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SUMS OF SIXTEEN SQUARES

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Abstract

For positive integers n and k, we let $r_k(n)$ denote the number of representations of n as the sum of k squares. In 1987 Ewell used modular functions to give a formula for $r_{16}(n)$. In 1996 Milne used elliptic functions to give a different formula for $r_{16}(n)$. In this paper, we give elementary arithmetic proofs of both of these formulae.

1. Introduction

Let $\mathbb{N}$ denote the set of all positive integers, $\mathbb{Z}$ the set of all integers, and $\mathbb{Q}$ the set of all rational numbers. For $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$ we let $r_k(n)$ denote the number of representations of n as the sum of k squares, that is

$$r_k(n) = \sum_{\substack{(x_1, \ldots, x_k) \in \mathbb{Z}^k \\ x_1^2 + \cdots + x_k^2 = n}} 1,$$

so that $r_k(0) = 1$. The following formulae for $r_2(n)$, $r_4(n)$ and $r_8(n)$ $(n \in \mathbb{N})$ are classical:

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$$r_2(n) = 4\sum_{\substack{d \mid n \\ 2 \nmid d}} (-1)^{(d-1)/2}, \tag{1.1}$$

$$r_4(n) = 8\sum_{\substack{d \mid n \\ 4 \nmid d}} d, \tag{1.2}$$

$$r_8(n) = 16(-1)^n \sum_{d \mid n} (-1)^d d^3. \tag{1.3}$$

Each of (1.1), (1.2) and (1.3) can be proved by entirely elementary means, see for example [2], [5], [6] and [7]. For $k \in \mathbb{N}$ and $x \in \mathbb{Q}$ we define

$$\sigma_k(x) := \begin{cases} \sum_{d \mid x} d^k, & \text{if } x \in \mathbb{N}, \\ 0, & \text{if } x \in \mathbb{Q},\ x \notin \mathbb{N}, \end{cases}$$

and

$$\sigma(x) := \sigma_1(x).$$

With this notation we can rewrite (1.2) and (1.3) as

$$r_4(n) = 8\left(\sigma(n) - 4\sigma\left(\frac{n}{4}\right)\right), \tag{1.4}$$

$$r_8(n) = 16(-1)^{n-1}\left(\sigma_3(n) - 16\sigma_3\left(\frac{n}{2}\right)\right). \tag{1.5}$$

Formulae for $r_{16}(n)$ have been given by Ewell [1] and Milne [4, formula (2) and Theorem 1.4]. Their proofs use modular functions and elliptic functions respectively and so are not elementary.

Ewell's formula. Let $n \in \mathbb{N}$. Define $\beta(n) \in \mathbb{N} \cup \{0\}$ and $\gamma(n) \in \mathbb{N}$ by

$$n = 2^{\beta(n)}\gamma(n), \quad 2 \nmid \gamma(n).$$

Then

$$r_{16}(n) = \frac{32}{17}\sigma_7(n) - \frac{64}{17}\sigma_7\left(\frac{n}{2}\right) + \frac{8192}{17}\sigma_7\left(\frac{n}{4}\right)$$

$$+(-1)^{n-1}\frac{512}{17}2^{3\beta(n)}\sigma_3(\gamma(n))$$

$$+(-1)^{n-1}\frac{8192}{17}\sum_{d=1}^{n-1}(-1)^d d^3 \sum_{k<n/d} 2^{3\beta(n-kd)}\sigma_3(\gamma(n-kd)). \quad (1.6)$$

Milne's formula. Let $n \in \mathbb{N}$. Then

$$r_{16}(n) = \frac{32}{3}(-1)^{n-1}\left(\sum_{d|n}(-1)^{d+(n/d)}d + \sum_{d|n}(-1)^{d+(n/d)}d^3 + \sum_{d|n}(-1)^{d+(n/d)}d^5\right)$$

$$+\frac{256}{3}(-1)^n\left(\sum_{ax+by=n}(-1)^{a+b+x+y}ab^5 - \sum_{ax+by=n}(-1)^{a+b+x+y}a^3b^3\right), \quad (1.7)$$

where the latter two sums are over all positive integers a, b, x, y satisfying $ax + by = n$.

In this paper, we show that both Ewell's formula (1.6) and Milne's formula (1.7) can be proved by entirely elementary means. The main tool used in doing this is the following recent identity due to Huard, Ou, Spearman and Williams [2, Theorem 1], the proof of which involves nothing more than the manipulation of finite sums.

Proposition. *Let $f : \mathbb{Z}^4 \to \mathbb{C}$ be such that*

$$f(a, b, x, y) - f(x, y, a, b) = f(-a, -b, x, y) - f(x, y, -a, -b) \quad (1.8)$$

for all integers a, b, x and y. Then

$$\sum_{ax+by=n}(f(a, b, x, -y) - f(a, -b, x, y) + f(a, a-b, x+y, y)$$

$$- f(a, a+b, y-x, y) + f(b-a, b, x, x+y) - f(a+b, b, x, x-y))$$

$$= \sum_{d|n}\sum_{x=1}^{d-1}(f(0, n/d, x, d) + f(n/d, 0, d, x) + f(n/d, n/d, d-x, -x)$$

$$- f(x, x-d, n/d, n/d) - f(x, d, 0, n/d) - f(d, x, n/d, 0)), \quad (1.9)$$

where the sum on the left hand side of (1.9) *is over all positive integers a, b, x, y satisfying $ax + by = n$.*

2. Elementary Lemmas

In this section, we state without proof three easily-proved elementary lemmas.

Lemma 1. *Let $n \in \mathbb{N}$. Then*

$$2^{3\beta(n)}\sigma_3(\gamma(n)) = \sigma_3(n) - \sigma_3\left(\frac{n}{2}\right).$$

Lemma 2. *Let $e, n \in \mathbb{N}$. Then*

$$\sigma_e\left(\frac{n}{4}\right) = \frac{(2^e+1)}{2^e}\sigma_e\left(\frac{n}{2}\right) - \frac{(1+(-1)^n)}{2^{e+1}}\sigma_e(n).$$

Lemma 3. *Let $e, n \in \mathbb{N}$. Then*

$$\sum_{d \mid n} (-1)^d d^e = 2^{e+1}\sigma_e\left(\frac{n}{2}\right) - \sigma_e(n),$$

$$\sum_{d \mid n} (-1)^{n/d} d^e = 2\sigma_e\left(\frac{n}{2}\right) - \sigma_e(n),$$

$$\sum_{d \mid n} (-1)^{d+(n/d)} d^e = (2^{e+1}+2)\sigma_e\left(\frac{n}{2}\right) - (1+2(-1)^n)\sigma_e(n),$$

$$\sum_{\substack{d \mid n \\ 2 \mid d}} d^e = 2^e\sigma_e\left(\frac{n}{2}\right),$$

$$\sum_{\substack{d \mid n \\ 2 \mid d}} (-1)^{n/d} d^e = (2^e+2)\sigma_e\left(\frac{n}{2}\right) - (1+(-1)^n)\sigma_e(n).$$

3. The Sums $S_{e,f}(n)$, $A_{e,f}(n)$ and $B_{e,f}(n)$

For $e, f, n \in \mathbb{N}$, we define

$$S_{e,f}(n) := \sum_{m=1}^{n-1} \sigma_e(m)\sigma_f(n-m) = \sum_{ax+by=n} a^e b^f. \tag{3.1}$$

Clearly

$$S_{e,f}(n) = S_{f,e}(n) = \sum_{a=1}^{n-1} a^e \sum_{m<n/a} \sigma_f(n-am), \tag{3.2}$$

where m runs through the positive integers satisfying $m < n/a$. Also, for $e, f, n \in \mathbb{N}$, we define

$$A_{e,f}(n) := \sum_{m<n/2} \sigma_e(m)\sigma_f(n-2m) = \sum_{2ax+by=n} a^e b^f, \tag{3.3}$$

where m runs through the positive integers satisfying $m < n/2$. We note that

$$\sum_{a<n/2} a^e \sum_{m<n/2a} \sigma_f(n-2am) = A_{e,f}(n) \tag{3.4}$$

and

$$\sum_{a=1}^{n-1} a^e \sum_{m<n/a} \sigma_f\left(\frac{n-am}{2}\right) = A_{f,e}(n). \tag{3.5}$$

The next theorem is elementary and its proof omitted. It is a simple application of the inclusion-exclusion principle.

Theorem 1. *Let $e, f, n \in \mathbb{N}$. Then*

$$\sum_{ax+by=n} (-1)^a a^e b^f = 2^{e+1} A_{e,f}(n) - S_{e,f}(n),$$

$$\sum_{ax+by=n} (-1)^x a^e b^f = 2A_{e,f}(n) - S_{e,f}(n),$$

$$\sum_{ax+by=n} (-1)^{a+b} a^e b^f = 2^{e+f+2} S_{e,f}\left(\frac{n}{2}\right) - 2^{e+1} A_{e,f}(n) - 2^{f+1} A_{f,e}(n) + S_{e,f}(n),$$

$$\sum_{ax+by=n} (-1)^{a+y} a^e b^f = 2^{e+2} S_{e,f}\left(\frac{n}{2}\right) - 2^{e+1} A_{e,f}(n) - 2A_{f,e}(n) + S_{e,f}(n),$$

$$\sum_{ax+by=n} (-1)^{x+y} a^e b^f = 4S_{e,f}\left(\frac{n}{2}\right) - 2A_{e,f}(n) - 2A_{f,e}(n) + S_{e,f}(n).$$

The sum $\sum_{ax+by=n} (-1)^{a+x} a^e b^f$ can be treated in a similar fashion but applying the inclusion-exclusion principle to it leads to sums of the type

$$B_{e,f}(n) := \sum_{m<n/4} \sigma_e(m)\sigma_f(n-4m) = \sum_{4ax+by=n} a^e b^f.$$

Some results concerning these sums are given in Section 6.

The sums $S_{e,f}(n)$ can be evaluated in an elementary manner for $e, f \in \mathbb{N}$ satisfying

$$e \equiv f \equiv 1 \pmod 2, \quad e + f = 2, 4, 6, 8, 12,$$

by taking particular choices of $f(a, b, x, y)$ in the Proposition, see [2]. We need the values of $S_{1,5}(n)$ and $S_{3,3}(n)$.

Theorem 2. *Let* $n \in \mathbb{N}$. *Then*

$$S_{1,5}(n) = \frac{5}{126}\sigma_7(n) + \frac{1}{24}(1-2n)\sigma_5(n) + \frac{1}{504}\sigma(n),$$

$$S_{3,3}(n) = \frac{1}{120}\sigma_7(n) - \frac{1}{120}\sigma_3(n).$$

The values of $A_{1,1}(n)$, $A_{1,3}(n)$ and $A_{3,1}(n)$ were derived in [2, Theorems 2 and 6] in an elementary manner from the Proposition. The values of $A_{1,5}(n)$, $A_{3,3}(n)$ and $A_{5,1}(n)$ are not known explicitly, however two linear relations between them were proved in [2, Theorem 15] in an elementary fashion from the Proposition. These relations are given in Theorem 3 in a slightly rearranged form.

Theorem 3. *For* $n \in \mathbb{N}$, *we have*

$$A_{1,5}(n) = \frac{1}{90}\sigma_7(n) + \frac{1}{24}(1-n)\sigma_5(n) - \frac{1}{90}\sigma_3(n)$$

$$+ \frac{16}{315}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{90}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{504}\sigma\left(\frac{n}{2}\right) - \frac{8}{3}A_{3,3}(n)$$

and

$$A_{5,1}(n) = \frac{1}{1260}\sigma_7(n) - \frac{1}{360}\sigma_3(n) + \frac{1}{504}\sigma(n)$$

$$+ \frac{2}{45}\sigma_7\left(\frac{n}{2}\right) + \frac{1}{24}(1-2n)\sigma_5\left(\frac{n}{2}\right) - \frac{1}{360}\sigma_3\left(\frac{n}{2}\right) - \frac{2}{3}A_{3,3}(n).$$

From Theorems 1, 2 and 3, we obtain the following evaluations in terms of $A_{3,3}(n)$.

Theorem 4. *Let* $n \in \mathbb{N}$. *Then*

$$\sum_{ax+by=n} (-1)^{b+x} ab^5 = -\frac{1}{30}\sigma_7(n) - \frac{1}{24}\sigma_5(n) + \frac{1}{5}\sigma_3(n) - \frac{1}{8}\sigma(n) + \frac{32}{15}\sigma_7\left(\frac{n}{2}\right)$$

$$+ \frac{8}{3}\sigma_5\left(\frac{n}{2}\right) + \frac{1}{5}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{4}\sigma\left(\frac{n}{2}\right) + 48A_{3,3}(n),$$

$$\sum_{ax+by=n} (-1)^{b+x} a^3b^3 = \frac{1}{120}\sigma_7(n) - \frac{1}{120}\sigma_3(n) + \frac{4}{15}\sigma_7\left(\frac{n}{2}\right)$$

$$- \frac{4}{15}\sigma_3\left(\frac{n}{2}\right) - 18A_{3,3}(n),$$

$$\sum_{ax+by=n} (-1)^{b+x} a^5b = -\frac{2}{315}\sigma_7(n) + \frac{(2n-3)}{24}\sigma_5(n) + \frac{1}{20}\sigma_3(n) - \frac{1}{504}\sigma(n)$$

$$+ \frac{8}{315}\sigma_7\left(\frac{n}{2}\right) + \frac{(3-2n)}{12}\sigma_5\left(\frac{n}{2}\right) + \frac{1}{20}\sigma_3\left(\frac{n}{2}\right)$$

$$+ \frac{1}{126}\sigma\left(\frac{n}{2}\right) + 12A_{3,3}(n).$$

In order to prove Milne's formula we express the last two sums in (1.7) in terms of $A_{3,3}(n)$ by means of the Proposition.

Theorem 5. *For* $n \in \mathbb{N}$, *we have*

$$\sum_{ax+by=n} (-1)^{a+b+x+y} ab^5 = \left(-\frac{1}{8} - \frac{1}{4}(-1)^n\right)\sigma_5(n)$$

$$-\frac{1}{4}\sigma_3(n)+\left(-\frac{1}{8}-\frac{1}{4}(-1)^n\right)\sigma(n)$$

$$+8\sigma_7\left(\frac{n}{2}\right)+\frac{33}{4}\sigma_5\left(\frac{n}{2}\right)-\frac{1}{4}\sigma_3\left(\frac{n}{2}\right)$$

$$+\frac{3}{4}\sigma\left(\frac{n}{2}\right)-60A_{3,3}(n)$$

and

$$\sum_{ax+by=n}(-1)^{a+b+x+y}a^3b^3=-\frac{1}{40}\sigma_7(n)+\left(\frac{11}{40}+\frac{1}{4}(-1)^n\right)\sigma_3(n)+\frac{8}{5}\sigma_7\left(\frac{n}{2}\right)$$

$$-\frac{21}{10}\sigma_3\left(\frac{n}{2}\right)+36A_{3,3}(n).$$

Proof. To prove the first of the two equalities we take $f(a, b, x, y) = (-1)^{a+b+x+y}ab^5$ in the Proposition, and to prove the second we take $f(a, b, x, y) = (-1)^{a+b+x+y}a^3b^3$. Both choices satisfy condition (1.8). We just give the details in the first case. In this case, the first two terms on the left hand side of (1.9) give

$$2\sum_{ax+by=n}(-1)^{a+b+x+y}ab^5.$$

The third and fourth terms on the left hand side of (1.9) give by Theorem 4,

$$\sum_{ax+by=n}(-1)^{b+x}a((a-b)^5-(a+b)^5)$$

$$=-2\sum_{ax+by=n}(-1)^{b+x}ab^5-20\sum_{ax+by=n}(-1)^{b+x}a^3b^3-10\sum_{ax+by=n}(-1)^{b+x}a^5b$$

$$=-\frac{23}{630}\sigma_7(n)+\frac{(8-5n)}{6}\sigma_5(n)-\frac{11}{15}\sigma_3(n)+\frac{17}{63}\sigma(n)-\frac{3104}{315}\sigma_7\left(\frac{n}{2}\right)$$

$$+\frac{(10n-47)}{6}\sigma_5\left(\frac{n}{2}\right)+\frac{133}{30}\sigma_3\left(\frac{n}{2}\right)-\frac{73}{126}\sigma\left(\frac{n}{2}\right)+144A_{3,3}(n).$$

The fifth and sixth terms on the left hand side of (1.9) give by Theorem 4,

$$\sum_{ax+by=n} (-1)^{a+y}((b-a)-(a+b))\,b^5$$

$$= -2 \sum_{ax+by=n} (-1)^{a+y} ab^5$$

$$= -2 \sum_{ax+by=n} (-1)^{b+x} a^5 b$$

$$= \frac{4}{315}\sigma_7(n) + \frac{(3-2n)}{12}\sigma_5(n) - \frac{1}{10}\sigma_3(n) + \frac{1}{252}\sigma(n) - \frac{16}{315}\sigma_7\left(\frac{n}{2}\right)$$

$$+ \frac{(2n-3)}{6}\sigma_5\left(\frac{n}{2}\right) - \frac{1}{10}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{63}\sigma\left(\frac{n}{2}\right) - 24A_{3,3}(n).$$

The first two terms on the right hand side of (1.9) vanish. The third term yields by Lemma 3,

$$\sum_{d\mid n}\sum_{x=1}^{d-1}(-1)^d\left(\frac{n}{d}\right)^6 = \sum_{d\mid n}(-1)^d\left(\frac{n}{d}\right)^6(d-1)$$

$$= n\sum_{d\mid n}(-1)^{n/d}d^5 - \sum_{d\mid n}(-1)^{n/d}d^6$$

$$= n\left(2\sigma_5\left(\frac{n}{2}\right) - \sigma_5(n)\right) - \left(2\sigma_6\left(\frac{n}{2}\right) - \sigma_6(n)\right).$$

The fourth term is by Bernoulli's formula for the sum $\sum_{x=1}^{d-1} x^k$ $(k = 1, \ldots, 6)$

and Lemma 3,

$$-\sum_{d\mid n}\sum_{x=1}^{d-1}(-1)^d x(x-d)^5$$

$$= -\sum_{d\mid n}(-1)^d\sum_{x=1}^{d-1}(x^6 - 5x^5d + 10x^4d^2 - 10x^3d^3 + 5x^2d^4 - xd^5)$$

$$= \frac{1}{42}\sum_{d|n}(-1)^d d^7 - \frac{1}{12}\sum_{d|n}(-1)^d d^5 + \frac{1}{12}\sum_{d|n}(-1)^d d^3 - \frac{1}{42}\sum_{d|n}(-1)^d d$$

$$= -\frac{1}{42}\sigma_7(n) + \frac{1}{12}\sigma_5(n) - \frac{1}{12}\sigma_3(n) + \frac{1}{42}\sigma(n)$$

$$+ \frac{128}{21}\sigma_7\left(\frac{n}{2}\right) - \frac{16}{3}\sigma_5\left(\frac{n}{2}\right) + \frac{4}{3}\sigma_3\left(\frac{n}{2}\right) - \frac{2}{21}\sigma\left(\frac{n}{2}\right).$$

The fifth term is by Lemmas 2 and 3,

$$-\sum_{d|n}(-1)^{d+(n/d)}d^5\sum_{x=1}^{d-1}(-1)^x x$$

$$= -\sum_{\substack{d|n\\2|d}}(-1)^{d+(n/d)}d^5\left(-\frac{d}{2}\right) - \sum_{\substack{d|n\\2\nmid d}}(-1)^{d+(n/d)}d^5\left(\frac{d-1}{2}\right)$$

$$= \frac{1}{2}\sum_{d|n}(-1)^{n/d}d^6 - \frac{1}{2}\sum_{\substack{d|n\\2\nmid d}}(-1)^{n/d}d^5$$

$$= \frac{1}{2}\left(2\sigma_6\left(\frac{n}{2}\right) - \sigma_6(n)\right) - \frac{1}{2}\left(2\sigma_5\left(\frac{n}{2}\right) - \sigma_5(n) - \left(34\sigma_5\left(\frac{n}{2}\right) - (1+(-1)^n)\sigma_5(n)\right)\right)$$

$$= -\frac{1}{2}\sigma_6(n) - \frac{(-1)^n}{2}\sigma_5(n) + \sigma_6\left(\frac{n}{2}\right) + 16\sigma_5\left(\frac{n}{2}\right).$$

The sixth term is by Lemma 3 and Euler's formula for the sum $\sum_{x=1}^{d-1}(-1)^x x^k$,

$$-\sum_{d|n}(-1)^{d+(n/d)}d\sum_{x=1}^{d-1}(-1)^x x^5$$

$$= -\sum_{\substack{d|n\\2|d}}(-1)^{d+(n/d)}d\left(-\frac{1}{2}d^5 + \frac{5}{4}d^4 - \frac{5}{4}d^2\right)$$

$$-\sum_{\substack{d|n\\2\nmid d}}(-1)^{d+(n/d)}d\left(\frac{1}{2}d^5 - \frac{5}{4}d^4 + \frac{5}{4}d^2 - \frac{1}{2}\right)$$

$$= \sum_{d\mid n} (-1)^{n/d} d\left(\frac{1}{2}d^5 - \frac{5}{4}d^4 + \frac{5}{4}d^2\right) - \frac{1}{2}\sum_{\substack{d\mid n \\ 2\nmid d}} (-1)^{n/d} d$$

$$= \frac{1}{2}\sum_{d\mid n} (-1)^{n/d} d^6 - \frac{5}{4}\sum_{d\mid n} (-1)^{n/d} d^5 + \frac{5}{4}\sum_{d\mid n} (-1)^{n/d} d^3$$

$$-\frac{1}{2}\sum_{d\mid n} (-1)^{n/d} d + \frac{1}{2}\sum_{\substack{d\mid n \\ 2\mid d}} (-1)^{n/d} d$$

$$= -\frac{1}{2}\sigma_6(n) + \frac{5}{4}\sigma_5(n) - \frac{5}{4}\sigma_3(n) - \frac{(-1)^n}{2}\sigma(n)$$

$$+ \sigma_6\left(\frac{n}{2}\right) - \frac{5}{2}\sigma_5\left(\frac{n}{2}\right) + \frac{5}{2}\sigma_3\left(\frac{n}{2}\right) + \sigma\left(\frac{n}{2}\right).$$

The first equality of the theorem now follows by the Proposition.

Our final theorem of this section makes use of (1.5) and Theorem 2 to express $r_{16}(n)$ in terms of $A_{3,3}(n)$.

Theorem 6. *Let* $n \in \mathbb{N}$. *Then*

$$r_{16}(n) = \frac{32}{15}(-1)^n \sigma_7(n) + \frac{512}{15}(-1)^{n-1}\sigma_3(n) + \frac{8192}{15}\sigma_7\left(\frac{n}{2}\right)$$

$$- \frac{512}{15}\sigma_3\left(\frac{n}{2}\right) + 8192(-1)^{n-1}A_{3,3}(n).$$

Proof. Clearly

$$r_{16}(n) = \sum_{k=0}^{n} r_8(k) r_8(n-k)$$

so that

$$r_{16}(n) = 2r_8(n) + \sum_{k=1}^{n-1} r_8(k) r_8(n-k)$$

as $r_8(0) = 1$. Hence, by (1.5), we have

$$r_{16}(n) = 32(-1)^{n-1}\sigma_3(n) + 512(-1)^n\sigma_3\left(\frac{n}{2}\right)$$

$$+ 256(-1)^n\sum_{k=1}^{n-1}\left(\sigma_3(k) - 16\sigma_3\left(\frac{k}{2}\right)\right)\left(\sigma_3(n-k) - 16\sigma_3\left(\frac{n-k}{2}\right)\right)$$

$$= 32(-1)^{n-1}\sigma_3(n) + 512(-1)^n\sigma_3\left(\frac{n}{2}\right)$$

$$+ 256(-1)^n\left(S_{3,3}(n) - 32A_{3,3}(n) + 256S_{3,3}\left(\frac{n}{2}\right)\right).$$

The result now follows on appealing to Theorem 2 for the values of $S_{3,3}(n)$ and $S_{3,3}\left(\frac{n}{2}\right)$, and noting that $(-1)^n\sigma_e\left(\frac{n}{2}\right) = \sigma_e\left(\frac{n}{2}\right)$.

4. Elementary Proof of Milne's Formula

By Theorem 5, we have

$$\sum_{ax+by=n}(-1)^{a+b+x+y}ab^5 - \sum_{ax+by=n}(-1)^{a+b+x+y}a^3b^3$$

$$= \frac{1}{40}\sigma_7(n) + \left(-\frac{1}{8} - \frac{1}{4}(-1)^n\right)\sigma_5(n) + \left(-\frac{21}{40} - \frac{1}{4}(-1)^n\right)\sigma_3(n)$$

$$+\left(-\frac{1}{8} - \frac{1}{4}(-1)^n\right)\sigma(n) + \frac{32}{5}\sigma_7\left(\frac{n}{2}\right) + \frac{33}{4}\sigma_5\left(\frac{n}{2}\right)$$

$$+ \frac{37}{20}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{4}\sigma\left(\frac{n}{2}\right) - 96A_{3,3}(n).$$

By Lemma 3, we have

$$\sum_{d\mid n}(-1)^{d+(n/d)}d + \sum_{d\mid n}(-1)^{d+(n/d)}d^3 + \sum_{d\mid n}(-1)^{d+(n/d)}d^5$$

$$= -(1+2(-1)^n)\sigma_5(n) - (1+2(-1)^n)\sigma_3(n) - (1+2(-1)^n)\sigma(n)$$

$$+ 66\sigma_5\left(\frac{n}{2}\right) + 18\sigma_3\left(\frac{n}{2}\right) + 6\sigma\left(\frac{n}{2}\right).$$

Hence

$$\frac{32}{3}(-1)^{n-1}\left(\sum_{d|n}(-1)^{d+(n/d)}d+\sum_{d|n}(-1)^{d+(n/d)}d^3+\sum_{d|n}(-1)^{d+(n/d)}d^5\right)$$

$$+\frac{256}{3}(-1)^{n}\left(\sum_{ax+by=n}(-1)^{a+b+x+y}ab^5-\sum_{ax+by=n}(-1)^{a+b+x+y}a^3b^3\right)$$

$$=\frac{32}{15}(-1)^{n}\sigma_7(n)+\frac{512}{15}(-1)^{n-1}\sigma_3(n)+\frac{8192}{15}\sigma_7\left(\frac{n}{2}\right)$$

$$-\frac{512}{15}\sigma_3\left(\frac{n}{2}\right)+8192(-1)^{n-1}A_{3,3}(n)$$

$$=r_{16}(n),$$

by Theorem 6.

5. Elementary Proof of Ewell's Formula

Appealing to Lemma 1, (3.2), (3.4), (3.5) and Theorem 2, we obtain

$$\sum_{d=1}^{n-1}(-1)^d d^3\sum_{k<n/d}2^{3\beta(n-kd)}\sigma_3(\gamma(n-kd))$$

$$=\sum_{d=1}^{n-1}(-1)^d d^3\sum_{k<n/d}\left(\sigma_3(n-kd)-\sigma_3\left(\frac{n-kd}{2}\right)\right)$$

$$=2\sum_{\substack{d=1\\2|d}}^{n-1}d^3\sum_{k<n/d}\left(\sigma_3(n-kd)-\sigma_3\left(\frac{n-kd}{2}\right)\right)$$

$$-\sum_{d=1}^{n-1}d^3\sum_{k<n/d}\left(\sigma_3(n-kd)-\sigma_3\left(\frac{n-kd}{2}\right)\right)$$

$$=16\sum_{d<n/2}d^3\sum_{k<n/2d}\left(\sigma_3(n-2kd)-\sigma_3\left(\frac{n}{2}-kd\right)\right)$$

$$-\sum_{d=1}^{n-1} d^3 \sum_{k<n/d} \left(\sigma_3(n-kd) - \sigma_3\left(\frac{n-kd}{2}\right)\right)$$

$$= -S_{3,3}(n) - 16S_{3,3}\left(\frac{n}{2}\right) + 17A_{3,3}(n)$$

$$= -\frac{1}{120}\sigma_7(n) + \frac{1}{120}\sigma_3(n) - \frac{2}{15}\sigma_7\left(\frac{n}{2}\right) + \frac{2}{15}\sigma_3\left(\frac{n}{2}\right) + 17A_{3,3}(n).$$

Then, appealing to Lemma 1, we obtain as $(-1)^n\sigma_e\left(\frac{n}{2}\right) = \sigma_e\left(\frac{n}{2}\right)$,

$$\frac{32}{17}\sigma_7(n) - \frac{64}{17}\sigma_7\left(\frac{n}{2}\right) + \frac{8192}{17}\sigma_7\left(\frac{n}{4}\right) + (-1)^{n-1}\frac{512}{17}2^{3\beta(n)}\sigma_3(\gamma(n))$$

$$+ (-1)^{n-1}\frac{8192}{17}\sum_{d=1}^{n-1}(-1)^d d^3 \sum_{k<n/d} 2^{3\beta(n-kd)}\sigma_3(\gamma(n-kd))$$

$$= \frac{32}{15}(-1)^n\sigma_7(n) + \frac{512}{15}(-1)^{n-1}\sigma_3(n) + \frac{8192}{15}\sigma_7\left(\frac{n}{2}\right)$$

$$-\frac{512}{15}\sigma_3\left(\frac{n}{2}\right) + 8192(-1)^{n-1}A_{3,3}(n).$$

Ewell's formula now follows by Theorem 6.

6. The Sums $B_{1,5}(n)$, $B_{3,3}(n)$ and $B_{5,1}(n)$

The sum $B_{1,1}(n)$ is determined in [2, Theorem 4] in an elementary way using the Proposition. A linear relation between $B_{1,3}(n)$ and $B_{3,1}(n)$ can also be deduced from the Proposition. The case when n is odd is treated in [3, formula (2.11)]. In this section, we obtain the rather surprising result that $B_{1,5}(n) + 16B_{5,1}(n)$ and $B_{3,3}(n)$ can be expressed in terms of $A_{3,3}(n)$ and $A_{3,3}\left(\frac{n}{2}\right)$.

Similarly to Theorem 1, we can prove

Theorem 7. *For $e, f, n \in \mathbb{N}$, we have*

$$\sum_{ax+by=n} (-1)^{a+b+x+y} a^e b^f$$

$$= 2^{e+f+4} S_{e,f}\left(\frac{n}{4}\right) - 2^{e+3}(2^f+1) A_{e,f}\left(\frac{n}{2}\right) - 2^{f+3}(2^e+1) A_{f,e}\left(\frac{n}{2}\right)$$

$$+ 2^2(2^e+1)(2^f+1) S_{e,f}\left(\frac{n}{2}\right) + 2^{e+2} B_{e,f}(n) + 2^{f+2} B_{f,e}(n)$$

$$- 2(2^{e}+1) A_{e,f}(n) - 2(2^f+1) A_{f,e}(n) + S_{e,f}(n).$$

Proof. We make use of the identity

$$(-1)^{a_1+a_2+a_3+a_4} = \prod_{i=1}^{4}(1+(-1)^{a_i}) - \sum_{j=1}^{4}\prod_{\substack{i=1\\ i\neq j}}^{4}(1+(-1)^{a_i})$$

$$+ \sum_{1\le j<k\le 4}\prod_{\substack{i=1\\ i\neq j,k}}^{4}(1+(-1)^{a_i}) - \sum_{i=1}^{4}(1+(-1)^{a_i}) + 1.$$

Using this identity for $(-1)^{a+b+x+y}$ in the sum $\sum_{ax+by=n} (-1)^{a+b+x+y} a^e b^f$, we obtain sixteen sums. We just evaluate two of them. The rest can be done similarly.

We have

$$\sum_{ax+by=n} (1+(-1)^a)(1+(-1)^b)(1+(-1)^x)(1+(-1)^y) a^e b^f$$

$$= 2^4 \sum_{\substack{ax+by=n\\ 2|a,\ 2|b,\ 2|x,\ 2|y}} a^e b^f$$

$$= 2^{e+f+4} \sum_{ax+by=n/4} a^e b^f$$

$$= 2^{e+f+4} S_{e,f}\left(\frac{n}{4}\right)$$

and

$$\sum_{ax+by=n} (1+(-1)^a)(1+(-1)^b)(1+(-1)^x)a^e b^f$$

$$= 2^3 \sum_{\substack{ax+by=n \\ 2|a,\ 2|b,\ 2|x}} a^e b^f$$

$$= 2^{e+f+3} \sum_{2ax+by=n/2} a^e b^f$$

$$= 2^{e+f+3} A_{e,f}\left(\frac{n}{2}\right).$$

This completes the proof.

Appealing to Theorem 7 with $(e, f) = (1, 5)$ and $(3, 3)$, we obtain

$$\sum_{ax+by=n} (-1)^{a+b+x+y} ab^5 = 1024 S_{1,5}\left(\frac{n}{4}\right) + 396 S_{1,5}\left(\frac{n}{2}\right) + S_{1,5}(n)$$

$$- 528 A_{1,5}\left(\frac{n}{2}\right) - 768 A_{5,1}\left(\frac{n}{2}\right) - 6A_{1,5}(n)$$

$$- 66 A_{5,1}(n) + 8B_{1,5}(n) + 128 B_{5,1}(n)$$

and

$$\sum_{ax+by=n} (-1)^{a+b+x+y} a^3 b^3 = 1024 S_{3,3}\left(\frac{n}{4}\right) + 324 S_{3,3}\left(\frac{n}{2}\right) + S_{3,3}(n)$$

$$- 1152 A_{3,3}\left(\frac{n}{2}\right) - 36 A_{3,3}(n) + 64 B_{3,3}(n).$$

Then, using Theorems 5, 2 and 3 and Lemma 2, we obtain the following evaluations in terms of $A_{3,3}(n)$ and $A_{3,3}\left(\frac{n}{2}\right)$.

Theorem 8. *For* $n \in \mathbb{N}$,

$$B_{1,5}(n) + 16 B_{5,1}(n)$$

$$= -\frac{5}{504}(-1)^n \sigma_7(n) + \left(\frac{1}{32} - (-1)^n \left(\frac{1}{96} - \frac{1}{48} n\right)\right) \sigma_5(n) + \frac{1}{16}(-1)^n \sigma_3(n)$$

$$+\frac{1}{16}(-1)^n\sigma_3(n)+\left(\frac{1}{32}-\frac{1}{2016}(-1)^n\right)\sigma(n)+\frac{59}{21}\sigma_7\left(\frac{n}{2}\right)$$

$$+\left(\frac{11}{16}-\frac{11}{8}n\right)\sigma_5\left(\frac{n}{2}\right)-\frac{35}{16}\sigma_3\left(\frac{n}{2}\right)+\frac{1}{336}\sigma\left(\frac{n}{2}\right)$$

$$-15A_{3,3}(n)-240A_{3,3}\left(\frac{n}{2}\right)$$

and

$$B_{3,3}(n)=\frac{1}{1920}(-1)^n\sigma_7(n)-\left(\frac{1}{256}+\frac{17}{3840}(-1)^n\right)\sigma_3(n)$$

$$-\frac{97}{640}\sigma_7\left(\frac{n}{2}\right)+\frac{51}{320}\sigma_3\left(\frac{n}{2}\right)+\frac{9}{8}A_{3,3}(n)+18A_{3,3}\left(\frac{n}{2}\right).$$

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