \end{cases} \tag{2.3}$$

$$D_i = a_1^{i-2}(a_1, \dots, a_i), \quad i = 1, \dots, n. \tag{2.4}$$

We observe that

$$D_1 = a_1^{-1} a_1 = 1, \tag{2.5}$$

$$D_2 = (a_1, a_2) = d_2, \tag{2.6}$$

$$D_n = a_1^{n-2}(a_1, \dots, a_n) = a_1^{n-2}, \tag{2.7}$$

$$D_i | a_1^{i-1}, \quad i = 1, \dots, n, \tag{2.8}$$

$$D_i | a_1^{i-2} a_j, \quad j = 2, \dots, i. \tag{2.9}$$

For $i = 3, \dots, n$ we have by (2.2)–(2.4)

$$\begin{aligned} \left(\frac{a_1}{d_i}, \frac{a_1^{i-2}}{D_{i-1}} \right) &= \left(\frac{a_1}{(a_1, a_i)}, \frac{a_1}{(a_1, \dots, a_{i-1})} \right) \\ &= \frac{a_1}{[(a_1, a_i), (a_1, \dots, a_{i-1})]} \\ &= \frac{a_1}{(a_1, [a_2, \dots, a_i])} \\ &= d_i'. \end{aligned}$$

For $i = 2$ we have

$$\left(\frac{a_1}{d_i}, \frac{a_1^{i-2}}{D_{i-1}} \right) = \left(\frac{a_1}{d_2}, 1 \right) = 1 = d_2'.$$

Hence

$$d_i' = \left(\frac{a_1}{d_i}, \frac{a_1^{i-2}}{D_{i-1}} \right), \quad i = 2, \dots, n. \tag{2.10}$$

As a consequence of (2.10) we see that

$$d_i d_i' | a_1, \quad i = 2, \dots, n. \tag{2.11}$$

Next, for $i = 3, \ldots, n$, we have

$$\begin{aligned}\frac{a_1}{d_i'} &= (a_1, [a_2, \ldots, a_i]) \quad \text{(by (2.3))}\\ &= [(a_1, a_i), (a_1, a_2, \ldots, a_{i-1})]\\ &= \frac{(a_1, a_i)(a_1, a_2, \ldots, a_{i-1})}{(a_1, a_2, \ldots, a_i)}\\ &= \frac{d_i(D_{i-1}/a_1^{i-3})}{(D_i/a_1^{i-2})} \quad \text{(by (2.2) and (2.4))}\\ &= a_1 d_i \frac{D_{i-1}}{D_i},\end{aligned}$$

so that $D_i = d_i d_i' D_{i-1}$ for $i = 3, \ldots, n$. Also $D_2 = d_2 = d_2 d_2' D_1$ by (2.3), (2.5) and (2.6). Hence

$$D_i = d_i d_i' D_{i-1}, \quad i = 2, \ldots, n. \tag{2.12}$$

From (2.5) and (2.12) we deduce that

$$D_i = d_2 d_2' d_3 d_3' \cdots d_i d_i', \quad i = 2, \ldots, n. \tag{2.13}$$

Finally, from (2.2) and (2.10), we see that

$$\left(\frac{a_1}{d_i}, \frac{a_i}{d_i}\frac{a_1^{i-2}}{D_{i-1}}\right) = \left(\frac{a_1}{d_i}, \frac{a_1^{i-2}}{D_{i-1}}\right) = d_i', \quad i = 2, \ldots, n, \tag{2.14}$$

so that we can choose integers u_i ($i = 2, \ldots, n$) and v_i ($i = 2, \ldots, n$) such that

$$d_i' = \frac{a_1}{d_i} u_i - \frac{a_i}{d_i}\frac{a_1^{i-2}}{D_{i-1}} v_i, \quad i = 2, \ldots, n. \tag{2.15}$$

We are now ready to prove the following lemma.

Lemma *Let $a_1, \ldots, a_n$ ($n \geq 2$) be nonzero integers satisfying (2.1). Define d_i ($i = 2, \ldots, n$), d_i' ($i = 2, \ldots, n$) and D_i ($i = 1, 2, \ldots, n$) as in (2.2)–(2.4). Fix integers u_i ($i = 2, \ldots, n$) and v_i ($i = 2, \ldots, n$) satisfying (2.15). Let y_i ($i = 1, 2, \ldots, n$) and z_i ($i = 2, \ldots, n$) be integers such that*

$$a_1 y_i - a_i y_1 = d_i z_i, \quad i = 2, \ldots, n. \tag{2.16}$$

Then there exists integers x_i ($i = 1, 2, \ldots, n$) such that

$$y_1 = \sum_{k=1}^{n} \frac{a_1^{k-1}}{D_k} v_{k+1} x_k, \tag{2.17}$$

$$y_i = u_i x_{i-1} + \sum_{k=i}^{n} \frac{a_i a_1^{k-2}}{D_k} v_{k+1} x_k, \quad i = 2, \ldots, n, \tag{2.18}$$

$$z_i = -\frac{a_i}{d_i} \sum_{k=1}^{i-2} \frac{a_1^{k-1}}{D_k} v_{k+1} x_k + d_i' x_{i-1}, \quad i = 2, \ldots, n, \tag{2.19}$$

where $v_{n+1} = 1$. *In particular we have*

$$\begin{pmatrix} z_2 \\ \vdots \\ z_n \end{pmatrix} = P \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}, \tag{2.20}$$

where P *is a lower triangular integral matrix with*

$$\det(P) = d_2' d_3' \cdots d_n'. \tag{2.21}$$

Proof The proof shows how to construct recursively the required integers $x_1, \dots, x_n$. The first step determines integers x_1 and t_1 in terms of y_1, y_2, z_2 such that

$$\begin{cases} y_1 = v_2 x_1 + \frac{a_1}{D_2} t_1, \\ y_2 = u_2 x_1 + \frac{a_2}{D_2} t_1, \\ z_2 = d_2' x_1. \end{cases} \tag{2.22}$$

The second step determines integers x_2 and t_2 in terms of y_3 and t_1 such that

$$\begin{cases} y_1 = v_2 x_1 + \frac{a_1}{D_2} v_3 x_2 + \frac{a_1^2}{D_3} t_2, \\ y_2 = u_2 x_1 + \frac{a_2}{D_2} v_3 x_2 + \frac{a_2 a_1}{D_3} t_2, \\ y_3 = u_3 x_2 + \frac{a_3 a_1}{D_3} t_2, \\ z_2 = d_2' x_1, \\ z_3 = -\frac{a_3}{d_3} v_2 x_1 + d_3' x_2. \end{cases} \tag{2.23}$$

The i-th step ($i = 2, \dots, n-1$) determines integers x_i and t_i in terms of y_{i+1} and t_{i-1} such that

$$\begin{cases} y_1 = \sum\limits_{k=1}^{i} \frac{a_1^{k-1}}{D_k} v_{k+1} x_k + \frac{a_1^i}{D_{i+1}} t_i, \\ y_r = u_r x_{r-1} + \sum\limits_{k=r}^{i} \frac{a_r a_1^{k-2}}{D_k} v_{k+1} x_k + \frac{a_r a_1^{i-1}}{D_{i+1}} t_i, \quad r = 2, \dots, i+1, \\ z_r = -\frac{a_r}{d_r} \sum\limits_{k=1}^{r-2} \frac{a_1^{k-1}}{D_k} v_{k+1} x_k + d_r' x_{r-1}, \quad r = 2, \dots, i+1. \end{cases} \tag{2.24}$$

Step 1 The first step determines x_1 and t_1. We choose

$$x_1 = z_2.$$

From

$$\frac{a_1}{d_2} y_2 - \frac{a_2}{d_2} y_1 = z_2 = x_1 = \left(\frac{a_1}{d_2} u_2 - \frac{a_2}{d_2} v_2 \right) x_1$$

we obtain

$$\frac{a_1}{d_2}(y_2 - u_2 x_1) = \frac{a_2}{d_2}(y_1 - v_2 x_1).$$

As $(\frac{a_1}{d_2}, \frac{a_2}{d_2}) = 1$ there exists an integer t_1 such that $y_1 - v_2 x_1 = \frac{a_1}{d_2} t_1$ and $y_2 - u_2 x_1 = \frac{a_2}{d_2} t_1$. Recalling that $d_2 = D_2$ and $d_2' = 1$ we obtain (2.22). This completes the first step.

Step 2 The second step determines x_2 and t_2. We choose

$$x_2 = \frac{a_1}{d_3 d_3'} y_3 - \frac{a_3 a_1}{D_3} t_1.$$

From

$$\frac{a_1}{d_3} y_3 - \frac{a_3}{d_3}\frac{a_1}{D_2} t_1 = d_3' x_2 = \left(\frac{a_1}{d_3} u_3 - \frac{a_3}{d_3}\frac{a_1}{D_2} v_3\right) x_2$$

we obtain

$$\frac{a_1}{d_3}(y_3 - u_3 x_2) = \frac{a_3}{d_3}\frac{a_1}{D_2}(t_1 - v_3 x_2).$$

As $(\frac{a_1}{d_3}, \frac{a_3}{d_3}\frac{a_1}{D_2}) = d_3'$ there exists an integer t_2 such that $t_1 - v_3 x_2 = \frac{a_1}{d_3 d_3'} t_2$ and $y_3 - u_3 x_2 = \frac{a_3 a_1}{D_3} t_2$. We now have y_1, y_2, y_3 and z_2 in the form given in (2.23). Finally

$$\begin{aligned}
z_3 &= \frac{1}{d_3}(a_1 y_3 - a_3 y_1) \\
&= \frac{1}{d_3}\left(a_1\left(u_3 x_2 + \frac{a_3 a_1}{D_3} t_2\right) - a_3\left(v_2 x_1 + \frac{a_1}{D_2} v_3 x_2 + \frac{a_1^2}{D_3} t_2\right)\right) \\
&= -\frac{a_3 v_2}{d_3} x_1 + \left(\frac{a_1 u_3}{d_3} - \frac{a_3}{d_3}\frac{a_1}{D_2} v_3\right) x_2 \\
&= -\frac{a_3}{d_3} v_2 x_1 + d_3' x_2.
\end{aligned}$$

This completes the second step.

Step i $(i = 2, \ldots, n-1)$ This step determines x_i and t_i. From step $i-1$ we have

$$y_1 = \sum_{k=1}^{i-1} \frac{a_1^{k-1}}{D_k} v_{k+1} x_k + \frac{a_1^{i-1}}{D_i} t_{i-1},$$

$$y_r = u_r x_{r-1} + \sum_{k=r}^{i-1} \frac{a_r a_1^{k-2}}{D_k} v_{k+1} x_k + \frac{a_r a_1^{i-2}}{D_i} t_{i-1}, \quad r = 2, \ldots, i,$$

$$z_r = -\frac{a_r}{d_r} \sum_{k=1}^{r-2} \frac{a_1^{k-1}}{D_k} v_{k+1} x_k + d_r' x_{r-1}, \quad r = 2, \ldots, i.$$

We choose

$$x_i = \frac{a_1}{d_{i+1} d_{i+1}'} y_{i+1} - \frac{a_{i+1} a_1^{i-1}}{D_{i+1}} t_{i-1}.$$

From

$$\frac{a_1}{d_{i+1}} y_{i+1} - \frac{a_{i+1}}{d_{i+1}}\frac{a_1^{i-1}}{D_i} t_{i-1} = d_{i+1}' x_i = \left(\frac{a_1}{d_{i+1}} u_{i+1} - \frac{a_{i+1}}{d_{i+1}}\frac{a_1^{i-1}}{D_i} v_{i+1}\right) x_i$$

we obtain

$$\frac{a_1}{d_{i+1}}(y_{i+1} - u_{i+1}x_i) = \frac{a_{i+1}}{d_{i+1}}\frac{a_1^{i-1}}{D_i}(t_{i-1} - v_{i+1}x_i).$$

As $(\frac{a_1}{d_{i+1}}, \frac{a_{i+1}}{d_{i+1}}\frac{a_1^{i-1}}{D_i}) = d'_{i+1}$ there exists an integer t_i such that

$$t_{i-1} - v_{i+1}x_i = \frac{a_1}{d_{i+1}d'_{i+1}}t_i$$

and

$$y_{i+1} - u_{i+1}x_i = \frac{a_{i+1}a_1^{i-1}}{D_{i+1}}t_i.$$

Hence

$$\begin{aligned} y_1 &= \sum_{k=1}^{i-1}\frac{a_1^{k-1}}{D_k}v_{k+1}x_k + \frac{a_1^{i-1}}{D_i}\left(v_{i+1}x_i + \frac{a_1}{d_{i+1}d'_{i+1}}t_i\right) \\ &= \sum_{k=1}^{i}\frac{a_1^{k-1}}{D_k}v_{k+1}x_k + \frac{a_1^{i}}{D_{i+1}}t_i. \end{aligned}$$

Also for $r = 2, \ldots, i$ we have

$$\begin{aligned} y_r &= u_r x_{r-1} + \sum_{k=r}^{i-1}\frac{a_r a_1^{k-2}}{D_k}v_{k+1}x_k + a_r\frac{a_1^{i-2}}{D_i}\left(v_{i+1}x_i + \frac{a_1}{d_{i+1}d'_{i+1}}t_i\right) \\ &= u_r x_{r-1} + \sum_{k=r}^{i}\frac{a_r a_1^{k-2}}{D_k}v_{k+1}x_k + \frac{a_r a_1^{i-1}}{D_{i+1}}t_i, \end{aligned}$$

which also holds for $r = i + 1$ in view of

$$y_{i+1} = u_{i+1}x_i + \frac{a_{i+1}a_1^{i-1}}{D_{i+1}}t_i.$$

Further

$$\begin{aligned} d_{i+1}z_{i+1} &= a_1 y_{i+1} - a_{i+1}y_i \\ &= a_1\left(u_{i+1}x_i + \frac{a_{i+1}a_1^{i-1}}{D_{i+1}}t_i\right) - a_{i+1}\left(\sum_{k=1}^{i}\frac{a_1^{k-1}}{D_k}v_{k+1}x_k + \frac{a_1^{i}}{D_{i+1}}t_i\right) \\ &= -a_{i+1}\sum_{k=1}^{i-1}\frac{a_1^{k-1}}{D_k}v_{k+1}x_k + \left(a_1 u_{i+1} - a_{i+1}\frac{a_1^{i-1}}{D_i}v_{i+1}\right)x_i \\ &= -a_{i+1}\sum_{k=1}^{i-1}\frac{a_1^{k-1}}{D_k}v_{k+1}x_k + d_{i+1}d'_{i+1}x_i \end{aligned}$$

so that

$$z_{i+1} = -\frac{a_{i+1}}{d_{i+1}} \sum_{k=1}^{i-1} \frac{a_1^{k-1}}{D_k} v_{k+1} x_k + d'_{i+1} x_i.$$

This concludes the i-th step.

After $n-1$ steps we have determined integers $x_1, \dots, x_{n-1}$ and t_{n-1} such that

$$y_1 = \sum_{k=1}^{n-1} \frac{a_1^{k-1}}{D_k} v_{k+1} x_k + \frac{a_1^{n-1}}{D_n} t_{n-1},$$

$$y_r = u_r x_{r-1} + \sum_{k=r}^{n-1} \frac{a_r a_1^{k-2}}{D_k} v_{k+1} x_k + \frac{a_r a_1^{n-2}}{D_n} t_{n-1}, \quad r = 2, \dots, n,$$

$$z_r = -\frac{a_r}{d_r} \sum_{k=1}^{r-2} \frac{a_1^{k-1}}{D_k} v_{k+1} x_k + d'_r x_{r-1}, \quad r = 2, \dots, n.$$

Setting $x_n = t_{n-1}$ and $v_{n+1} = 1$ we obtain the assertion of the lemma. ■

3 A Preliminary Proposition

In this section we make use of the lemma proved in the previous section to prove the following result from which our theorem will be deduced in Section 4.

Proposition *Let $\phi_1, \dots, \phi_n$ ($n \geq 2$) be nonzero integers such that*

(i) (1.1) is solvable in integers not all zero,
(ii) every solution $(x_1, \dots, x_n) \neq (0, \dots, 0)$ of (1.1) has $x_i \neq 0$ ($i = 1, \dots, n$).

Then (1.1) has a solution $(x_1, \dots, x_n)$ satisfying

$$0 < \sum_{i=1}^{n} |\phi_i| x_i^2 \leq 2^{4-n} \gamma_{n-1}^{n-1} |\phi_1 \phi_2 \cdots \phi_n|.$$

Proof Let $(x_1, \dots, x_n) \neq (0, \dots, 0)$ be a solution in integers of (1.1). Such a solution exists by assumption (i). By assumption (ii) $x_1 \neq 0, \dots, x_n \neq 0$. Clearly at least one of the ϕ_i is positive and at least one of the ϕ_i is negative. Suppose that exactly r of the ϕ_i are positive so that

$$1 \leq r \leq n-1.$$

Relabelling the ϕ_i, if necessary, we may suppose that

$$\phi_1 > 0, \dots, \phi_r > 0, \quad \phi_{r+1} < 0, \dots, \phi_n < 0. \tag{3.1}$$

We set $a_i = x_i/(x_1, \dots, x_n)$ ($i = 1, \dots, n$) so that $(a_1, \dots, a_n)$ is a solution of (1.1) satisfying

$$a_i \neq 0 \quad (i = 1, \dots, n), \ (a_1, \dots, a_n) = 1. \tag{3.2}$$

Let $y_1, \ldots, y_n$ and $d(\neq 0)$ be integers which will be chosen later. Set

$$u = \sum_{i=1}^{n} \phi_i y_i^2, \quad v = -2\sum_{i=1}^{n} \phi_i a_i y_i. \tag{3.3}$$

Choose d such that

$$d \mid u, \quad d \mid v. \tag{3.4}$$

Set

$$b_i = \frac{ua_i + vy_i}{d} \quad (i = 1, \ldots, n). \tag{3.5}$$

We will choose $y_1, \ldots, y_n, d$ such that $(b_1, \ldots, b_n) \neq (0, \ldots, 0)$. The b_i are integers such that

$$\sum_{i=1}^{n} \phi_i b_i^2 = 0 \tag{3.6}$$

since

$$\begin{aligned} d^2 \sum_{i=1}^{n} \phi_i b_i^2 &= \sum_{i=1}^{n} \phi_i (ua_i + vy_i)^2 \\ &= u^2 \sum_{i=1}^{n} \phi_i a_i^2 + 2uv \sum_{i=1}^{n} \phi_i a_i y_i + v^2 \sum_{i=1}^{n} \phi_i y_i^2 \\ &= u^2 \cdot 0 + uv(-v) + v^2 u = 0. \end{aligned}$$

Set

$$A = \sum_{i=1}^{r} |\phi_i| a_i^2 = \sum_{i=1}^{r} \phi_i a_i^2 = -\sum_{i=r+1}^{n} \phi_i a_i^2 = \sum_{i=r+1}^{n} |\phi_i| a_i^2. \tag{3.7}$$

If $A \leq 2^{3-n} \gamma_{n-1}^{n-1} |\phi_1 \cdots \phi_n|$ then $(a_1, \ldots, a_n)$ is a solution of (1.1) satisfying

$$0 < \sum_{i=1}^{n} |\phi_i| a_i^2 = 2\sum_{i=1}^{r} \phi_i a_i^2 = 2A \leq 2^{4-n} \gamma_{n-1}^{n-1} |\phi_1 \cdots \phi_n|$$

establishing the proposition in this case. We therefore suppose that

$$A > 2^{3-n} \gamma_{n-1}^{n-1} |\phi_1 \cdots \phi_n| \tag{3.8}$$

and show how to choose $y_1, \ldots, y_n$ and d (with $d|u$ and $d|v$) so that $(b_1, \ldots, b_n)$ is a solution of (1.1) satisfying

$$0 < \sum_{i=1}^{r} \phi_i b_i^2 < A. \tag{3.9}$$

If $\sum_{i=1}^{r} \phi_i b_i^2 \leq 2^{3-n}\gamma_{n-1}^{n-1}|\phi_1 \cdots \phi_n|$ then $(b_1, \ldots, b_n)$ is a solution of (1.1) satisfying

$$0 < \sum_{i=1}^{n} |\phi_i| b_i^2 = 2\sum_{i=1}^{r} \phi_i b_i^2 \leq 2^{4-n}\gamma_{n-1}^{n-1}|\phi_1 \cdots \phi_n|$$

as required. If $\sum_{i=1}^{r} \phi_i b_i^2 > 2^{3-n}\gamma_{n-1}^{n-1}|\phi_1 \cdots \phi_n|$ we repeat the process on the solution $(b_1, \ldots, b_n)$. Continuing in this way, after a finite number of steps, we obtain a solution satisfying the inequalities given in the proposition.

The remainder of the proof is devoted to showing how to choose $y_1, \ldots, y_n$ and d. First we introduce some notation. We set

$$B = \sum_{i=1}^{r} \phi_i a_i y_i, \tag{3.10}$$

$$C = \sum_{i=1}^{r} \phi_i y_i^2, \tag{3.11}$$

$$t_i = a_1 y_i - a_i y_1 \quad (i = 1, \ldots, n), \text{ so that } t_1 = 0, \tag{3.12}$$

$$B_1 = \sum_{i=1}^{r} \phi_i a_i t_i = \sum_{i=2}^{r} \phi_i a_i t_i, \tag{3.13}$$

$$C_1 = \sum_{i=1}^{r} \phi_i t_i^2 = \sum_{i=2}^{r} \phi_i t_i^2, \tag{3.14}$$

$$L = \sum_{i=1}^{n} \phi_i t_i^2 = \sum_{i=2}^{n} \phi_i t_i^2, \tag{3.15}$$

$$M = \sum_{i=1}^{n} \phi_i a_i t_i = \sum_{i=2}^{n} \phi_i a_i t_i. \tag{3.16}$$

Next we deduce some relations between the quantities in (3.10)–(3.16).

From (3.7), (3.10), (3.12) and (3.13), we obtain

$$\begin{aligned} B &= \sum_{i=1}^{r} \phi_i a_i \left(\frac{a_i y_1 + t_i}{a_1} \right) \\ &= \frac{y_1}{a_1} \sum_{i=1}^{r} \phi_i a_i^2 + \frac{1}{a_1} \sum_{i=1}^{r} \phi_i a_i t_i, \end{aligned}$$

so that

$$B = \frac{A}{a_1} y_1 + \frac{B_1}{a_1}. \tag{3.17}$$

From (3.7) and (3.11)-(3.14), we have

$$C = \sum_{i=1}^{r} \phi_i \left(\frac{a_i y_1 + t_i}{a_1} \right)^2 = \frac{y_1^2}{a_1^2} \sum_{i=1}^{r} \phi_i a_i^2 + \frac{2y_1}{a_1^2} \sum_{i=1}^{r} \phi_i a_i t_i + \frac{1}{a_1^2} \sum_{i=1}^{r} \phi_i t_i^2,$$

so that

$$C = \frac{A}{a_1^2} y_1^2 + \frac{2B_1}{a_1^2} y_1 + \frac{C_1}{a_1^2}. \tag{3.18}$$

From (3.17) and (3.18), we deduce that

$$C - \frac{1}{A} B^2 = \frac{1}{a_1^2} \left(C_1 - \frac{1}{A} B_1^2 \right). \tag{3.19}$$

Next, from (3.3), (3.12), (3.15) and (3.16), we obtain

$$u = \sum_{i=1}^{n} \phi_i \left(\frac{a_i y_1 + t_i}{a_1} \right)^2 = \frac{y_1^2}{a_1^2} \sum_{i=1}^{n} \phi_i a_i^2 + \frac{2y_1}{a_1^2} \sum_{i=1}^{n} \phi_i a_i t_i + \frac{1}{a_1^2} \sum_{i=1}^{n} \phi_i t_i^2,$$

so that

$$u = \frac{2y_1}{a_1^2} M + \frac{1}{a_1^2} L. \tag{3.20}$$

From (3.3), (3.12) and (3.16), we have

$$v = -2 \sum_{i=1}^{n} \phi_i a_i \left(\frac{a_i y_1 + t_i}{a_1} \right) = \frac{-2y_1}{a_1} \sum_{i=1}^{n} \phi_i a_i^2 - \frac{2}{a_1} \sum_{i=1}^{n} \phi_i a_i t_i,$$

so that

$$v = -\frac{2}{a_1} M. \tag{3.21}$$

From (3.17), (3.20) and (3.21), we obtain

$$u + \frac{B}{A} v = \frac{2y_1}{a_1^2} M + \frac{1}{a_1^2} L - \frac{2B}{a_1 A} M = \frac{1}{a_1^2} L + \frac{2}{a_1^2 A} (Ay_1 - a_1 B) M,$$

so that

$$u + \frac{B}{A} v = \frac{1}{a_1^2} L - \frac{2B_1}{a_1^2 A} M. \tag{3.22}$$

Next, from (3.7), (3.13) and (3.16), we have

$$\sum_{i=r+1}^{n} \phi_i a_i \left(t_i - \frac{B_1}{A} a_i \right) = \sum_{i=r+1}^{n} \phi_i a_i t_i - \frac{B_1}{A} \sum_{i=r+1}^{n} \phi_i a_i^2 = M - B_1 - \frac{B_1}{A}(-A) = M,$$

and, from (3.7) and (3.13)-(3.16), we have

$$\begin{aligned}\sum_{i=r+1}^{n} \phi_i a_i \left(t_i - \frac{B_1}{A} a_i\right)^2 &= \sum_{i=r+1}^{n} \phi_i t_i^2 - \frac{2B_1}{A} \sum_{i=r+1}^{n} \phi_i a_i t_i + \frac{B_1^2}{A^2} \sum_{i=r+1}^{n} \phi_i a_i^2 \\ &= (L - C_1) - \frac{2B_1}{A}(M - B_1) + \frac{B_1^2}{A^2}(-A) \\ &= L - \frac{2MB_1}{A} - \left(C_1 - \frac{1}{A}B_1^2\right).\end{aligned}$$

Thus

$$L = \sum_{i=r+1}^{n} \phi_i \left(t_i - \frac{B_1}{A} a_i\right)^2 + \frac{2B_1}{A} \sum_{i=r+1}^{n} \phi_i a_i \left(t_i - \frac{B_1}{A} a_i\right) + \left(C_1 - \frac{1}{A}B_1^2\right) \tag{3.23}$$

and

$$M = \sum_{i=r+1}^{n} \phi_i a_i \left(t_i - \frac{B_1}{A} a_i\right). \tag{3.24}$$

Hence, from (3.21)–(3.24), we deduce that

$$v = -\frac{2}{a_1} \sum_{i=r+1}^{n} \phi_i a_i \left(t_i - \frac{B_1}{A} a_i\right) = \frac{2}{a_1} \sum_{i=r+1}^{n} |\phi_i| a_i \left(t_i - \frac{B_1}{A} a_i\right) \tag{3.25}$$

and

$$u + \frac{B}{A} v = \frac{1}{a_1^2} \sum_{i=r+1}^{n} \phi_i \left(t_i - \frac{B_1}{A} a_i\right)^2 + \frac{1}{a_1^2} \left(C_1 - \frac{1}{A}B_1^2\right). \tag{3.26}$$

We are now ready to examine $\sum_{i=1}^{r} \phi_i b_i^2$. Appealing to (3.5), (3.7), (3.10), (3.19), (3.25) and (3.26), we have

$$\begin{aligned}d^2 \sum_{i=1}^{r} \phi_i b_i^2 &= \sum_{i=1}^{r} \phi_i (u a_i + v y_i)^2 \\ &= u^2 \sum_{i=1}^{r} \phi_i a_i^2 + 2uv \sum_{i=1}^{r} \phi_i a_i y_i + v^2 \sum_{i=1}^{r} \phi_i y_i^2 \\ &= Au^2 + 2Buv + Cv^2 \\ &= A\left(u + \frac{B}{A}v\right)^2 + \left(C - \frac{1}{A}B^2\right) v^2 \\ &= \frac{A}{a_1^4} \left(\sum_{i=r+1}^{n} |\phi_i| \left(t_i - \frac{B_1}{A} a_i\right)^2 - \left(C_1 - \frac{1}{A}B_1^2\right)\right)^2 \\ &\quad + \frac{4}{a_1^4} \left(\sum_{i=r+1}^{n} |\phi_i| a_i \left(t_i - \frac{B_1}{A} a_i\right)\right)^2 \left(C_1 - \frac{1}{A}B_1^2\right).\end{aligned}$$

Now, by Cauchy's inequality, we have

$$\left(\sum_{i=r+1}^{n} |\phi_i| a_i \left(t_i - \frac{B_1}{A} a_i\right)\right)^2 \le \left(\sum_{i=r+1}^{n} |\phi_i| a_i^2\right) \left(\sum_{i=r+1}^{n} |\phi_i| \left(t_i - \frac{B_1}{A} a_i\right)^2\right),$$

so that by (3.7) we have

$$\begin{aligned} d^2 \sum_{i=1}^{r} \phi_i b_i^2 &\le \frac{A}{a_1^4} \left(\sum_{i=r+1}^{n} |\phi_i| \left(t_i - \frac{B_1}{A} a_i\right)^2 - \left(C_1 - \frac{1}{A} B_1^2\right)\right)^2 \\ &\quad + \frac{4A}{a_1^4} \left(\sum_{i=r+1}^{n} |\phi_i| \left(t_i - \frac{B_1}{A} a_i\right)^2\right) \left(C_1 - \frac{1}{A} B_1^2\right) \\ &= \frac{A}{a_1^4} \left(\sum_{i=r+1}^{n} |\phi_i| \left(t_i - \frac{B_1}{A} a_i\right)^2 + \left(C_1 - \frac{1}{A} B_1^2\right)\right)^2. \end{aligned}$$

From (3.13) and (3.14) we have

$$C_1 - \frac{1}{A} B_1^2 = \sum_{i=2}^{r} \phi_i \left(1 - \frac{\phi_i a_i^2}{A}\right) t_i^2 - \sum_{\substack{2 \le i,j \le r \\ i \ne j}} \frac{\phi_i \phi_j a_i a_j}{A} t_i t_j.$$

Hence

$$\begin{aligned} f(t_2, \ldots, t_n) &:= \sum_{i=r+1}^{n} |\phi_i| \left(t_i - \frac{a_i}{A} B_1\right)^2 + \frac{1}{A}(AC_1 - B_1^2) \\ &= \sum_{i=r+1}^{n} |\phi_i| \left(t_i - \frac{a_i}{A} B_1\right)^2 + \frac{1}{A}\left(\left(\sum_{i=1}^{r} \phi_i a_i^2\right)\left(\sum_{k=2}^{r} \phi_k t_k^2\right) - \left(\sum_{k=2}^{r} \phi_k a_k t_k\right)^2\right) \\ &= \sum_{i=r+1}^{n} |\phi_i| \left(t_i - \frac{a_i}{A} B_1\right)^2 + \frac{\phi_1 a_1^2}{A} \sum_{k=2}^{r} \phi_k t_k^2 \\ &\quad + \frac{1}{A}\left(\sum_{i=2}^{r} \sum_{k=2}^{r} \phi_i a_i^2 \phi_k t_k^2 - \sum_{i=2}^{r} \sum_{k=2}^{r} \phi_i a_i t_i \phi_k a_k t_k\right) \\ &= \sum_{i=r+1}^{n} |\phi_i| \left(t_i - \frac{a_i}{A} B_1\right)^2 + \frac{\phi_1 a_1^2}{A} \sum_{k=2}^{r} \phi_k t_k^2 \\ &\quad + \frac{1}{2A} \sum_{i,k=2}^{r} (\phi_i a_i^2 \phi_k t_k^2 + \phi_k a_k^2 \phi_i t_i^2 - 2\phi_i a_i t_i \phi_k a_k t_k) \end{aligned}$$

$$= \sum_{i=r+1}^{n} |\phi_i| \left(t_i - \frac{a_i}{A} B_1\right)^2 + \frac{\phi_1 a_1^2}{A} \sum_{k=2}^{r} \phi_k t_k^2 + \frac{1}{2A} \sum_{\substack{i,k=2 \\ i \neq k}}^{r} \phi_i \phi_k (a_i t_k - a_k t_i)^2.$$

Thus $f(t_2, \ldots, t_n)$ is a positive-definite quadratic form satisfying

$$d^2 \sum_{i=1}^{r} \phi_i b_i^2 \leq \frac{A}{a_1^4} \left(f(t_2, \ldots, t_n)\right)^2. \tag{3.27}$$

It is shown in Section 5 that

$$\det(f) = \frac{a_1^2}{A} |\phi_1 \phi_2 \ldots \phi_n|. \tag{3.28}$$

Next, with the notation of Section 2, we have by the Lemma

$$t_i = d_i z_i, \quad i = 2, \ldots, n,$$

and

$$\begin{pmatrix} z_2 \\ \vdots \\ z_n \end{pmatrix} = P \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix},$$

where P is a lower triangular integral matrix with

$$\det P = d_2' \cdots d_n'.$$

In addition

$$y_i = u_i x_{i-1} + \sum_{k=i}^{n-1} \frac{a_i a_1^{k-2}}{D_k} v_{k+1} x_k + \frac{a_i a_1^{n-2}}{D_n} t_{n-1}, \quad i = 2, \ldots, n,$$

$$y_1 = \sum_{k=1}^{n-1} \frac{a_1^{k-1}}{D_k} v_{k+1} x_k + \frac{a_1^{n-1}}{D_n} t_{n-1}.$$

Thus

$$\sum_{i=1}^{n} \phi_i y_i^2 \equiv \sum_{i=1}^{n} \phi_i y_i \equiv B_1 x_1 + B_2 x_2 + \cdots + B_{n-1} x_{n-1} + t_{n-1} \frac{a_1^{n-2}}{D_n} \sum_{i=1}^{n} \phi_i a_i \pmod 2$$

for some integers B_i. Since $0 = \sum_{i=1}^{n} \phi_i a_i^2 \equiv \sum_{i=1}^{n} \phi_i a_i \pmod 2$ we have $u = \sum_{i=1}^{n} \phi_i y_i^2 \equiv B_1 x_1 + B_2 x_2 + \cdots + B_{n-1} x_{n-1} \pmod 2$. If B_{n-1} is odd, we choose $x_{n-1} =$

$B_1x_1 + B_2x_2 + \cdots + B_{n-2}x_{n-2} + 2x'_{n-1}$ so that $u \equiv 0 \pmod 2$ and in variables $x'_1 = x_1, \ldots, x'_{n-2} = x_{n-2}, x'_{n-1}$, we have

$$\begin{pmatrix} z_2 \\ \vdots \\ z_n \end{pmatrix} = P' \begin{pmatrix} x'_1 \\ \vdots \\ x'_{n-1} \end{pmatrix},$$

where P' is another lower triangular integral matrix with

$$\det(P') = 2\det(P).$$

If B_{n-1} is even and B_{n-2} is odd, we choose $x_{n-2} = B_1x_1 + \cdots + B_{n-3}x_{n-3} + 2x'_{n-2}$ so that $u \equiv 0 \pmod 2$ and in variables $x'_1 = x_1, \ldots, x'_{n-3} = x_{n-3}, x'_{n-2}, x'_{n-1} = x_{n-1}$, we have the same result as above. Continuing in this way, if one of the B_i is odd, we obtain a lower triangular integral matrix P' such that

$$\begin{pmatrix} z_2 \\ \vdots \\ z_n \end{pmatrix} = P' \begin{pmatrix} x'_1 \\ \vdots \\ x'_{n-1} \end{pmatrix}, \quad \det(P') = 2d'_2d'_3 \cdots d'_n,$$

and, for any integers x'_i, we have $u = \sum_{i=1}^{n} \phi_i y_i^2 \equiv 0 \pmod 2$. If all of the B_i are even we have such a P' with $\det(P') = \det(P) = d'_2d'_3 \cdots d'_n$.

Now $f(t_2, \ldots, t_n)$ is a positive definite quadratic form in the variables $x'_1, x'_2, \ldots, x'_{n-1}$, and its determinant is by (3.28)

$$\text{(3.29)} \qquad \det\big(f(x'_1, x'_2, \ldots, x'_{n-1})\big) = \frac{a_1^2}{A}|\phi_1\phi_2\cdots\phi_n|(2d'_2\cdots d'_n\, d_2\cdots d_n)^2 \quad \text{or}$$
$$\frac{a_1^2}{A}|\phi_1\phi_2\cdots\phi_n|(d'_2\cdots d'_n\, d_2\cdots d_n)^2.$$

By the definition of γ_{n-1}, there exist integers $x'_1, \ldots, x'_{n-1}$, not all zero, such that

$$\text{(3.30)} \qquad 0 < f(t_2, \ldots, t_n) \le \gamma_{n-1}\left[\frac{4a_1^2}{A}|\phi_1\phi_2\cdots\phi_n|D_n^2\right]^{1/n-1}.$$

So taking $d = 2$, we have by (2.7), (3.27) and (3.30)

$$\sum_{i=1}^{r} \phi_i b_i^2 \le \frac{1}{4}\frac{A}{a_1^4}\gamma_{n-1}^2\left[\frac{4a_1^{2(n-1)}}{A}|\phi_1\phi_2\cdots\phi_n|\right]^{2/n-1}$$
$$= A\left[\frac{2^{3-n}}{A}\gamma_{n-1}^{n-1}|\phi_1\phi_2\cdots\phi_n|\right]^{2/n-1} < A.$$

Moreover, as $(x_1', \dots, x_{n-1}') \neq (0, \dots, 0)$, we have $(t_2, \dots, t_n) \neq (0, \dots, 0)$. We now show that $b_1 \neq 0$. For if $b_1 = 0$ then, by (1.1), (3.3), (3.5) and (3.12), we have

$$\begin{aligned}
\sum_{i=2}^{n} \phi_i t_i^2 = \sum_{i=1}^{n} \phi_i t_i^2 = \sum_{i=1}^{n} \phi_i (a_1 y_i - a_i y_1)^2 \\
&= a_1^2 \sum_{i=1}^{n} \phi_i y_i^2 - 2a_1 y_1 \sum_{i=1}^{n} \phi_i a_i y_i + y_1^2 \sum_{i=1}^{n} \phi_i a_i^2 \\
&= a_1^2 u - 2a_1 y_1 \left(\frac{-v_1}{2} \right) \\
&= a_1 (u a_1 + v y_1) \\
&= d a_1 b_1 \\
&= 0.
\end{aligned}$$

Hence (1.1) has the solution $(0, t_2, \dots, t_n) \neq (0, \dots, 0)$ contradicting assumption (ii). Thus $b_1 \neq 0$ and

$$0 < \sum_{i=1}^{r} \phi_i b_i^2 < A.$$

Hence (3.9) holds and the proposition follows. ∎

4 Proof of Theorem

Let $\phi_1, \dots, \phi_n$ $(n \geq 2)$ be nonzero integers such that the equation

$$\phi_1 x_1^2 + \cdots + \phi_n x_n^2 = 0 \tag{4.1}$$

is solvable in integers $x_1, \dots, x_n$ not all zero. Let l be the largest integer for which there exists a solution $(x_1, \dots, x_n) \neq (0, \dots, 0)$ in Z^n of (4.1) with l of the x_i equal to 0. (If every nonzero solution in Z^n of (4.1) has $x_i \neq 0$ $(i = 1, \dots, n)$ then $l = 0$). Clearly

$$0 \leq l \leq n - 2. \tag{4.2}$$

Relabelling $\phi_1, \dots, \phi_n$, if necessary, we may suppose that such a solution has

$$x_{n-l+1} = \cdots = x_n = 0. \tag{4.3}$$

Set $k = n - l$ so that from (4.2) we have

$$2 \leq k \leq n. \tag{4.4}$$

Then the equation

$$\phi_1 x_1^2 + \cdots + \phi_k x_k^2 = 0 \tag{4.5}$$

is solvable in integers not all zero, and moreover, by the maximality of l, every solution $(x_1, \ldots, x_k) \neq (0, \ldots, 0)$ of (4.5) has $x_i \neq 0$ $(i = 1, 2, \ldots, k)$. Reordering $\phi_1, \ldots, \phi_k$, if necessary, we may suppose that $\phi_1 > 0$ and $\phi_2 < 0$. Suppose $k \geq 6$. It is known (see for example [2, pp. 69–70]), that there exist integers $y_1, \ldots, y_5$ not all zero such that

$$\phi_1 y_1^2 + \cdots + \phi_5 y_5^2 = 0.$$

Then the equation (4.5) has the solution

$$(x_1, \ldots, x_k) = (y_1, \ldots, y_5, 0, \ldots, 0) \neq (0, \ldots, 0),$$

a contradiction. Hence $k \leq 5$ so that (4.4) can be improved to

$$2 \leq k \leq \min(5, n).$$

If $k = 2$ or 3 then, by the remarks in Section 1, (4.5) has a solution satisfying

$$\begin{cases} 0 < |\phi_1|x_1^2 + |\phi_2|x_2^2 \leq 2|\phi_1\phi_2|, & \text{if } k = 2, \\ 0 < |\phi_1|x_1^2 + |\phi_2|x_2^2 + |\phi_3|x_3^2 \leq 2|\phi_1\phi_2\phi_3|, & \text{if } k = 3, \end{cases}$$

and thus

$$(x_1, \ldots, x_n) = \begin{cases} (x_1, x_2, 0, \ldots, 0), & \text{if } k = 2, \\ (x_1, x_2, x_3, 0, \ldots, 0), & \text{if } k = 3, \end{cases}$$

is a solution of (4.1) satisfying

$$0 < |\phi_1|x_1^2 + \cdots + |\phi_n|x_n^2 \leq 2|\phi_1 \cdots \phi_n|. \tag{4.6}$$

If $k = 4$ or 5 then, by the Proposition of Section 3, (4.5) has a solution satisfying

$$0 < |\phi_1|x_1^2 + \cdots + |\phi_4|x_4^2 \leq \gamma_3^3|\phi_1\phi_2\phi_3\phi_4| = 2|\phi_1\phi_2\phi_3\phi_4|, \quad \text{if } k = 4,$$

$$0 < |\phi_1|x_1^2 + \cdots + |\phi_5|x_5^2 \leq 2^{-1}\gamma_4^4|\phi_1\phi_2\phi_3\phi_4\phi_5| = 2|\phi_1\phi_2\phi_3\phi_4\phi_5|, \quad \text{if } k = 5,$$

by (1.3). Then, exactly as for $k = 2$ or 3, (4.1) has a solution satisfying (4.6). ■

5 Calculation of $\det(f)$

Recall from (3.13) that $B_1 = \sum_{i=2}^{r} \phi_i a_i t_i$. Under the transformation

$$T_i = \begin{cases} t_i, & i = 2, \ldots, r, \\ t_i - \frac{a_i}{A}B_1, & i = r+1, \ldots, n, \end{cases}$$

the form

$$f(t_2, \ldots, t_n) = \sum_{i=r+1}^{n} |\phi_i| \left(t_i - \frac{a_i}{A}B_1\right)^2 + \sum_{i=2}^{r} \phi_i \left(1 - \frac{\phi_i a_i^2}{A}\right) t_i^2 - \sum_{\substack{i,j=2 \\ i \neq j}}^{r} \frac{\phi_i \phi_j a_i a_j}{A} t_i t_j,$$

which was defined in Section 3, becomes the form

$$g(T_2,\dots,T_n)=\sum_{i=2}^{r}\phi_i\left(1-\frac{\phi_i a_i^2}{A}\right)T_i^2+\sum_{i=r+1}^{n}|\phi_i|T_i^2-\sum_{\substack{i,j=2\\ i\neq j}}^{r}\frac{\phi_i\phi_j a_i a_j}{A}T_iT_j.$$

Clearly we have

$$\begin{pmatrix}T_2\\ \vdots\\ T_n\end{pmatrix}=S\begin{pmatrix}t_2\\ \vdots\\ t_n\end{pmatrix},$$

where the $(n-1)\times(n-1)$ matrix S is given by

$$S=\begin{pmatrix} I_{r-1} & \vdots & 0\\ \cdots\cdots & \cdots\cdots & \cdots\cdots\\ \star & \vdots & I_{n-r}\end{pmatrix}.$$

Thus $\det S=1$ and so

(5.1) $$\det(f)=\deg(g).$$

Now

$$\det(g)=\det\begin{pmatrix} C & \vdots & 0\\ \cdots\cdots & \cdots\cdots & \cdots\cdots\\ 0 & \vdots & C'\end{pmatrix},$$

where C is the matrix of the form

$$\sum_{i=2}^{r}\phi_i\left(1-\frac{\phi_i a_i^2}{A}\right)T_i^2-\sum_{\substack{i,j=2\\ i\neq j}}^{r}\frac{\phi_i\phi_j a_i a_j}{A}T_iT_j,$$

and C' is the matrix of the form

$$\sum_{i=r+1}^{n}|\phi_i|T_i^2.$$

Hence

(5.2) $$\det(g)=\det C\det C'=|\phi_{r+1}|\cdots|\phi_n|\det C.$$

Now

$$C=(c_{ij})_{i,j=2,\dots,r},$$

where

$$c_{ii}=\phi_i\left(1-\frac{\phi_i a_i^2}{A}\right),\quad i=2,\dots,r,$$

$$c_{ij}=-\frac{\phi_i\phi_j a_i a_j}{A},\quad i,j=2,\dots,r,\quad i\neq j.$$

Removing a common factor of $\phi_i a_i/A$ from the i-th row of C for $i = 2, \ldots, r$, we obtain

$$\det C = \frac{(\phi_2 \cdots \phi_r)(a_2 \cdots a_r)}{A^{r-1}} \det D, \tag{5.3}$$

where $D = (d_{ij})_{i,j=2,\ldots,r}$ is given by

$$d_{ii} = \frac{A}{a_i} - \phi_i a_i, \quad i = 2, \ldots, r,$$
$$d_{ij} = -\phi_j a_j, \quad i, j = 2, \ldots, r,\ i \neq j.$$

Removing a common factor $\phi_j a_j$ from the j-th column of D for $j = 2, \ldots, r$, we have

$$\det D = (\phi_2 \cdots \phi_r)(a_2 \cdots a_r) \det E, \tag{5.4}$$

where $E = (e_{ij})_{i,j=2,\ldots,r}$ is given by

$$e_{ii} = \frac{A}{\phi_i a_i^2} - 1, \quad i = 2, \ldots, r,$$
$$e_{ij} = -1, \quad i, j = 2, \ldots, r,\ i \neq j.$$

Next we define a $r \times r$ matrix $F = (f_{ij})_{i,j=1,\ldots,r}$ by

$$f_{1j} = 1, \quad j = 1, \ldots, r,$$
$$f_{i1} = 0, \quad i = 2, \ldots, r,$$
$$f_{ij} = e_{ij}, \quad i, j = 2, \ldots, r.$$

Clearly F is formed from E by adjoining a first column $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and a first row $(1\ 1 \ldots 1)$. Hence

$$\det E = \det F. \tag{5.5}$$

Adding the first row of F to each of the other rows, we obtain

$$\det F = \det G, \tag{5.6}$$

where $G = (g_{ij})_{i,j=1,\ldots,r}$ is given by

$$\begin{aligned} g_{i1} &= 1, & i &= 1, \ldots, r, \\ g_{1j} &= 1, & j &= 2, \ldots, r, \\ g_{ii} &= \tfrac{A}{\phi_i a_i^2}, & i &= 2, \ldots, r, \\ g_{ij} &= 0, & i, j &= 2, \ldots, r,\ i \neq j. \end{aligned}$$

Forming a new first column of G as

$$(\text{col } 1) - \frac{\phi_2 a_2^2}{A}(\text{col } 2) - \cdots - \frac{\phi_r a_r^2}{A}(\text{col } r),$$

we obtain

$$\det G = \det H, \tag{5.7}$$

where $H = (h_{ij})$ is given by

$$\begin{aligned} h_{11} &= 1 - \tfrac{\phi_2 a_2^2}{A} - \cdots - \tfrac{\phi_r a_r^2}{A}, \\ h_{i1} &= 0, && i = 2, \ldots, r, \\ h_{1j} &= 1, && j = 2, \ldots, r, \\ h_{ii} &= \tfrac{A}{\phi_i a_i^2}, && i = 2, \ldots, r, \\ h_{ij} &= 0, && i, j = 2, \ldots, r,\ i \neq j. \end{aligned}$$

Clearly, as H is upper triangular, we have

$$\det H = \left(1 - \frac{\phi_2 a_2^2}{A} - \cdots - \frac{\phi_r a_r^2}{A}\right)\left(\frac{A}{\phi_2 a_2^2}\right)\cdots\left(\frac{A}{\phi_r a_r^2}\right),$$

that is

$$\det H = \left(\frac{\phi_1 a_1^2}{A}\right) A^{r-1}(\phi_1 \cdots \phi_r)^{-1}(a_2 \cdots a_r)^{-2}. \tag{5.8}$$

From (5.1)–(5.8) we deduce that

$$\det(f) = \frac{a_1^2}{A}|\phi_1 \cdots \phi_n|, \tag{5.9}$$

as asserted in (3.28). ∎

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