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The Index of a Cyclic Quartic Field

By

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Abstract. Let K be a cyclic quartic field. Let $i(K)$ denote the index of K. It is known that $i(K) \in \{1, 2, 3, 4, 6, 12\}$. In Part 1 of this paper we show that $i(K)$ assumes all of these values and we give necessary and sufficient conditions for each to occur. In Part 2 an asymptotic formula is given for the number of cyclic quartic fields with discriminant $\leqslant x$ and $i(K) = i$ for each $i \in \{1, 2, 3, 4, 6, 12\}$.

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Part 1: Characterization of Cyclic Quartic Fields of Given Index

1.1. Introduction. Let K be an algebraic number field of degree n over the rational field $\mathbb{Q}$. Let O_K denote the ring of integers of the field K. Let $\{\omega_1(=1), \omega_2, \ldots, \omega_n\}$ be an integral basis of K. The discriminant of the field K is denoted by $d(K)$. If $\alpha \in K$ we denote the minimal polynomial of α over $\mathbb{Q}$ by f_α and set $d(\alpha) = \operatorname{disc}(f_\alpha)$. An element $\alpha \in O_K$ is called a generator of K if $K = \mathbb{Q}(\alpha)$. It is known that $\alpha \in O_K$ is a generator of K if and only if $d(\alpha) \neq 0$. For α a generator of K, the index $i(\alpha)$ of α is the positive integer given by

$$i(\alpha) = \sqrt{\frac{d(\alpha)}{d(K)}}.$$

If $\alpha \in O_K$ is not a generator of K, we set $i(\alpha) = 0$. Thus

$$d(\alpha) = i(\alpha)^2 d(K) \text{ for all } \alpha \in O_K.$$

The index of K is defined by

$$i(K) = \gcd\{i(\alpha) |\ \alpha \in O_K\} = \gcd\{i(\alpha) |\ \alpha \text{ a generator of } K\}.$$

Next we define the index form $i(x_1, \ldots, x_{n-1})$ of the algebraic number field K with respect to the integral basis $\{\omega_1, \ldots, \omega_n\}$. Let $x_1, \ldots, x_n \in \mathbb{Z}$. Set

$$\theta = x_1\omega_2 + \cdots + x_{n-1}\omega_n \in O_K.$$

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Hence we can define $f_{i1},\ldots,f_{in}\in\mathbb{Z}$ $(i=0,1,\ldots,n-1)$ by

$$\theta^i = f_{i1}\omega_1 + f_{i2}\omega_2 + \cdots + f_{in}\omega_n, \quad i=0,1,\ldots,n-1,$$

so that

$$f_{01}=1,\ f_{02}=0,\ \ldots,\ f_{0n}=0,$$

$$f_{11}=0,\ f_{12}=x_1,\ \ldots,\ f_{1n}=x_{n-1}.$$

Each nonzero f_{ij} is a homogeneous polynomial in $x_1,\ \ldots,x_{n-1}$ with coefficients in $\mathbb{Z}$ of degree i. We define the $n\times n$ matrix F by

$$F=\begin{bmatrix} f_{01} & f_{02} & \cdots & f_{0n} \\ f_{11} & f_{12} & \cdots & f_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ f_{n-11} & f_{n-12} & \cdots & f_{n-1n} \end{bmatrix}.$$

The index form of K with respect to the integral basis $\{\omega_1,\ \ldots,\omega_n\}$ is defined by

$$i(x_1,\ \ldots,x_{n-1}) = \det F = \begin{vmatrix} x_1 & x_2 & \cdots & x_{n-1} \\ f_{22} & f_{23} & \cdots & f_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ f_{n-12} & f_{n-13} & \cdots & f_{n-1n} \end{vmatrix}.$$

Thus $i(x_1,\ \ldots,x_{n-1})$ is a homogeneous polynomial in $x_1,\ldots,x_{n-1}$ of degree $(n-1)n/2$ with coefficients in $\mathbb{Z}$. Next we observe that

$$[1\ \theta\ \cdots\ \theta^{n-1}]^T = F[\omega_1\ \omega_2\ \cdots\ \omega_n]^T,$$

so that

$$i(\theta)^2 d(K) = d(\theta) = (\det F)^2 d(\omega_1,\ \ldots,\omega_n) = i(x_1,\ \ldots,x_{n-1})^2 d(K).$$

Hence $i(\theta) = |\,i(x_1,\ \ldots,x_{n-1})\,|$. Now let $x_0\in\mathbb{Z}$ and set

$$\alpha = x_0\omega_1 + x_1\omega_2 + \cdots + x_{n-1}\omega_n = x_0 + \theta \in O_K.$$

As $d(\alpha) = d(\alpha+k)$ for any integer k, we have $d(\alpha) = d(\alpha - x_0) = d(\theta)$, so that

$$i(\alpha)^2 d(K) = d(\alpha) = d(\theta) = i(\theta)^2 d(K)$$

and thus $i(\alpha) = i(\theta)$. Hence we have

$$i(x_0\omega_1 + \cdots + x_{n-1}\omega_n) = |\,i(x_1,\ \ldots,x_{n-1})\,| \qquad (1.1.1)$$

for all integers $x_0,\ x_1,\ \ldots,\ x_{n-1}$, and

$$i(K) = \gcd\{i(x_1,\ \ldots,x_{n-1})\ |\ x_1,\ \ldots,\ x_{n-1}\in\mathbb{Z}\}. \qquad (1.1.2)$$

If K is a quadratic field (so that $n = 2$) it is easy to show that $i(K) = 1$. If K is a cubic field $(n = 3)$ Engstrom has shown that $i(K) = 1$ or 2 [2, p. 234]. Cubic fields with index 2 have been discussed by the authors in [17]. If K is a quartic field $(n = 4)$ Engstrom has shown that

$$i(K) = 1,\ 2,\ 3,\ 4,\ 6 \text{ or } 12. \tag{1.1.3}$$

Funakura [3, Theorem 5, p. 36] has shown that in the case of a pure quartic field $i(K) = 1$ or 2. Gaál, Pethö and Pohst proved in [4] for bicyclic quartic fields that the index $i(K)$ can take each of the values in (1.1.3) and they gave necessary and sufficient conditions for each case, see also [5–7].

In Part 1 of this paper we show for cyclic quartic fields K that $i(K)$ assumes each of the values 1, 2, 3, 4, 6, 12. In addition we determine necessary and sufficient conditions for each to occur, see Theorem A in Section 1.2.

In Part 2 we determine an asymptotic formula for the number of cyclic quartic fields K with $d(K) \leqslant x$ and $i(K) = i$ for $i \in \{1, 2, 3, 4, 6, 12\}$, see Theorem B in Section 2.1.

1.2. Representation of a cyclic quartic field. Hardy, Hudson, Richman, Williams and Holtz [8] have shown that each cyclic quartic field K can be expressed uniquely in the form

$$K = \mathbb{Q}\left(\sqrt{A(D + B\sqrt{D})}\right), \tag{1.2.1}$$

where A, B, C, D are integers such that

$$A \text{ is squarefree and odd}, \tag{1.2.2}$$

$$D = B^2 + C^2 \text{ is squarefree},\ B > 0,\ C > 0, \tag{1.2.3}$$

$$(A, D) = 1. \tag{1.2.4}$$

Moreover each field of the form (1.2.1) satisfying (1.2.2) – (1.2.4) is a cyclic quartic field. It is convenient to distinguish five cases as follows (see [11, Theorem, p. 146]):

case (i) $D \equiv 0 \pmod 2$,
case (ii) $D \equiv 1 \pmod 2$, $B \equiv 1 \pmod 2$,
case (iii) $D \equiv 1 \pmod 2$, $B \equiv 0 \pmod 2$, $A + B \equiv 3 \pmod 4$,
case (iv) $D \equiv 1 \pmod 2$, $B \equiv 0 \pmod 2$, $A + B \equiv 1 \pmod 4$, $A \equiv C \pmod 4$,
case (v) $D \equiv 1 \pmod 2$, $B \equiv 0 \pmod 2$, $A + B \equiv 1 \pmod 4$, $A \equiv -C \pmod 4$.

We observe that

$$\begin{array}{lll} D \equiv 2 \pmod 4, & & \text{in case (i)}, \\ D \equiv 1 \pmod 4, & C \equiv 0 \pmod 2, & \text{in case (ii)}, \\ D \equiv 1 \pmod 4, & C \equiv 1 \pmod 2, & \text{in cases (iii), (iv), (v)}. \end{array}$$

The main result of Part 1 is the following theorem which gives necessary and sufficient conditions for each of $i(K) = 1, 2, 3, 4, 6, 12$.

Theorem A.

A	B	D	$i(K)$
1 (12)	±1, ±2, ±5 (12)		1
3 (12)			1
5 (12)	±1, ±2, ±5 (12)	2 (3)	1
5 (12)	±3, 6 (12)		1
7 (12)	±1 (3)		1
9 (12)	±1, 2 (4)		1
11 (12)	0 (3)		1
11 (12)	±1 (3)	2 (3)	1
1 (24)	±4 (24)		2
5 (24)	0 (24)		2
5 (24)	±8 (24)	2 (3)	2
9 (24)	4 (8)		2
13 (24)	±8 (24)		2
17 (24)	±4 (24)	2 (3)	2
17 (24)	12 (24)		2
21 (24)	0 (8)		2
1 (12)	±3, 6 (12)		3
5 (12)	±1, ±2, ±5 (12)	1 (3)	3
7 (12)	0 (3)		3
11 (12)	±1 (3)	1 (3)	3
1 (24)	±8 (24)		4
5 (24)	±4 (24)	2 (3)	4
5 (24)	12 (24)		4
9 (24)	0 (8)		4
13 (24)	±4 (24)		4
17 (24)	0 (24)		4
17 (24)	±8 (24)	2 (3)	4
21 (24)	4 (8)		4
1 (24)	12 (24)		6
5 (24)	±8 (24)	1 (3)	6
13 (24)	0 (24)		6
17 (24)	±4 (24)	1 (3)	6
1 (24)	0 (24)		12
5 (24)	±4 (24)	1 (3)	12
13 (24)	12 (24)		12
17 (24)	±8 (24)	1 (3)	12

Note that in the table a (mod b) has been abbreviated to a (b). Results needed for the proof of Theorem A are given in the remainder of this section as well as Section 1.3, and the proof completed in Section 1.4.

Our first task is to find the index form $i(x, y, z)$ of K. It is shown in [8] (see also [16]) that

$$d(K) = 2^{\lambda} A^2 D^3, \tag{1.2.5}$$

where

$$\lambda = \begin{cases} 8, & \text{case (i)}, \\ 6, & \text{case (ii)}, \\ 4, & \text{case (iii)}, \\ 0, & \text{cases (iv), (v)}. \end{cases} \tag{1.2.6}$$

Set

$$\alpha = \sqrt{A(D + B\sqrt{D})}, \quad \beta = \sqrt{A(D - B\sqrt{D})}. \tag{1.2.7}$$

Hudson and Williams [11, Theorem, p. 146] have shown that an integral basis $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ for K is given as follows:

$$\{1, \sqrt{D}, \alpha, \beta\}, \text{case (i)},$$

$$\left\{1, \frac{1}{2}(1 + \sqrt{D}), \alpha, \beta\right\}, \text{case (ii)},$$

$$\left\{1, \frac{1}{2}(1 + \sqrt{D}), \frac{1}{2}(\alpha + \beta), \frac{1}{2}(\alpha - \beta)\right\}, \text{case (iii)},$$

$$\left\{1, \frac{1}{2}(1+\sqrt{D}), \frac{1}{4}(1+\sqrt{D}+\alpha+\beta), \frac{1}{4}(1-\sqrt{D}+\alpha-\beta)\right\}, \text{case (iv)},$$

$$\left\{1, \frac{1}{2}(1+\sqrt{D}), \frac{1}{4}(1+\sqrt{D}+\alpha-\beta), \frac{1}{4}(1-\sqrt{D}+\alpha+\beta)\right\}, \text{case (v)}. \qquad (1.2.8)$$

Using MAPLE we find that the index form $i(x, y, z)$ of K with respect to the integral basis $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ given in (1.2.8) is given by

$$\begin{aligned}
i(x,y,z) &= (Cy^2-2Byz-Cz^2)(D(2x^2-Ay^2-Az^2)^2 \\
&\quad - A^2(Cy^2-2Byz-Cz^2)^2), \text{case (i)}, \\
i(x,y,z) &= \frac{1}{2}(Cy^2-2Byz-Cz^2)(D(x^2-2Ay^2-2Az^2)^2 \\
&\quad - 4A^2(Cy^2-2Byz-Cz^2)^2), \text{case (ii)}, \\
i(x,y,z) &= \frac{1}{2}(-By^2+2Cyz+Bz^2)(D(x^2-Ay^2-Az^2)^2 \\
&\quad - A^2(-By^2+2Cyz+Bz^2)^2), \text{case (iii)}, \\
i(x,y,z) &= \frac{1}{32}(-By^2+2Cyz+Bz^2)(D(4x^2+(1-A)y^2 \\
&\quad + (1-A)z^2+4xy-4xz-2yz)^2 \\
&\quad - A^2(-By^2+2Cyz+Bz^2)^2), \text{case (iv)}, \\
i(x,y,z) &= \frac{1}{32}(By^2+2Cyz-Bz^2)(D(4x^2+(1-A)y^2 \\
&\quad + (1-A)z^2+4xy-4xz-2yz)^2 \\
&\quad - A^2(By^2+2Cyz-Bz^2)^2), \text{case (v)}.
\end{aligned}$$

For $x, y, z \in \mathbb{Z}$, we define

$$R = R(x,y,z) = \begin{cases} Cy^2-2Byz-Cz^2, & \text{case (i)}, \\ 2Cy^2-4Byz-2Cz^2, & \text{case (ii)}, \\ -By^2+2Cyz+Bz^2, & \text{cases (iii), (iv)}, \\ By^2+2Cyz-Bz^2, & \text{case (v)}, \end{cases} \qquad (1.2.9)$$

and

$$S = S(x,y,z) = \begin{cases} 2x^2-Ay^2-Az^2, & \text{case (i)}, \\ x^2-2Ay^2-2Az^2, & \text{case (ii)}, \\ x^2-Ay^2-Az^2, & \text{case (iii)}, \\ 4x^2+(1-A)y^2+(1-A)z^2 & \\ \quad +4xy-4xz-2yz, & \text{cases (iv), (v)}. \end{cases} \qquad (1.2.10)$$

Then

$$i(x,y,z) = kR(DS^2-A^2R^2), \qquad (1.2.11)$$

where

$$k = \begin{cases} 1, & \text{case (i)}, \\ \frac{1}{4}, & \text{case (ii)}, \\ \frac{1}{2}, & \text{case (iii)}, \\ \frac{1}{32}, & \text{cases (iv), (v)}. \end{cases} \qquad (1.2.12)$$

From (1.1.2) we see that the index $i(K)$ of K is given by

$$i(K) = \gcd\{i(x,y,z) \mid x,y,z \in \mathbb{Z}\}. \tag{1.2.13}$$

1.3. Conditions for divisibility of $i(K)$ by $2, 3$ and 4. Let K be a cyclic quartic field given in the form (1.2.1). First we prove the following lemma.

Lemma 1.3.1. *If* $2 \mid d(K)$ *then* $2 \nmid i(K)$.

Proof. Suppose $2 \mid d(K)$. Then, by Dedekind's theorem, 2 ramifies in O_K. If $2 \mid i(K)$ then by [2, p. 234] either $2O_K = P_1P_2P_3P_4$ or P_1P_2, where P_1, P_2, P_3, P_4 are distinct prime ideals. This contradicts that 2 ramifies in O_K. Hence $2 \nmid i(K)$. □

We are now ready to give a necessary and sufficient condition for $i(K)$ to be divisible by 2.

Theorem 1.3.2. $2 \mid i(K) \Longleftrightarrow A \equiv 1 \pmod 4, B \equiv 0 \pmod 4$.

Proof. Suppose that $2 \mid i(K)$. Then, by Lemma 1.3.1, we have $2 \nmid d(K)$ so that by (1.2.5) either case (iv) or case (v) holds. Then, by (1.2.9)–(1.2.12), we have

$$i(0,1,0) = -\frac{\epsilon B}{32}(D(1-A)^2 - A^2B^2), \tag{1.3.1}$$

where

$$\epsilon = \begin{cases} 1, & \text{case (iv)}, \\ -1, & \text{case (v)}. \end{cases} \tag{1.3.2}$$

Next we define integers M, N and T by

$$B = 2M, \quad C = \epsilon A + 4T, \quad A = 1 + 4N - 2M.$$

Using these together with $D = B^2 + C^2$ in (1.3.1), we obtain in both cases

$$i(0,1,0) \equiv MN^2 + M^2N + M^5 \pmod 2$$

so that

$$i(0,1,0) \equiv M \pmod 2.$$

As $2 \mid i(K)$, by (1.2.13) we have $i(0,1,0) \equiv 0 \pmod 2$ so that $M \equiv 0 \pmod 2$, and thus

$$A \equiv 1 \pmod 4, \quad B \equiv 0 \pmod 4. \tag{1.3.3}$$

Now suppose that (1.3.3) holds. As $(B,C) = 1$ we have $C \equiv 1 \pmod 2$. Hence $D = B^2 + C^2 \equiv 1 \pmod 2$. Since $B \equiv 0 \pmod 2$ and $A + B \equiv 1 \pmod 4$, either case (iv) or case (v) holds. We define integers T, U and V by

$$A = 1 + 4T, \quad B = 4U, \quad C = \epsilon A + 4V,$$

where ϵ is defined in (1.3.2). Substituting these and $D = B^2 + C^2$ into $i(x,y,z)$ and reducing the coefficients modulo 2 and the powers by $x^n \equiv x \pmod 2$ $(n = 1, 2, \ldots)$, we obtain

$$i(x,y,z) \equiv 0 \pmod 2 \text{ for all integers } x, y, z.$$

Hence

$$i(K) \equiv 0 \pmod{2}.$$

This completes the proof of the theorem. □

Now we turn to the divisibility of $i(K)$ by 3. We require some preliminary results.

Lemma 1.3.3. *If* $i(K) \equiv 0 \pmod{3}$ *then* $A \not\equiv 0 \pmod{3}$.

Proof. Suppose that $i(K) \equiv 0 \pmod{3}$. If $A \equiv 0 \pmod{3}$ then $d(K) \equiv 0 \pmod{3}$ by (1.2.5), so that, by Dedekind's theorem, $3O_K$ ramifies, contradicting [2, p. 234] that $3O_K$ splits completely. Hence $A \not\equiv 0 \pmod{3}$. □

Lemma 1.3.4. *If* $i(K) \equiv 0 \pmod{3}$ *then either* $B \equiv 0 \pmod{3}$ *and* $C \not\equiv 0 \pmod{3}$ *or* $B \not\equiv 0 \pmod{3}$ *and* $C \equiv 0 \pmod{3}$.

Proof. Suppose that $i(K) \equiv 0 \pmod{3}$. Then $i(x,y,z) \equiv 0 \pmod{3}$ for all x, y, $z \in \mathbb{Z}$. Thus in particular we have

$$\begin{aligned} &i(0,1,0) \equiv 0 \pmod{3}, \text{cases (i), (ii), (iii)},\\ &i(0,1,1) \equiv 0 \pmod{3}, \text{cases (iv), (v)}. \end{aligned}$$

Now

$$\begin{aligned} R(0,1,0) &= \begin{cases} C, & \text{case (i)},\\ 2C, & \text{case (ii)},\\ -B, & \text{case (iii)}, \end{cases}\\ R(0,1,1) &= 2C, \quad \text{cases (iv), (v)},\\ S(0,1,0) &= \begin{cases} -A, & \text{case (i)},\\ -2A, & \text{case (ii)},\\ -A, & \text{case (iii)}, \end{cases}\\ S(0,1,1) &= -2A, \quad \text{cases (iv), (v)}, \end{aligned}$$

so that modulo 3 we have

$$\begin{aligned} &R(0,1,0)(DS(0,1,0)^2 - A^2R(0,1,0)^2)\\ &\equiv \begin{cases} C(DA^2 - A^2C^2), & \text{case (i)},\\ 2C(4DA^2 - 4A^2C^2), & \text{case (ii)},\\ -B(DA^2 - A^2C^2), & \text{case (iii)}, \end{cases}\\ &\equiv \begin{cases} A^2B^2C, & \text{case (i)},\\ -A^2B^2C, & \text{case (ii)},\\ -A^2BC^2, & \text{case (iii)}, \end{cases} \end{aligned}$$

and in cases (iv) and (v)

$$R(0,1,1)(DS(0,1,1)^2 - A^2R(0,1,1)^2) \equiv 2C(4DA^2 - 4A^2C^2) \equiv -A^2B^2C.$$

Hence by (1.2.11) in all cases we have $ABC \equiv 0 \pmod 3$. But by Lemma 1.3.3 we have $A \not\equiv 0 \pmod 3$. Thus $BC \equiv 0 \pmod 3$. But $(B, C) = 1$ so that either $B \equiv 0 \pmod 3$, $C \not\equiv 0 \pmod 3$ or $B \not\equiv 0 \pmod 3$, $C \equiv 0 \pmod 3$. □

Lemma 1.3.5. *If* $i(K) \equiv 0 \pmod 3$ *and* $B \equiv 0 \pmod 3$ *then we have* $A \equiv 1 \pmod 3$.

Proof. Suppose that $i(K) \equiv 0 \pmod 3$ and $B \equiv 0 \pmod 3$. Then $C \not\equiv 0 \pmod 3$ and $D = B^2 + C^2 \equiv 1 \pmod 3$. Also, by (1.2.13), we have

$$\begin{aligned} &i(1,1,0) \equiv 0 \pmod 3, && \text{cases (i), (ii)},\\ &i(1,1,1) \equiv 0 \pmod 3, && \text{cases (iii), (iv), (v)}. \end{aligned}$$

Now by (1.2.9) and (1.2.10) we have

$$\begin{aligned} &R(1,1,0) = C, && \text{case (i)},\\ &R(1,1,0) = 2C, && \text{case (ii)},\\ &R(1,1,1) = 2C, && \text{cases (iii), (iv), (v)}, \end{aligned}$$

and

$$\begin{aligned} &S(1,1,0) = 2 - A, && \text{case (i)},\\ &S(1,1,0) = 1 - 2A, && \text{case (ii)},\\ &S(1,1,1) = 1 - 2A, && \text{case (iii)},\\ &S(1,1,1) = 4 - 2A, && \text{cases (iv), (v)}. \end{aligned}$$

Hence, by (1.2.11), in all cases we have

$$(2 - A)^2 - A^2 \equiv 0 \pmod 3,$$

so that

$$A \equiv 1 \pmod 3,$$

as asserted. □

Lemma 1.3.6. *If* $i(K) \equiv 0 \pmod 3$ *and* $C \equiv 0 \pmod 3$ *then we have* $A \equiv 2 \pmod 3$.

Proof. Suppose that $i(K) \equiv 0 \pmod 3$ and $C \equiv 0 \pmod 3$. Then $B \not\equiv 0 \pmod 3$ and $D = B^2 + C^2 \equiv 1 \pmod 3$. By (1.2.13) we have

$$\begin{aligned} &i(1,1,1) \equiv 0 \pmod 3, && \text{cases (i), (ii)},\\ &i(1,1,0) \equiv 0 \pmod 3, && \text{case (iii)},\\ &i(0,1,0) \equiv 0 \pmod 3, && \text{cases (iv), (v)}. \end{aligned}$$

Now, by (1.2.9) and (1.2.10), we have

$$\begin{aligned} &R(1,1,1) = -2B, && \text{case (i)},\\ &R(1,1,1) = -4B, && \text{case (ii)},\\ &R(1,1,0) = -B, && \text{case (iii)},\\ &R(0,1,0) = -B, && \text{case (iv)},\\ &R(0,1,0) = B, && \text{case (v)}, \end{aligned}$$

and

$$\begin{aligned} S(1,1,1) &= 2-2A, && \text{case (i)},\\ S(1,1,1) &= 1-4A, && \text{case (ii)},\\ S(1,1,0) &= 1-A, && \text{case (iii)},\\ S(0,1,0) &= 1-A, && \text{case (iv)},\\ S(0,1,0) &= 1-A, && \text{case (v)}. \end{aligned}$$

Hence, by (1.2.11), in all cases we have

$$(1-A)^2 - A^2 \equiv 0 \pmod 3,$$

and thus

$$A \equiv 2 \pmod 3,$$

as claimed. □

Lemma 1.3.7. *If* $A \equiv 1 \pmod 3$ *and* $B \equiv 0 \pmod 3$ *then we have* $i(K) \equiv 0 \pmod 3$.

Proof. Suppose that $A \equiv 1 \pmod 3$ and $B \equiv 0 \pmod 3$ so that $C \not\equiv 0 \pmod 3$ and $D = B^2 + C^2 \equiv 1 \pmod 3$.

Cases (i), (ii). We have

$$R(x,y,z) \equiv \pm C(y^2 - z^2) \pmod 3, \quad S(x,y,z) \equiv \mp(x^2 + y^2 + z^2) \pmod 3.$$

If $y^2 \equiv z^2 \pmod 3$ then $R(x,y,z) \equiv 0 \pmod 3$ and by (1.2.11) $i(x,y,z) \equiv 0 \pmod 3$. If $y^2 \not\equiv z^2 \pmod 3$ then $R(x,y,z) \not\equiv 0 \pmod 3$ and $S(x,y,z) \equiv \mp(x^2+1) \equiv \pm 1 \not\equiv 0 \pmod 3$ so that $DS(x,y,z)^2 - A^2R(x,y,z)^2 \equiv 1 - 1 \equiv 0 \pmod 3$ and thus by (1.2.11) $i(x,y,z) \equiv 0 \pmod 3$.

Case (iii). We have

$$R(x,y,z) \equiv -Cyz \pmod 3, \quad S(x,y,z) \equiv x^2 - y^2 - z^2 \pmod 3.$$

If $yz \equiv 0 \pmod 3$ then $R(x,y,z) \equiv 0 \pmod 3$ and by (1.2.11) we have $i(x,y,z) \equiv 0 \pmod 3$. If $yz \not\equiv 0 \pmod 3$ then $y^2 \equiv z^2 \equiv 1 \pmod 3$ so that $R(x,y,z) \not\equiv 0 \pmod 3$ and $S(x,y,z) \equiv x^2 - 2 \not\equiv 0 \pmod 3$ so that $DS(x,y,z)^2 - A^2R(x,y,z)^2 \equiv 1 - 1 \equiv 0 \pmod 3$ and thus by (1.2.11) $i(x,y,z) \equiv 0 \pmod 3$.

Cases (iv), (v). We have

$$R(x,y,z) \equiv -Cyz \pmod 3, \quad S(x,y,z) \equiv x^2 + xy - xz + yz \pmod 3.$$

If $yz \equiv 0 \pmod 3$ then $R(x,y,z) \equiv 0 \pmod 3$ and by (1.2.11) we see that $i(x,y,z) \equiv 0 \pmod 3$. If $yz \not\equiv 0 \pmod 3$ then $R(x,y,z) \not\equiv 0 \pmod 3$ and

$$S(x,y,z) \equiv \begin{cases} x^2 + 1 \not\equiv 0 \pmod 3, & \text{if } y \equiv z \pmod 3,\\ (x+y)^2 + 1 \not\equiv 0 \pmod 3, & \text{if } y \equiv -z \pmod 3, \end{cases}$$

so that

$$DS(x,y,z)^2 - A^2R(x,y,z)^2 \equiv 1 - 1 \equiv 0 \pmod 3$$

and thus by (1.2.11) $i(x,y,z) \equiv 0 \pmod 3$.

Hence in all five cases we have shown that $i(x, y, z) \equiv 0 \pmod 3$ for all integers x, y, z so that by (1.2.13) $i(K) \equiv 0 \pmod 3$. □

Lemma 1.3.8. *If* $A \equiv 2 \pmod 3$ *and* $C \equiv 0 \pmod 3$ *then we have* $i(K) \equiv 0 \pmod 3$.

Proof. Suppose that $A \equiv 2 \pmod 3$ and $C \equiv 0 \pmod 3$ so that $B \not\equiv 0 \pmod 3$ and $D = B^2 + C^2 \equiv 1 \pmod 3$.

Cases (i), (ii). We have

$$R(x, y, z) \equiv \pm Byz \pmod 3, \quad S(x, y, z) \equiv \pm(-x^2 + y^2 + z^2) \pmod 3.$$

If $yz \equiv 0 \pmod 3$ then $R(x, y, z) \equiv 0 \pmod 3$ and by (1.2.11) we have $i(x, y, z) \equiv 0 \pmod 3$. If $yz \not\equiv 0 \pmod 3$ then $R(x, y, z) \not\equiv 0 \pmod 3$ and $S(x, y, z) \equiv \mp(x^2 + 1) \not\equiv 0 \pmod 3$ so that $DS(x, y, z)^2 - A^2R(x, y, z)^2 \equiv 1 - 1 \equiv 0 \pmod 3$ and thus by (1.2.11) $i(x, y, z) \equiv 0 \pmod 3$.

Case (iii). We have

$$R(x, y, z) \equiv -B(y^2 - z^2) \pmod 3, \quad S(x, y, z) \equiv x^2 + y^2 + z^2 \pmod 3.$$

If $y^2 \equiv z^2 \pmod 3$ then $R(x, y, z) \equiv 0 \pmod 3$ and by (1.2.11) we have $i(x, y, z) \equiv 0 \pmod 3$. If $y^2 \not\equiv z^2 \pmod 3$ then $R(x, y, z) \not\equiv 0 \pmod 3$ and $S(x, y, z) \equiv x^2 + 1 \not\equiv 0 \pmod 3$ so that $DS(x, y, z)^2 - A^2R(x, y, z)^2 \equiv 1 - 1 \equiv 0 \pmod 3$ and thus by (1.2.11) $i(x, y, z) \equiv 0 \pmod 3$.

Cases (iv), (v). We have

$$R(x, y, z) \equiv \mp B(y^2 - z^2) \pmod 3,$$
$$S(x, y, z) \equiv x^2 - y^2 - z^2 + xy - xz + yz \pmod 3.$$

If $y^2 \equiv z^2 \pmod 3$ then $R(x, y, z) \equiv 0 \pmod 3$ and by (1.2.11) we have $i(x, y, z) \equiv 0 \pmod 3$. If $y^2 \not\equiv z^2 \pmod 3$ then $R(x, y, z) \not\equiv 0 \pmod 3$ and

$$S(x, y, z) \equiv \begin{cases} (x + z)^2 + 1 \pmod 3, & \text{if } y \equiv 0 \pmod 3, \\ (x - y)^2 + 1 \pmod 3, & \text{if } z \equiv 0 \pmod 3, \end{cases}$$

so that $S(x, y, z) \not\equiv 0 \pmod 3$. Hence $DS(x, y, z)^2 - A^2R(x, y, z)^2 \equiv 1 - 1 \equiv 0 \pmod 3$ and thus by (1.2.11) $i(x, y, z) \equiv 0 \pmod 3$.

Hence in all five cases we have shown that $i(x, y, z) \equiv 0 \pmod 3$ for all integers x, y, z so that by (1.2.13) we have $i(K) \equiv 0 \pmod 3$. □

We can now give a necessary and sufficient condition for $i(K)$ to be divisible by 3.

Theorem 1.3.9.

$$3 \mid i(K) \Longleftrightarrow A \equiv 1 \pmod 3,\ B \equiv 0 \pmod 3 \text{or } A \equiv 2 \pmod 3,\ C \equiv 0 \pmod 3.$$

Proof. This result follows immediately from Lemmas 1.3.3–1.3.8. □

As

$$D \equiv 1 \pmod 3 \Longleftrightarrow B^2 + C^2 \equiv 1 \pmod 3$$
$$\Longleftrightarrow \text{exactly one of } B \text{ or } C \equiv 0 \pmod 3,$$

we can reformulate Theorem 1.3.9 as

Theorem 1.3.9.

$$3 \mid i(K) \Longleftrightarrow A \equiv 1 \pmod 3, \quad B \equiv 0 \pmod 3 \quad \textit{or}$$
$$A \equiv 2 \pmod 3, \quad B \equiv \pm 1 \pmod 3, \quad D \equiv 1 \pmod 3.$$

Next we determine a necessary and sufficient condition for $i(K)$ to be divisible by 4.

Theorem 1.3.10.

$$4 \mid i(K) \Longleftrightarrow A \equiv 1 \pmod 8, \quad B \equiv 0 \pmod 8 \quad \textit{or}$$
$$A \equiv 5 \pmod 8, \quad B \equiv 4 \pmod 8.$$

Proof. Assume $4 \mid i(K)$. By Theorem 1.3.2 we have $A \equiv 1 \pmod 4$, $B \equiv 0 \pmod 4$ so that either case (iv) or case (v) holds. Define integers M, N and T by

$$A = 1 + 4M, \quad B = 4N, \quad C = \epsilon A + 4T, \tag{1.3.4}$$

where ϵ is defined in (1.3.2). From (1.2.9) and (1.2.10) we obtain

$$R(0,1,0) = -\epsilon B, \quad S(0,1,0) = 1 - A,$$

so that by (1.2.11) we have

$$i(0,1,0) = -\frac{\epsilon B}{32}\left(D(1-A)^2 - A^2B^2\right).$$

Appealing to (1.3.4) we obtain

$$i(0,1,0) = -2\epsilon N(DM^2 - A^2N^2) \equiv 2N(M^2 - N^2) \equiv 2N(M+1) \pmod 4.$$

Also from (1.2.9) and (1.2.10) we obtain

$$R(0,1,2) = 3\epsilon B + 4C, \quad S(0,1,2) = 1 - 5A,$$

so that by (1.2.11) we have

$$i(0,1,2) = \frac{(3\epsilon B + 4C)}{32}\left(D(1-5A)^2 - A^2(3\epsilon B + 4C)^2\right).$$

Appealing to (1.3.4) we obtain

$$\begin{aligned} i(0,1,2) &= 2(3\epsilon N + C)(D(A+M)^2 - A^2(3\epsilon N + C)^2) \\ &\equiv 2(\epsilon N + \epsilon)(A + M - A(\epsilon N + C)) \pmod 4 \\ &\equiv 2(N+1)(1 + M - N - 1) \pmod 4 \\ &\equiv 2M(N+1) \pmod 4. \end{aligned}$$

As $4 \mid i(K)$ we have by (1.2.13) $i(0,1,0) \equiv i(0,1,2) \equiv 0 \pmod 4$ so that

$$(M+1)N \equiv M(N+1) \equiv 0 \pmod 2,$$

that is

$$M \equiv N \equiv 0 \pmod 2 \text{ or } M \equiv N \equiv 1 \pmod 2.$$

Thus by (1.3.4) we have

$$A \equiv 1 \pmod 8, \quad B \equiv 0 \pmod 8 \text{ or } A \equiv 5 \pmod 8, \quad B \equiv 4 \pmod 8.$$

Conversely suppose that

$$A \equiv 1 \pmod 8, \quad B \equiv 0 \pmod 8 \text{ or } A \equiv 5 \pmod 8, \quad B \equiv 4 \pmod 8.$$

As $B \equiv 0 \pmod 4$ and $(B, C) = 1$ we have $C \equiv 1 \pmod 2$ so that $D = B^2 + C^2 \equiv 1 \pmod 8$. Also $A + B \equiv 1 \pmod 8$. Hence either case (iv) or case (v) occurs. We can define integers T, U and V by

$$A = 1 + 4T, \quad B = 8U - 4T, \quad C = \epsilon(1 + 4T) + 4V,$$

where ϵ is defined in (1.3.2).

First we determine D modulo 16. We have

$$D = B^2 + C^2 = (8U - 4T)^2 + (\epsilon + 4\epsilon T + 4V)^2 \equiv 1 + 8(T + V) \pmod{16}.$$

Secondly

$$\begin{aligned} R = R(x, y, z) &= -\epsilon B y^2 + 2Cyz + \epsilon B z^2 \\ &= -\epsilon(8U - 4T)y^2 + 2(\epsilon + 4\epsilon T + 4V)yz + \epsilon(8U - 4T)z^2 \\ &= 2\epsilon yz + 4\epsilon T(y^2 - z^2) + 8(\epsilon U(z^2 - y^2) + (\epsilon T + V)yz), \end{aligned}$$

so that

$$\frac{R}{2} = \epsilon yz + 2\epsilon T(y^2 - z^2) + 4(\epsilon U(z^2 - y^2) + (\epsilon T + V)yz),$$

$$\begin{aligned} \left(\frac{R}{2}\right)^2 &\equiv y^2z^2 + 4(T^2(y^2 - z^2)^2 + T(y^2 - z^2)yz) \\ &\quad + 8\epsilon yz(\epsilon U(z^2 - y^2) + (\epsilon T + V)yz) \pmod{16} \\ &\equiv y^2z^2 + 4T(y^2 - z^2)(T(y^2 - z^2) + yz) + 8y^2z^2(T + V) \pmod{16}, \end{aligned}$$

as $yz(z^2 - y^2) \equiv 0 \pmod 2$. Thirdly

$$\begin{aligned} S = S(x, y, z) &= 4x^2 + (1 - A)y^2 + (1 - A)z^2 + 4xy - 4xz - 2yz \\ &= 4x^2 - 4Ty^2 - 4Tz^2 + 4xy - 4xz - 2yz \\ &= -2yz + 4(x^2 - Ty^2 - Tz^2 + xy - xz) \end{aligned}$$

so that

$$\frac{S}{2} = -yz + 2(x^2 - Ty^2 - Tz^2 + xy - xz),$$

$$\begin{aligned} \left(\frac{S}{2}\right)^2 &= y^2z^2 - 4yz(x^2 - Ty^2 - Tz^2 + xy - xz) + 4(x^2 - Ty^2 - Tz^2 + xy - xz)^2 \\ &= y^2z^2 + 4(x^2 - Ty^2 - Tz^2 + xy - xz)(x^2 - Ty^2 - Tz^2 + xy - xz - yz). \end{aligned}$$

Fourthly

$$D\left(\frac{S}{2}\right)^2 \equiv y^2z^2 + 4(x^2 - Ty^2 - Tz^2 + xy - xz)(x^2 - Ty^2 - Tz^2 + xy - xz - yz) + 8(T+V)y^2z^2 \pmod{16}.$$

Fifthly

$$\begin{aligned} A^2\left(\frac{R}{2}\right)^2 &\equiv (1+8T)(y^2z^2 + 4T(y^2 - z^2)(T(y^2 - z^2) + yz) \\ &\quad + 8y^2z^2(T+V)) \pmod{16} \\ &\equiv y^2z^2 + 4T(y^2 - z^2)(T(y^2 - z^2) + yz) + 8Vy^2z^2 \pmod{16}. \end{aligned}$$

Hence

$$\begin{aligned} & D\left(\frac{S}{2}\right)^2 - A^2\left(\frac{R}{2}\right)^2 \\ &\quad \equiv 4((x^2 - Ty^2 - Tz^2 + xy - xz)(x^2 - Ty^2 - Tz^2 + xy - xz - yz) \\ &\quad\quad - T(y^2 - z^2)(T(y^2 - z^2) + yz) + 2Ty^2z^2) \pmod{16} \\ &\quad \equiv 4(x^4 - 2Tx^2y^2 - 2Tx^2z^2 - 2Txy^3 + 2Tyz^3 + 2Txz^3 - 2Txyz^2 \\ &\quad\quad + 2Txy^2z + 2x^3y - 2x^3z - 2x^2yz + x^2y^2 + x^2z^2 + xyz^2 \\ &\quad\quad - x^2yz - xy^2z + 2Ty^2z^2) \pmod{16} \end{aligned}$$

so that

$$\begin{aligned} & \frac{1}{4}\left(D\left(\frac{S}{2}\right)^2 - A^2\left(\frac{R}{2}\right)^2\right) \\ &\quad \equiv (x^4 + x^2y^2 + x^2z^2 + xyz^2 - x^2yz - xy^2z) \\ &\quad\quad + 2T(-x^2y^2 - x^2z^2 + y^2z^2 - xy^3 + yz^3 + xz^3 - xyz^2 + xy^2z) \\ &\quad\quad + 2(x^3y - x^3z - x^2yz) \pmod{4}. \end{aligned}$$

Then, by (1.2.11), we have

$$\begin{aligned} i(x,y,z) &= \frac{1}{4}\left(\frac{R}{2}\right)\left(D\left(\frac{S}{2}\right)^2 - A^2\left(\frac{R}{2}\right)^2\right) \\ &\equiv (\epsilon yz + 2\epsilon T(y^2 - z^2)) \times \\ &\quad (x^4 + x^2y^2 + x^2z^2 + xyz^2 - x^2yz - xy^2z + 2(x^3y - x^3z - x^2yz) \\ &\quad + 2T(-x^2y^2 - x^2z^2 + y^2z^2 - xy^3 + yz^3 + xz^3 - xyz^2 + xy^2z)) \\ &\equiv \epsilon(yz(x^4 + x^2y^2 + x^2z^2 + xyz^2 - x^2yz - xy^2z) \\ &\quad + 2yz(x^3y - x^3z - x^2yz) \\ &\quad + 2Tyz(-x^2y^2 - x^2z^2 + y^2z^2 - xy^3 + yz^3 + xz^3 - xyz^2 + xy^2z) \\ &\quad + 2T(y^2 - z^2)(x^4 + x^2y^2 + x^2z^2 + xyz^2 - x^2yz - xy^2z)) \pmod{4}. \end{aligned}$$

Now, as

$$yz(y \pm z), \quad yz(y^2 \pm z^2), \quad (y^2 - z^2)(x^4 + x^2y^2 + x^2z^2)$$

are all even, we have

$$\begin{aligned} &2Tyz(-x^2y^2 - x^2z^2 + y^2z^2 - xy^3 + yz^3 + xz^3 - xyz^2 + xy^2z) \\ &\quad = (-2Tx^2)yz(y^2 + z^2) + (2Tyz^2)yz(y + z) - (2Txy^2)yz(y - z) \\ &\qquad - (2Txz^2)yz(y - z) \\ &\quad \equiv 0 \pmod 4) \end{aligned}$$

and

$$\begin{aligned} &2T(y^2 - z^2)(x^4 + x^2y^2 + x^2z^2 + xyz^2 - x^2yz - xy^2z) \\ &\quad = (2T)((y^2 - z^2)(x^4 + x^2y^2 + x^2z^2)) + (2Tx(z - x - y))((yz)(y^2 - z^2)) \\ &\quad \equiv 0 \pmod 4. \end{aligned}$$

Hence

$$\begin{aligned} i(x, y, z) &\equiv \epsilon xyz(x^3 + xy^2 + xz^2 + yz^2 - xyz - y^2z + 2x^2y - 2x^2z - 2xyz) \\ &\equiv \epsilon xyz(x^3 + 2(y + z)x^2 + (y^2 + yz + z^2)x + (yz^2 - y^2z)) \\ &\equiv \epsilon xyz((x + y)(x + z)(x + y + z) - 2yz(x - y)) \\ &\equiv 0 \pmod 4 \end{aligned}$$

as

$$xyz(x + y)(x + z)(x + y + z) \equiv 0 \pmod 4$$

and

$$xy(x - y) \equiv 0 \pmod 2.$$

Thus, by (1.2.13), we have $4 \mid i(K)$.

This completes the proof of Theorem 1.3.10. □

1.4. Conditions for $i(K) = 1, 2, 3, 4, 6, 12$: Proof of Theorem A. Theorems 1.4.1, 1.4.2 and 1.4.3 are immediate consequences of Theorems 1.3.2, 1.3.9 and 1.3.10.

Theorem 1.4.1.

$$\begin{aligned} 2 \nmid i(K) \Longleftrightarrow A &\equiv 1 \pmod 4,\ B \equiv 1,\ 2,\ 3 \pmod 4 \ \textit{or} \\ A &\equiv 3 \pmod 4. \end{aligned}$$

Theorem 1.4.2.

$$\begin{aligned} 2 \parallel i(K) \Longleftrightarrow A &\equiv 1 \pmod 8,\ B \equiv 4 \pmod 8,\ \textit{or} \\ A &\equiv 5 \pmod 8,\ B \equiv 0 \pmod 8. \end{aligned}$$

Theorem 1.4.3.

$$\begin{aligned} 3 \nmid i(K) \Longleftrightarrow A &\equiv 0 \pmod 3 \ \textit{or} \\ A &\equiv 1 \pmod 3,\ B \equiv \pm 1 \pmod 3 \ \textit{or} \\ A &\equiv 2 \pmod 3,\ C \equiv \pm 1 \pmod 3. \end{aligned}$$

If $D \equiv 1 \pmod 3$ then

$$B \equiv 0 \pmod 3,\ C \equiv \pm 1 \pmod 3 \text{ or } B \equiv \pm 1 \pmod 3,\ C \equiv 0 \pmod 3,$$

and if $D \equiv 2 \pmod 3$ then

$$B \equiv \pm 1 \pmod 3,\ C \equiv \pm 1 \pmod 3.$$

Thus Theorem 1.4.3 can be reformulated as

Theorem 1.4.3.

$$\begin{aligned} 3 \nmid i(K) \Longleftrightarrow & A \equiv 0 \pmod 3 \ \textit{or} \\ & A \equiv 1 \pmod 3,\ B \equiv \pm 1 \pmod 3 \ \textit{or} \\ & A \equiv 2 \pmod 3,\ B \equiv 0 \pmod 3 \ \textit{or} \\ & A \equiv 2 \pmod 3,\ B \equiv \pm 1 \pmod 3,\ D \equiv 2 \pmod 3. \end{aligned}$$

Proof of Theorem A. We recall from (1.1.3) that for a cyclic quartic field K we have $i(K) = 1$, 2, 3, 4, 6 or 12. These possibilities can be recognized as follows:

$$\begin{aligned} i(K) = 1 &\Longleftrightarrow 2 \nmid i(K),\ 3 \nmid i(K), \\ i(K) = 2 &\Longleftrightarrow 2 \parallel i(K),\ 3 \nmid i(K), \\ i(K) = 3 &\Longleftrightarrow 2 \nmid i(K),\ 3 \mid i(K), \\ i(K) = 4 &\Longleftrightarrow 4 \mid i(K),\ 3 \nmid i(K), \\ i(K) = 6 &\Longleftrightarrow 2 \parallel i(K),\ 3 \mid i(K), \\ i(K) = 12 &\Longleftrightarrow 4 \mid i(K),\ 3 \mid i(K). \end{aligned}$$

Appealing to Theorems 1.3.9, 1.3.10, 1.4.1, 1.4.2 and 1.4.3, we obtain Theorem A. □

Part 2: Asymptotic Number of Cyclic Quartic Fields with Discriminant $\leqslant x$ and Given Index

2.1. Introduction. We define for a positive integer i

$$N(x;i) = \text{number of cyclic quartic fields } K \text{ with}$$

$$d(K) \leqslant x \text{ and } i(K) = i. \tag{2.1.1}$$

By Engstrom's theorem [2, p. 234] we have

$$N(x;i) = 0 \text{ for } i \neq 1,\ 2,\ 3,\ 4,\ 6,\ 12.$$

In Section 2.6 we prove the following theorem after some preliminary results in Sections 2.2–2.5.

Theorem B. *For* $i = 1, 2, 3, 4, 6, 12$ *we have*

$$N(x;i) = \alpha_i x^{\frac{1}{2}} + O(x^{\frac{1}{3}} \log^3 x),$$

where

$$\begin{aligned}
\alpha_1 &= \frac{1}{128\pi^2}\big((208+13\sqrt{2})c(3/2)-(48-3\sqrt{2})c(3/2,-3)-104c(3/2,8)\\
&\qquad\qquad +24c(3/2,-24)+16\sqrt{2}\big),\\
\alpha_2 &= \frac{1}{32\pi^2}(13c(3/2)+13c(3/2,8)-3c(3/2,-3)-3c(3/2,-24)),\\
\alpha_3 &= \frac{3}{128\pi^2}\big((16+\sqrt{2})c(3/2)+(16-\sqrt{2})c(3/2,-3)-8c(3/2,8)\\
&\qquad\qquad -8c(3/2,-24)\big),\\
\alpha_4 &= \frac{1}{32\pi^2}(13c(3/2)+13c(3/2,8)-3c(3/2,-3)-3c(3/2,-24)),\\
\alpha_6 &= \frac{3}{32\pi^2}(c(3/2)+c(3/2,-3)+c(3/2,8)+c(3/2,-24)),\\
\alpha_{12} &= \frac{3}{32\pi^2}(c(3/2)+c(3/2,-3)+c(3/2,8)+c(3/2,-24)),
\end{aligned}$$

where the constants $c(\alpha)$ and $c(\alpha,\Delta)$ are defined in (2.3.2) *and* (2.3.3) *respectively.*

We note that

$$\begin{aligned}
&\alpha_1 \approx 0.0970153, \quad \alpha_2 \approx 0.0067627, \quad \alpha_3 \approx 0.0101764,\\
&\alpha_4 \approx 0.0067627, \quad \alpha_6 \approx 0.0006321, \quad \alpha_{12} \approx 0.0006321.
\end{aligned}$$

2.2. Some estimates involving $d(n)$. In this section we give some estimates for certain sums involving the divisor function

$$d(n) = \sum_{d\mid n} 1,$$

which will be needed in Section 2.3.

Lemma 2.2.1. *Let $c_1, c_2, \ldots$ be a sequence of real numbers and set*

$$C(x) = \sum_{n \leqslant x} c_n, \quad x \geqslant 1. \tag{2.2.1}$$

Let $f(t)$ be a real-valued function of t which has a continuous derivative for $t \geqslant 1$. Then for $x \geqslant 1$

$$\sum_{n \leqslant x} c_n f(n) = C(x)f(x) - \int_1^x C(t)f'(t)dt.$$

Proof. This elementary lemma is well-known, see for example [9, Theorem 421, p. 346]. □

Our next lemma is a simple consequence of Lemma 2.2.1.

Lemma 2.2.2. *Let $c_1, c_2, \ldots$ be a sequence of real numbers and define $C(x)$ as in (2.2.1). Let $f(t)$ be a real-valued function of t such that*

(*a*) *$f'(t)$ exists and is continuous for $t \geqslant 1$,*
(*b*) $\lim_{x\to\infty} C(x)f(x) = 0$,
(*c*) *$\int_x^\infty C(t)f'(t)dt$ converges for $x \geqslant 1$.*

Then for $x \geqslant 1$

$$\sum_{n>x} c_n f(n) = -C(x)f(x) - \int_x^\infty C(t)f'(t)dt.$$

Proof. Let x and y be real numbers such that $1 \leqslant x < y$. In view of condition (a) we can apply Lemma 2.2.1 to obtain

$$\sum_{n\leqslant x} c_n f(n) = C(x)f(x) - \int_1^x C(t)f'(t)dt \tag{2.2.2}$$

and

$$\sum_{n\leqslant y} c_n f(n) = C(y)f(y) - \int_1^y C(t)f'(t)dt. \tag{2.2.3}$$

Subtracting (2.2.2) from (2.2.3) we deduce that

$$\sum_{x<n\leqslant y} c_n f(n) = C(y)f(y) - C(x)f(x) - \int_x^y C(t)f'(t)dt.$$

Letting $y \to +\infty$, in view of conditions (b) and (c), we obtain

$$\sum_{x<n} c_n f(n) = -C(x)f(x) - \int_x^\infty C(t)f'(t)dt$$

as asserted. □

Lemma 2.2.3. *Let $\epsilon > 0$. Then there exists a positive constant $A(\epsilon)$ such that*

$$d(n) \leqslant A(\epsilon)n^\epsilon$$

for all positive integers n.

Proof. See [9, Theorem 315, p. 260]. □

Lemma 2.2.4.

$$\sum_{n\leqslant x} d(n) = x\log x + (2\gamma - 1)x + O(x^{\frac{1}{2}}),$$

where γ is Euler's constant.

Proof. See [9, Theorem 320, p. 264]. □

Lemma 2.2.5. *For $\alpha > 1$*

$$\sum_{n>x} \frac{d(n)}{n^\alpha} = O_\alpha\left(\frac{\log x}{x^{\alpha-1}}\right),$$

where the constant implied by the O-symbol depends only on α.

Proof. We choose $c_n = d(n)$ and $f(t) = \frac{1}{t^\alpha}$. With this choice condition (a) of Lemma 2.2.2 is satisfied. By Lemma 2.2.4 we have

$$C(x) = \sum_{n \leqslant x} d(n) = x \log x + (2\gamma - 1)x + O(x^{\frac{1}{2}}). \tag{2.2.4}$$

Thus

$$\lim_{x\to\infty} C(x)f(x) = \lim_{x\to\infty} \left(\frac{\log x}{x^{\alpha-1}} + \frac{2\gamma-1}{x^{\alpha-1}} + O\left(\frac{1}{x^{\alpha-\frac{1}{2}}}\right)\right) = 0,$$

as $\alpha > 1$, so that condition (b) of Lemma 2.2.2 is satisfied. By (2.2.4) we have $C(t) = O(t \log t)$ so that

$$C(t)f'(t) = O_\alpha\left(\frac{\log t}{t^a}\right).$$

As $\int_x^\infty \frac{\log t}{t^\alpha} dt$ converges for $\alpha > 1$, the integral $\int_x^\infty C(t)f'(t)dt$ also converges. Thus condition (c) of Lemma 2.2.2 is satisfied. Now

$$\int_x^\infty \frac{\log t}{t^\alpha} dt = \frac{\log x}{(\alpha-1)x^{\alpha-1}} + \frac{1}{(\alpha-1)^2 x^{\alpha-1}}$$

so by Lemma 2.2.2 we have

$$\sum_{n>x} \frac{d(n)}{n^\alpha} = O\left(\frac{\log x}{x^{\alpha-1}}\right) + O_\alpha\left(\int_x^\infty \frac{\log t}{t^\alpha} dt\right) = O_\alpha\left(\frac{\log x}{x^{\alpha-1}}\right),$$

as asserted. □

Lemma 2.2.6.

$$\sum_{n \leqslant x} d^2(n) = O(x \log^3 x).$$

Proof. See [10, Theorem 5.3, p. 111], [15, p. 175]. □

Lemma 2.2.7. *Let* $\alpha > 0$. *Then*

$$\sum_{n \leqslant x} \frac{d^2(n)}{n^\alpha} = \begin{cases} O_\alpha(x^{1-\alpha} \log^3 x), & \text{if } \alpha \neq 1, \\ O(\log^4 x), & \text{if } \alpha = 1. \end{cases}$$

Proof. We choose $c_n = d^2(n)$ and $f(t) = \frac{1}{t^\alpha}$. By Lemma 2.2.6 we have $C(x) = O(x \log^3 x)$. Appealing to Lemma 2.2.1 we obtain

$$\sum_{n \leqslant x} \frac{d^2(n)}{n^\alpha} = O(x^{1-\alpha} \log^3 x) - \int_1^x O_\alpha\left(\frac{\log^3 t}{t^\alpha}\right) dt.$$

The asserted result follows as

$$\int_1^x O_\alpha\left(\frac{\log^3 t}{t^\alpha}\right) dt = O_\alpha\left(\int_1^x \frac{\log^3 t}{t^\alpha} dt\right) = O_\alpha\left(\log^3 x \int_1^x \frac{dt}{t^\alpha}\right)$$

and

$$\int_1^x \frac{dt}{t^\alpha} = \begin{cases} \frac{x^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha}, & \text{if } \alpha \neq 1, \\ \log x, & \text{if } \alpha = 1. \end{cases}$$

We remark that the case $\alpha = 1$ of Lemma 2.2.7 is a special case of [10, Theorem 5.3, p. 111]. □

2.3. Further estimates. Throughout this section Δ denotes a nonsquare integer such that $\Delta \equiv 0$ or 1 (mod 4) and $(\frac{\Delta}{n}) = (\Delta/n)$ is the Legendre-Jacobi-Kronecker symbol. We define

$$\wp = \{n \in \mathbb{N}, n > 1 \mid n = \text{product of distinct primes} \equiv 1 (\text{mod } 4)\}. \qquad (2.3.1)$$

We emphasize that $1 \notin \wp$. We write "n sqf" to indicate that n is squarefree and "$n = \square$" to denote that n is a perfect square. As usual p denotes a prime. $\phi(n)$ is Euler's phi function and $\mu(n)$ is the Möbius function.

For $\alpha > 1$ we have

$$\left| \frac{2}{p^{\alpha-1}(p+1)} \right| \leqslant \frac{2}{p^\alpha}, \left| \frac{2(\Delta/p)}{p^{\alpha-1}(p+1)} \right| \leqslant \frac{2}{p^\alpha},$$

so that both of the infinite series

$$\sum_{p \equiv 1 (\text{mod } 4)} \frac{2}{p^{\alpha-1}(p+1)}, \sum_{p \equiv 1 (\text{mod } 4)} \frac{2(\Delta/p)}{p^{\alpha-1}(p+1)},$$

converge absolutely. Hence both of the infinite products

$$\prod_{p \equiv 1 (\text{mod } 4)} \left(1 + \frac{2}{p^{\alpha-1}(p+1)}\right), \prod_{p \equiv 1 (\text{mod } 4)} \left(1 + \frac{2(\Delta/p)}{p^{\alpha-1}(p+1)}\right),$$

converge absolutely. We define

$$c(\alpha) = \prod_{p \equiv 1 (\text{mod } 4)} \left(1 + \frac{2}{p^{\alpha-1}(p+1)}\right) - 1, \qquad (2.3.2)$$

and

$$c(\alpha, \Delta) = \prod_{p \equiv 1 (\text{mod } 4)} \left(1 + \frac{2(\Delta/p)}{p^{\alpha-1}(p+1)}\right) - 1. \qquad (2.3.3)$$

We note that if $m \in \mathbb{N}$ is not divisible by a prime $\equiv 1 (\text{mod } 4)$ then

$$c(\alpha, \Delta) = c(\alpha, \Delta m^2).$$

Lemma 2.3.1. *For* $\alpha > 1$

(*a*) $$\sum_{\substack{n=1 \\ n \in \wp}}^{\infty} d(n) n^{-\alpha-1} \phi(n) \prod_{p \mid n} \left(1 - \frac{1}{p^2}\right)^{-1} = c(\alpha),$$

(*b*) $$\sum_{\substack{n=1 \\ n \in \wp}}^{\infty} \left(\frac{\Delta}{n}\right) d(n) n^{-\alpha-1} \phi(n) \prod_{p \mid n} \left(1 - \frac{1}{p^2}\right)^{-1} = c(\alpha, \Delta).$$

Before proving Lemma 2.3.1 we note that

$$0 < n^{-1}\phi(n) \prod_{p|n} \left(1 - \frac{1}{p^2}\right)^{-1} \leqslant 1 \tag{2.3.4}$$

for all positive integers n as

$$\phi(n) \prod_{p|n} \left(1 - \frac{1}{p^2}\right)^{-1} = n \prod_{p|n} \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p^2}\right)^{-1} = n \prod_{p|n} \left(1 + \frac{1}{p}\right)^{-1}.$$

Proof. We treat parts (a) and (b) together. We set

$$f(n) = \begin{cases} 1, & \text{part (a)}, \\ \left(\frac{\Delta}{n}\right), & \text{part (b)}, \end{cases} \tag{2.3.5}$$

so that $f(n)$ is a multiplicative function of n satisfying $|f(n)| \leqslant 1$ for all n. As $\alpha > 1$ we can choose ϵ so that $\alpha - 1 > \epsilon > 0$. Set $\beta = \alpha - \epsilon$ so that $\beta > 1$. By Lemma 2.2.3 we have $d(n) \leqslant A(\epsilon)n^{\epsilon}$. Then

$$\left| f(n)d(n)n^{-\alpha-1}\phi(n) \prod_{p|n} \left(1 - \frac{1}{p^2}\right)^{-1} \right| \leqslant \frac{A(\epsilon)}{n^{\beta}},$$

so that the infinite series

$$\sum_{\substack{n=1 \\ n \in \wp}}^{\infty} f(n)d(n)n^{-\alpha-1}\phi(n) \prod_{p|n} \left(1 - \frac{1}{p^2}\right)^{-1}$$

converges absolutely. As

$$f(n)d(n)n^{-\alpha-1}\phi(n) \prod_{p|n} \left(1 - \frac{1}{p^2}\right)^{-1}$$

is a multiplicative function of n, we have as $1 \notin \wp$,

$$\begin{aligned} &\sum_{\substack{n=1 \\ n \in \wp}}^{\infty} f(n)d(n)n^{-\alpha-1}\phi(n) \prod_{p|n} \left(1 - \frac{1}{p^2}\right)^{-1} \\ &= \prod_{p \equiv 1 (\mathrm{mod}\ 4)} \left(1 + f(p)d(p)p^{-\alpha-1}\phi(p)\left(1 - \frac{1}{p^2}\right)^{-1}\right) - 1 \\ &= \prod_{p \equiv 1 (\mathrm{mod}\ 4)} \left(1 + \frac{2f(p)}{p^{\alpha-1}(p+1)}\right) - 1 \\ &= \begin{cases} c(\alpha), & \text{part (a)}, \\ c(\alpha, \Delta), & \text{part (b)}, \end{cases} \end{aligned}$$

as asserted. □

Lemma 2.3.2. *For* $\alpha > 1$

$$(a)\quad \sum_{\substack{n>x\\ n\in\wp}} d(n)n^{-\alpha-1}\phi(n)\prod_{p|n}\left(1-\frac{1}{p^2}\right)^{-1} = O_\alpha\left(\frac{\log x}{x^{\alpha-1}}\right),$$

$$(b)\quad \sum_{\substack{n>x\\ n\in\wp}} \left(\frac{\Delta}{n}\right)d(n)n^{-\alpha-1}\phi(n)\prod_{p|n}\left(1-\frac{1}{p^2}\right)^{-1} = O_\alpha\left(\frac{\log x}{x^{\alpha-1}}\right).$$

Proof. We define $f(n)$ as in (2.3.5). By (2.3.4) and Lemma 2.2.5 we have

$$\left|\sum_{\substack{n>x\\ n\in\wp}} f(n)d(n)n^{-\alpha-1}\phi(n)\prod_{p|n}\left(1-\frac{1}{p^2}\right)^{-1}\right| \leqslant \sum_{n>x}\frac{d(n)}{n^\alpha} = O_\alpha\left(\frac{\log x}{x^{\alpha-1}}\right).$$

Lemma 2.3.3. *For* $\alpha > 1$

$$(a)\quad \sum_{\substack{n\leqslant x\\ n\in\wp}} d(n)n^{-\alpha-1}\phi(n)\prod_{p|n}\left(1-\frac{1}{p^2}\right)^{-1} = c(\alpha) + O_\alpha\left(\frac{\log x}{x^{\alpha-1}}\right),$$

$$(b)\quad \sum_{\substack{n\leqslant x\\ n\in\wp}} \left(\frac{\Delta}{n}\right)d(n)n^{-\alpha-1}\phi(n)\prod_{p|n}\left(1-\frac{1}{p^2}\right)^{-1} = c(\alpha,\Delta) + O_\alpha\left(\frac{\log x}{x^{\alpha-1}}\right).$$

Proof. This follows immediately from Lemmas 2.3.1 and 2.3.2. □

Lemma 2.3.4. *Let m be a positive integer. Then*

$$(a)\quad \sum_{\substack{n\leqslant x\\ n \text{ sqf}\\ (n,m)=1}} 1 = \frac{6}{\pi^2}x\frac{\phi(m)}{m}\prod_{p|m}\left(1-\frac{1}{p^2}\right)^{-1} + O\big(x^{\frac{1}{2}}d(m)\big),$$

$$(b)\quad \sum_{\substack{n\leqslant x\\ n \text{ sqf}\\ (n,m)=1}} \left(\frac{\Delta}{n}\right) = O\big(|\Delta|d(m)x^{\frac{1}{2}}\big).$$

We will use the following two well-known results

$$\sum_{d|n}\mu(d) = \begin{cases} 1, & \text{if } n=1,\\ 0, & \text{if } n>1,\end{cases} \tag{2.3.6}$$

and

$$\sum_{d^2|n}\mu(d) = \begin{cases} 1, & \text{if } n \text{ is squarefree},\\ 0, & \text{otherwise}.\end{cases} \tag{2.3.7}$$

Proof. We have by (2.3.7) and (2.3.6)

$$\begin{aligned}
\sum_{\substack{n \leqslant x \\ n \,\mathrm{sqf} \\ (n,m)=1}} 1 &= \sum_{\substack{n \leqslant x \\ (n,m)=1}} \sum_{d^2 \mid n} \mu(d) \\
&= \sum_{\substack{ad^2 \leqslant x \\ (ad^2,m)=1}} \mu(d) \\
&= \sum_{\substack{d \leqslant x^{\frac{1}{2}} \\ (d,m)=1}} \mu(d) \sum_{\substack{a \leqslant d^{-2}x \\ (a,m)=1}} 1 \\
&= \sum_{\substack{d \leqslant x^{\frac{1}{2}} \\ (d,m)=1}} \mu(d) \sum_{a \leqslant d^{-2}x} \sum_{e \mid (a,m)} \mu(e) \\
&= \sum_{\substack{d \leqslant x^{1/2} \\ (d,m)=1}} \mu(d) \sum_{e \mid m} \mu(e) \sum_{\substack{a \leqslant d^{-2}x \\ e \mid a}} 1 \\
&= \sum_{\substack{d \leqslant x^{\frac{1}{2}} \\ (d,m)=1}} \mu(d) \sum_{e \mid m} \mu(e) \sum_{b \leqslant d^{-2}e^{-1}x} 1 \\
&= \sum_{\substack{d \leqslant x^{\frac{1}{2}} \\ (d,m)=1}} \mu(d) \sum_{e \mid m} \mu(e) \left[\frac{x}{d^2 e} \right] \\
&= \sum_{\substack{d \leqslant x^{\frac{1}{2}} \\ (d,m)=1}} \mu(d) \sum_{e \mid m} \mu(e) \left(\frac{x}{d^2 e} + O(1) \right) \\
&= x \sum_{\substack{d \leqslant x^{\frac{1}{2}} \\ (d,m)=1}} \frac{\mu(d)}{d^2} \sum_{e \mid m} \frac{\mu(e)}{e} + O(x^{\frac{1}{2}} d(m)) \\
&= x \frac{\phi(m)}{m} \sum_{\substack{d \leqslant x^{\frac{1}{2}} \\ (d,m)=1}} \frac{\mu(d)}{d^2} + O(x^{\frac{1}{2}} d(m)).
\end{aligned}$$

Now

$$\begin{aligned}
\sum_{\substack{d \leqslant x^{1/2} \\ (d,m)=1}} \frac{\mu(d)}{d^2} &= \sum_{\substack{d=1 \\ (d,m)=1}}^{\infty} \frac{\mu(d)}{d^2} - \sum_{\substack{d > x^{1/2} \\ (d,m)=1}} \frac{\mu(d)}{d^2} \\
&= \prod_{p \nmid m} \left(1 - \frac{1}{p^2} \right) + O\left(\frac{1}{x^{\frac{1}{2}}} \right) \\
&= \frac{6}{\pi^2} \prod_{p \mid m} \left(1 - \frac{1}{p^2} \right)^{-1} + O\left(\frac{1}{x^{\frac{1}{2}}} \right).
\end{aligned}$$

Finally

$$\sum_{\substack{n \leqslant x \\ n\,\mathrm{sqf} \\ (n,m)=1}} 1 = x\frac{\phi(m)}{m}\left(\frac{6}{\pi^2}\prod_{p|n}\left(1-\frac{1}{p^2}\right)^{-1} + O\left(\frac{1}{x^{\frac{1}{2}}}\right)\right) + O(x^{\frac{1}{2}}d(m))$$

$$= \frac{6}{\pi^2}x\frac{\phi(m)}{m}\prod_{p|m}\left(1-\frac{1}{p^2}\right)^{-1} + O(x^{\frac{1}{2}}d(m)).$$

(b) By (2.3.7) we have

$$\sum_{\substack{n \leqslant x \\ n\,\mathrm{sqf} \\ (n,m)=1}} \left(\frac{\Delta}{n}\right) = \sum_{\substack{n \leqslant x \\ (n,m)=1}} \left(\frac{\Delta}{n}\right)\sum_{d^2|n}\mu(d)$$

$$= \sum_{\substack{ad^2 \leqslant x \\ (ad^2,m)=1}} \left(\frac{\Delta}{ad^2}\right)\mu(d)$$

$$= \sum_{\substack{d \leqslant x^{\frac{1}{2}} \\ (d,m)=1}} \mu(d)\left(\frac{\Delta}{d^2}\right)\sum_{\substack{a \leqslant x/d^2 \\ (a,m)=1}}\left(\frac{\Delta}{a}\right)$$

$$= \sum_{\substack{d \leqslant x^{\frac{1}{2}} \\ (d,\Delta m)=1}} \mu(d)\sum_{\substack{a \leqslant x/d^2 \\ (a,m)=1}}\left(\frac{\Delta}{a}\right).$$

Next, by (2.3.6), we have

$$\sum_{\substack{a \leqslant x/d^2 \\ (a,m)=1}} \left(\frac{\Delta}{a}\right) = \sum_{a \leqslant x/d^2}\left(\frac{\Delta}{a}\right)\sum_{e|(a,m)}\mu(e)$$

$$= \sum_{e|m}\mu(e)\sum_{\substack{a \leqslant x/d^2 \\ e|a}}\left(\frac{\Delta}{a}\right)$$

$$= \sum_{e|m}\mu(e)\left(\frac{\Delta}{e}\right)\sum_{b \leqslant d^{-2}e^{-1}x}\left(\frac{\Delta}{b}\right).$$

Now, as $\left(\frac{\Delta}{e}\right)$ is a character mod$|\Delta|$ ([10, Theorem 3.2, p. 305]), we have ([10, Theorem 2.3, p. 155])

$$\left|\sum_{b \leqslant d^{-2}e^{-1}x}\left(\frac{\Delta}{b}\right)\right| \leqslant |\Delta|,$$

so that

$$\left| \sum_{\substack{a \leqslant d^{-2}x \\ (a,m)=1}} \left(\frac{\Delta}{a} \right) \right| \leqslant |\Delta| d(m)$$

and

$$\left| \sum_{\substack{n \leqslant x \\ n \,\mathrm{sqf} \\ (n,m)=1}} \left(\frac{\Delta}{n} \right) \right| \leqslant |\Delta| d(m) x^{\frac{1}{2}}.$$

This completes the proof of Lemma 2.3.4. □

Lemma 2.3.5. *Let m be a positive integer with the following property*:

$$\text{if } p \mid m \text{ then } p \not\equiv 1 \pmod 4. \tag{2.3.8}$$

Let $\alpha > 1$ and set

$$\beta = \begin{cases} 3, & \text{if } \alpha \neq 2, \\ 4, & \text{if } \alpha = 2. \end{cases}$$

Then

(*a*) $$\sum_{\substack{D \leqslant y \\ D \in \wp}} d(D) \sum_{\substack{A \leqslant zD^{-\alpha} \\ A \,\mathrm{sqf} \\ (A,mD)=1}} 1 = \frac{6}{\pi^2} c(\alpha) z \frac{\phi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right)^{-1} + O_\alpha\left(\frac{z \log y}{y^{\alpha-1}}\right) + O_\alpha\left(d(m) z^{\frac{1}{2}} y^{1-\frac{\alpha}{2}} \log^\beta y\right),$$

(*b*) $$\sum_{\substack{D \leqslant y \\ D \in \wp}} \left(\frac{\Delta}{D}\right) d(D) \sum_{\substack{A \leqslant zD^{-\alpha} \\ A \,\mathrm{sqf} \\ (A,mD)=1}} 1 = \frac{6}{\pi^2} c(\alpha, \Delta) z \frac{\phi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right)^{-1} + O_\alpha\left(\frac{z \log y}{y^{\alpha-1}}\right) + O_\alpha\left(d(m) z^{\frac{1}{2}} y^{1-\frac{\alpha}{2}} \log^\beta y\right).$$

Proof. By (2.3.8) we see that

$$(m, D) = 1 \text{ for all } D \in \wp$$

so that

$$d(mD) = d(m) d(D) \text{ for all } D \in \wp.$$

Thus, for $D \in \wp$, we have by Lemma 2.3.4 (a)

$$\sum_{\substack{A \leqslant zD^{-\alpha} \\ A \text{ sqf} \\ (A,mD)=1}} 1 = \frac{6}{\pi^2} z \frac{\phi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right)^{-1} D^{-\alpha-1}\phi(D) \prod_{p|D} \left(1 - \frac{1}{p^2}\right)^{-1} + O\big(z^{\frac{1}{2}} D^{-\frac{\alpha}{2}} d(m) d(D)\big).$$

(a) Hence

$$\sum_{\substack{D \leqslant y \\ D \in \wp}} d(D) \sum_{\substack{A \leqslant zD^{-\alpha} \\ A \text{ sqf} \\ (A,mD)=1}} 1 = \frac{6}{\pi^2} z \frac{\phi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right)^{-1} \sum_{\substack{D \leqslant y \\ D \in \wp}} d(D) D^{-\alpha-1} \phi(D) \prod_{p|D} \left(1 - \frac{1}{p^2}\right)^{-1} + O\left(z^{\frac{1}{2}} d(m) \sum_{D \leqslant y} \frac{d^2(D)}{D^{\frac{\alpha}{2}}}\right).$$

Appealing to Lemma 2.3.3 (a), Lemma 2.2.7 and (2.3.4), we obtain

$$\begin{aligned}\sum_{\substack{D \leqslant y \\ D \in \wp}} d(D) \sum_{\substack{A \leqslant zD^{-\alpha} \\ A \text{ sqf} \\ (A,mD)=1}} 1 &= \frac{6}{\pi^2} z \frac{\phi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right)^{-1} \left(c(\alpha) + O_\alpha\left(\frac{\log\ y}{y^{\alpha-1}}\right)\right) \\ &\quad + O_\alpha\big(z^{\frac{1}{2}} d(m) y^{1-\frac{\alpha}{2}} \log^\beta y\big) \\ &= \frac{6}{\pi^2} c(\alpha) z \frac{\phi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right)^{-1} + O_\alpha\left(\frac{z \log\ y}{y^{\alpha-1}}\right) \\ &\quad + O_\alpha\big(d(m) z^{\frac{1}{2}} y^{1-\frac{\alpha}{2}} \log^\beta y\big).\end{aligned}$$

(b) The proof is exactly the same as for part (a) except that Lemma 2.3.3 (b) is used in place of Lemma 2.3.3 (a). □

Lemma 2.3.6. *Let r and s be real numbers. Let* Δ, Δ' *be nonsquare integers such that* $\Delta\Delta'$ *is nonsquare and* Δ, $\Delta' \equiv 0, 1 \pmod 4$. *Then*

(*a*)
$$\sum_{\substack{D \leqslant 2^{-r} x^{\frac{1}{3}} \\ D \in \wp}} d(D) \left(1 \pm \left(\frac{\Delta}{D}\right)\right) \sum_{\substack{1 \leqslant A \leqslant 2^{-s} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,2D)=1}} 1 = \frac{4}{2^s \pi^2} (c(3/2) \pm c(3/2, \Delta)) x^{\frac{1}{2}} + O_{r,s}(x^{\frac{1}{3}} \log^3 x).$$

$$(b)\quad \sum_{\substack{D\leqslant 2^{-r}x^{\frac13}\\ D\in\wp}} d(D)\left(1\pm\left(\frac{\Delta}{D}\right)\right)\sum_{\substack{1\leqslant A\leqslant 2^{-s}x^{\frac12}D^{-\frac32}\\ A\text{ sqf}\\ (A,6D)=1}} 1$$

$$=\frac{3}{2^s\pi^2}(c(3/2)\pm c(3/2,\Delta))x^{\frac12}+O_{r,s}(x^{\frac13}\log^3 x).$$

$$(c)\quad \sum_{\substack{D\leqslant x^{\frac13}\\ D\in\wp}} d(D)\left(1+\left(\frac{\Delta}{D}\right)\right)\left(1+\left(\frac{\Delta'}{D}\right)\right)\sum_{\substack{1\leqslant A\leqslant x^{\frac12}D^{-\frac32}\\ A\text{ sqf}\\ (A,6D)=1}} 1$$

$$=\frac{3}{\pi^2}(c(3/2)+c(3/2,\Delta)+c(3/2,\Delta')+c(3/2,\Delta\Delta'))x^{\frac12}+O(x^{\frac13}\log^3 x).$$

Proof. (a) Taking

$$y=2^{-r}x^{\frac13},\quad z=2^{-s}x^{\frac12},\quad \alpha=3/2 \text{ (so that } \beta=3),\quad m=2,$$

in Lemma 2.3.5 (a) (b), we obtain

$$\sum_{\substack{D\leqslant 2^{-r}x^{\frac13}\\ D\in\wp}} d(D)\sum_{\substack{1\leqslant A\leqslant 2^{-s}x^{\frac12}D^{-\frac32}\\ A\text{ sqf}\\ (A,2D)=1}} 1=\frac{4}{2^s\pi^2}c(3/2)x^{\frac12}+O_{r,s}\left(x^{\frac13}\log^3 x\right)$$

and

$$\sum_{\substack{D\leqslant 2^{-r}x^{\frac13}\\ D\in\wp}} \left(\frac{\Delta}{D}\right)d(D)\sum_{\substack{1\leqslant A\leqslant 2^{-s}x^{\frac12}D^{-\frac32}\\ A\text{ sqf}\\ (A,2D)=1}} 1=\frac{4}{2^s\pi^2}c(3/2,\Delta)x^{\frac12}+O_{r,s}\left(x^{\frac13}\log^3 x\right).$$

Adding and subtracting these equations, we obtain the asserted result.

(b) Taking

$$y=2^{-r}x^{\frac13},\quad z=2^{-s}x^{\frac12},\quad \alpha=3/2,\quad m=6,$$

in Lemma 2.3.5 (a) (b), we obtain in a similar manner the result of part (b).

(c) Taking

$$y=x^{\frac13},\quad z=x^{\frac12},\quad \alpha=3/2,\quad m=6,$$

in Lemma 2.3.5 (a) and in Lemma 2.3.5 (b) for Δ, Δ' and $\Delta\Delta'$ and adding the resulting equations, we obtain the result of part (c). □

Lemma 2.3.7. *Let r and s be real numbers. Let m be a positive integer and let Δ be a nonsquare integer $\equiv 0$ or $1 \pmod 4$. Then*

$$\sum_{D\leqslant 2^{-r}x^{\frac13}} d(D)\left|\sum_{\substack{1\leqslant A\leqslant 2^{-s}x^{\frac12}D^{-\frac32}\\ A\text{ sqf}\\ (A,mD)=1}}\left(\frac{\Delta}{A}\right)\right|=O_{\Delta,m,r,s}(x^{\frac13}\log^3 x).$$

Proof. By Lemma 2.3.4 (b) we have

$$\sum_{\substack{1 \leqslant A \leqslant 2^{-s} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,mD)=1}} \left(\frac{\Delta}{A}\right) = O(|\Delta| d(mD) 2^{-\frac{s}{2}} x^{\frac{1}{4}} D^{-\frac{3}{4}}) = O_{\Delta,m,s}(x^{\frac{1}{4}} d(D) D^{-\frac{3}{4}}),$$

as $d(mD) \leqslant d(m)d(D)$ ([10, Theorem 5.1, p. 111]). Then, appealing to Lemma 2.2.7 with $\alpha = \frac{3}{4}$, we obtain

$$\begin{aligned}\sum_{D \leqslant 2^{-r} x^{\frac{1}{3}}} d(D) \left| \sum_{\substack{1 \leqslant A \leqslant 2^{-s} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,mD)=1}} \left(\frac{\Delta}{A}\right) \right| &= O_{\Delta,m,s}\left(x^{\frac{1}{4}} \sum_{D \leqslant 2^{-r} x^{\frac{1}{3}}} \frac{d^2(D)}{D^{\frac{3}{4}}} \right) \\ &= O_{\Delta,m,r,s}(x^{\frac{1}{3}} \log^3 x),\end{aligned}$$

as asserted. □

2.4. Representations by binary quadratic forms. A binary quadratic form is an expression of the type $ax^2 + bxy + cy^2$, where a, b, c are real numbers. We write (a, b, c) for $ax^2 + bxy + cy^2$. The discriminant d of (a, b, c) is the real number $d = b^2 - 4ac$. The binary quadratic form (a, b, c) is said to be integral if a, b, c are integers in which case d is an integer $\equiv 0$ or $1 \pmod 4$. An integral binary quadratic form is said to be primitive if $\gcd(a, b, c) = 1$. A binary quadratic form is said to be positive-definite if it only assumes positive values for $(x, y) (\neq (0, 0)) \in \mathbb{R}^2$, equivalently $a > 0$ and $d < 0$. By a form we shall mean a positive-definite primitive integral binary quadratic form. Two forms (a, b, c) and (A, B, C) are said to be equivalent if there exist $p, q, r, s \in \mathbb{Z}$ with $ps - qr = 1$ such that

$$ax^2 + bxy + cy^2 = A(px + qy)^2 + B(px + qy)(rx + sy) + C(rx + sy)^2.$$

Equivalent forms have the same discriminant. The positive integer n is said to be represented by the form (a, b, c) if there exist $x, y \in \mathbb{Z}$ such that $n = ax^2 + bxy + cy^2$. Equivalent forms represent the same positive integers. A form (a, b, c) is said to be reduced [1, p. 68] if

$$-a < b \leqslant a \leqslant c \text{ with } b \geqslant 0 \text{ if } a = c.$$

A classical result of Gauss asserts that every form is equivalent to one and only one reduced form [1, p. 71]. We denote the finite set of reduced forms of discriminant d by $H(d)$. We also set

$$N_{(a,b,c)}(n) = \text{ number of } (x, y) \in \mathbb{Z}^2 \text{ such that } n = ax^2 + bxy + cy^2.$$

Dirichlet has evaluated the sum

$$\sum_{(a,b,c) \in H(d)} N_{(a,b,c)}(n).$$

Theorem 2.4.1. (Dirichlet) *If* $(n, d) = 1$ *then*

$$\sum_{(a,b,c) \in H(d)} N_{(a,b,c)}(n) = w(d) \sum_{k|n} \left(\frac{d}{k}\right),$$

where

$$w(d) = \begin{cases} 6, & \text{if } d = -3, \\ 4, & \text{if } d = -4, \\ 2, & \text{if } d < -4. \end{cases}$$

Proof. A proof of Dirichlet's theorem can be found for example in [1, Theorem 64, p. 78], [10, Theorem 4.1, p. 307], see also [12]. □

We now derive from Theorem 2.4.1 the results that we shall need for discriminants -36, -64, -144, -256, -576 and -2304. As the proofs are very similar we just prove the results for discriminants -36 and -2304.

Lemma 2.4.2. *Let D be a positive integer such that* $D \in \wp$ *and* $D \equiv 2 \pmod 3$. *Then*

$$N_{(1,0,9)}(2D) = N_{(2,2,5)}(D) = 2d(D).$$

Lemma 2.4.3. *Let D be a positive integer such that* $D \in \wp$ *and* $D \equiv 1 \pmod 8$. *Then*

$$N_{(1,0,16)}(D) = 2d(D).$$

Lemma 2.4.4. *Let D be a positive integer such that* $D \in \wp$ *and* $D \equiv 1 \pmod 3$. *Then*

$$N_{(1,0,36)}(D) + N_{(4,0,9)}(D) = 2d(D).$$

Lemma 2.4.5. *Let D be a positive integer such that* $D \in \wp$ *and* $D \equiv 1 \pmod 8$. *Then*

$$N_{(1,0,64)}(D) + N_{(4,4,17)}(D) = 2d(D).$$

Lemma 2.4.6. *Let D be a positive integer such that* $D \in \wp$ *and* $D \equiv 1 \pmod 3$. *Then*

$$N_{(1,0,144)}(D) + N_{(4,4,37)}(D) + N_{(9,0,16)}(D) + N_{(13,10,13)}(D) = 2d(D).$$

If in addition $D \equiv 1 \pmod 8$ *then*

$$N_{(1,0,144)}(D) + N_{(9,0,16)}(D) = 2d(D).$$

Lemma 2.4.7. *Let D be a positive integer such that* $D \in \wp$ *and* $D \equiv 1 \pmod{24}$. *Then*

$$N_{(1,0,576)}(D) + N_{(4,4,145)}(D) + N_{(9,0,64)}(D) + N_{(25,14,25)}(D) = 2d(D).$$

Proof of Lemma 2.4.2. Let

$$A = \{(x, y) \in \mathbb{Z}^2 \mid 2D = x^2 + 9y^2\}$$

and

$$B = \{(x, y) \in \mathbb{Z}^2 \mid D = 2x^2 + 2xy + 5y^2\}.$$

The mapping $f : A \to B$ given by

$$f((x, y)) = \left(\frac{x-y}{2}, y\right)$$

is a bijection. Hence

$$N_{(1,0,9)}(2D) = N_{(2,2,5)}(D).$$

The set of reduced forms of discriminant -36 is

$$H(-36) = \{(1,\ 0,\ 9), (2,\ 2,\ 5)\}.$$

Let n be a positive integer with $(n, -36) = 1$. Then, by Theorem 2.4.1, we have

$$N_{(1,0,9)}(n) + N_{(2,2,5)}(n) = 2\sum_{k|n}\left(\frac{-36}{k}\right).$$

Now

$$x^2 + 9y^2 \equiv x^2 \equiv 0 \text{ or } 1 \pmod 3$$

so that the form $(1,\ 0,\ 9)$ does not represent integers $\equiv 2 \pmod 3$. Hence

$$N_{(1,0,9)}(D) = 0$$

and so

$$N_{(2,2,5)}(D) = 2\sum_{k|D}\left(\frac{-36}{k}\right).$$

Now D is a product of distinct primes $\equiv 1 \pmod 4$ so that

$$\sum_{k|D}\left(\frac{-36}{k}\right) = \sum_{k|D}\left(\frac{-2^2 3^2}{k}\right) = \sum_{k|D}\left(\frac{-1}{k}\right) = \sum_{k|D} 1 = d(D).$$

Hence

$$N_{(2,2,5)}(D) = 2d(D),$$

as asserted. □

Proof of Lemma 2.4.7. The set of reduced forms of discriminant -2304 is

$$\begin{aligned} H(-2304) = \{&(1,\ 0,\ 576), (4,\ 4,\ 145), (5,\ 4,\ 116), (5,\ -4,\ 116),\\ &(9,\ 0,\ 64), (9,\ 6,\ 65), (9,\ -6,\ 65), (13,\ 6,\ 45),\\ &(13,\ -6,\ 45), (16,\ 8,\ 37), (16,\ -8,\ 37), (17,\ 12,\ 36),\\ &(17,\ -12,\ 36), (20,\ 4,\ 29), (20,\ -4,\ 29), (25,\ 14,\ 25)\}. \end{aligned}$$

Let n be a positive integer satisfying $(n, -2304) = 1$. Then, by Theorem 2.4.1, we obtain

$$N_{(1,0,576)}(n) + N_{(4,4,145)}(n) + \cdots + N_{(25,14,25)}(n) = 2\sum_{k|n}\left(\frac{-2304}{k}\right).$$

Clearly

$$\begin{aligned}
&9x^2 \pm 6xy + 65y^2 \equiv 0,\ 2 \pmod 3,\\
&17x^2 \pm 12xy + 36y^2 \equiv 0,\ 2 \pmod 3,\\
&5x^2 \pm 4xy + 116y^2 \equiv \frac{1}{5}((5x \pm 2y)^2 + 576y^2) \equiv 0,\ 2 \pmod 3,\\
&20x^2 \pm 4xy + 29y^2 \equiv \frac{1}{5}((10x \pm y)^2 + 144y^2) \equiv 0,\ 2 \pmod 3,
\end{aligned}$$

so that none of these forms represents an integer $\equiv 1 \pmod 3$. Hence

$$N_{(9,\ \pm 6, 65)}(D) = N_{(17,\ \pm 12, 36)}(D) = N_{(5,\ \pm 4, 116)}(D) = N_{(20,\ \pm 4, 29)}(D) = 0.$$

Also

$$\begin{aligned}
&16x^2 \pm 8xy + 37y^2 \equiv 0,\ 4,\ 5 \pmod 8,\\
&13x^2 \pm 6xy + 45y^2 \equiv \frac{1}{5}((x \pm 15y)^2 + 64x^2) \equiv 0,\ 4,\ 5 \pmod 8,
\end{aligned}$$

so that none of these forms represents an integer $\equiv 1 \pmod 8$. Hence

$$N_{(16,\ \pm 8, 37)}(D) = N_{(13,\ \pm 6, 45)}(D) = 0.$$

Thus

$$N_{(1,0,576)}(D) + N_{(4,4,145)}(D) + N_{(9,0,64)}(D) + N_{(25,14,25)}(D) = 2\sum_{k|D}\left(\frac{-2304}{k}\right).$$

As D is a product of distinct primes $\equiv 1 \pmod 4$, we have

$$\sum_{k|D}\left(\frac{-2304}{k}\right) = \sum_{k|D}\left(\frac{-2^8 3^2}{k}\right) = \sum_{k|D}\left(\frac{-1}{k}\right) = \sum_{k|D} 1 = d(D).$$

Finally

$$N_{(1,0,576)}(D) + N_{(4,4,145)}(D) + N_{(9,0,64)}(D) + N_{(25,14,25)}(D) = 2d(D),$$

as asserted. □

2.5. Number of cyclic quartic fields with discriminant $\leqslant x$ and index divisible by $2, 3, 4, 6$ **or** 12. Let a and b be integers with $0 \leqslant a < b$. We define for $x > 0$

$$N(x,a,b) = \text{ number of cyclic quartic fields } K \text{ with } d(K) \leqslant x \text{ and } i(K) \equiv a \pmod b. \tag{2.5.1}$$

First we determine an asymptotic formula for $N(x,0,2)$ valid for large x.

Theorem 2.5.1.

$$N(x,0,2) = \frac{1}{\pi^2}(c(3/2) + c(3/2,8))x^{\frac{1}{2}} + O(x^{\frac{1}{3}}\log^3 x),$$

where $c(\alpha)$ *and* $c(\alpha,\Delta)$ *are defined in* (2.3.2) *and* (2.3.3) *respectively.*

Proof. By (1.2.1)–(1.2.6) and Theorem 1.3.2 we have

$$N(x,0,2) = \text{number of } (A,B,D) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N} \text{ such that } \begin{array}{l} A^2D^3 \leqslant x \\ A \equiv 1 \pmod 4,\quad B \equiv 0 \pmod 4 \\ A \text{ sqf} \\ B \geqslant 1,\ D \geqslant 2,\ D \text{ sqf} \\ D - B^2 = \square \\ (A,D) = 1 \end{array}$$

so that

$$N(x,0,2) = \sum_{\substack{2 \leqslant D \leqslant x^{\frac{1}{3}} \\ D \text{ sqf}}} \sum_{\substack{|A| \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \equiv 1 (\text{mod } 4) \\ A \text{ sqf} \\ (A,D)=1}} \sum_{\substack{B>0 \\ B \equiv 0 (\text{mod } 4) \\ D-B^2=\square}} 1.$$

As $D \geqslant 2$ is squarefree and $D = B^2 + C^2$ with $B \equiv 0 \pmod 4$ and C odd, we see that $D \in \wp$ and $D \equiv 1 \pmod 8$ so that

$$N(x,0,2) = \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 (\text{mod } 8)}} \sum_{\substack{|A| \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \equiv 1 (\text{mod } 4) \\ A \text{ sqf} \\ (A,D)=1}} \sum_{\substack{B>0 \\ B \equiv 0 (\text{mod } 4) \\ D-B^2=\square}} 1.$$

Next

$$\begin{aligned} \sum_{\substack{B>0 \\ B \equiv 0 (\text{mod } 4) \\ D-B^2=\square}} 1 &= \sum_{\substack{B>0 \\ D-16B^2=\square}} 1 = \sum_{\substack{B<0 \\ D-16B^2=\square}} 1 \\ &= \frac{1}{2} \sum_{\substack{B \neq 0 \\ D-16B^2=\square}} 1 = \frac{1}{2} \sum_{\substack{B \\ D-16B^2=\square}} 1 \\ &= \frac{1}{4} \sum_{\substack{B,C \\ D-16B^2=C^2}} 1 = \frac{1}{4} N_{(1,0,16)}(D) \\ &= \frac{1}{2} d(D), \end{aligned}$$

by Lemma 2.4.2. Thus

$$N(x,0,2) = \frac{1}{2} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 (\text{mod } 8)}} d(D) \sum_{\substack{|A| \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \equiv 1\, (\text{mod}\, 4) \\ A \text{ sqf} \\ (A,D)=1}} 1.$$

For A odd, exactly one of A and $-A$ is $\equiv 1 \pmod 4$, so that

$$N(x,0,2) = \frac{1}{2} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod 8}} d(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,2D)=1}} 1.$$

For $D \in \wp$ we have $D \equiv 1 \pmod 4$ and

$$\left(\frac{8}{D}\right) = \left(\frac{2}{D}\right) = \begin{cases} 1, & \text{if } D \equiv 1 \pmod 8, \\ -1, & \text{if } D \equiv 5 \pmod 8, \end{cases}$$

so that

$$N(x,0,2) = \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp}} d(D) \left(1 + \left(\frac{8}{D}\right)\right) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,2D)=1}} 1.$$

Appealing to Lemma 2.3.6 (a) (with $r = 0$, $s = 0$ and $\Delta = 8$), we obtain

$$N(x,0,2) = \frac{1}{\pi^2} (c(3/2) + c(3/2,8)) x^{\frac{1}{2}} + O(x^{\frac{1}{3}} \log^3 x),$$

as asserted. □

Next we determine an asymptotic formula for $N(x,0,3)$.

Theorem 2.5.2.

$$N(x,0,3) = \frac{3}{128\pi^2} ((24+\sqrt{2})c(3/2) + (24-\sqrt{2})c(3/2,-3)) x^{\frac{1}{2}} + O(x^{\frac{1}{3}} \log^3 x).$$

Proof. By (1.2.1)–(1.2.6) and Theorem 1.3.9 we have

$N(x,0,3) =$ number of $(A, B, D) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$ such that

$2^\alpha A^2 D^3 \leqslant x$ with α given by (1.2.6),
A sqf and odd,
$B \geqslant 1,\ D \geqslant 2,\ D$ sqf,
$D - B^2 = \square$,
$(A,D) = 1$,
$A \equiv 1 \pmod 3,\ B \equiv 0 \pmod 3$, or
$A \equiv 2 \pmod 3,\ D - B^2 \equiv 0 \pmod 9$.

We observe that $D = 2$ does not contribute to $N(x,0,3)$. Taking the four possibilities for α $(= 0, 4, 6, 8)$ together with the two possibilities $A \equiv 1 \pmod 3$, $B \equiv 0 \pmod 3$ and $A \equiv 2 \pmod 3$, $D - B^2 \equiv 0 \pmod 9$, we can express $N(x,0,3)$ as the sum of eight subsums as follows:

$$N(x,0,3) = \sum_{i=1}^{8} S_i,$$

where

$$S_1 = \sum_{\substack{2 \leqslant D \leqslant x^{\frac{1}{3}} \\ D \text{ sqf} \\ D \equiv 1 \pmod 2}} \sum_{\substack{|A| \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,2D)=1 \\ A \equiv 1 \pmod 3}} \sum_{\substack{B>0 \\ D-B^2=\square \\ B \equiv 0 \pmod 2 \\ B \equiv 0 \pmod 3 \\ B \equiv 1-A \pmod 4}} 1,$$

$$S_2 = \sum_{\substack{2 \leqslant D \leqslant x^{\frac{1}{3}} \\ D \text{ sqf} \\ D \equiv 1 \pmod 2}} \sum_{\substack{|A| \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,2D)=1 \\ A \equiv 2 \pmod 3}} \sum_{\substack{B>0 \\ D-B^2=\square \equiv 0 \pmod 9 \\ B \equiv 0 \pmod 2 \\ B \equiv 1-A \pmod 4}} 1,$$

$$S_3 = \sum_{\substack{2 \leqslant D \leqslant 2^{-\frac{4}{3}} x^{\frac{1}{3}} \\ D \text{ sqf} \\ D \equiv 1 \pmod 2}} \sum_{\substack{|A| \leqslant 2^{-2} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,2D)=1 \\ A \equiv 1 \pmod 3}} \sum_{\substack{B>0 \\ D-B^2=\square \\ B \equiv 0 \pmod 2 \\ B \equiv 0 \pmod 3 \\ B \equiv 3-A \pmod 4}} 1,$$

$$S_4 = \sum_{\substack{2 \leqslant D \leqslant 2^{-\frac{4}{3}} x^{\frac{1}{3}} \\ D \text{ sqf} \\ D \equiv 1 \pmod 2}} \sum_{\substack{|A| \leqslant 2^{-2} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,2D)=1 \\ A \equiv 2 \pmod 3}} \sum_{\substack{B>0 \\ D-B^2=\square \equiv 0 \pmod 9 \\ B \equiv 0 \pmod 2 \\ B \equiv 3-A \pmod 4}} 1,$$

$$S_5 = \sum_{\substack{2 \leqslant D \leqslant 2^{-2} x^{\frac{1}{3}} \\ D \text{ sqf} \\ D \equiv 1 \pmod 2}} \sum_{\substack{|A| \leqslant 2^{-3} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{sqf} \\ (A,2D)=1 \\ A \equiv 1 \pmod 3}} \sum_{\substack{B>0 \\ D-B^2=\square \equiv 0 \pmod 4 \\ B \equiv 1 \pmod 2 \\ B \equiv 0 \pmod 3}} 1,$$

$$S_6 = \sum_{\substack{2 \leqslant D \leqslant 2^{-2} x^{\frac{1}{3}} \\ D \text{ sqf} \\ D \equiv 1 \pmod 2}} \sum_{\substack{|A| \leqslant 2^{-3} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,2D)=1 \\ A \equiv 2 \pmod 3}} \sum_{\substack{B>0 \\ D-B^2=\square \equiv 0 \pmod{36} \\ B \equiv 1 \pmod 2}} 1,$$

$$S_7 = \sum_{\substack{2 \leqslant D \leqslant 2^{-\frac{11}{3}} x^{\frac{1}{3}} \\ D \text{ sqf} \\ D \equiv 1 \pmod 2}} \sum_{\substack{|A| \leqslant 2^{-\frac{11}{2}} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,2D)=1 \\ A \equiv 1 \pmod 3}} \sum_{\substack{B>0 \\ 2D-B^2=\square \\ B \equiv 0 \pmod 3}} 1,$$

$$S_8 = \sum_{\substack{2 \leqslant D \leqslant 2^{-\frac{11}{3}} x^{\frac{1}{3}} \\ D \text{ sqf} \\ D \equiv 1 \pmod 2}} \sum_{\substack{|A| \leqslant 2^{-\frac{11}{2}} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,2D)=1 \\ A \equiv 2 \pmod 3}} \sum_{\substack{B>0 \\ 2D-B^2=\square \equiv 0 \pmod 9}} 1,$$

where in the sums S_7 and S_8 we replaced D by $2D$.

Next we examine the inner sums over B for each of $S_1, \ldots, S_8$. First we consider the inner sum for S_1.

$$\sum_{\substack{B>0 \\ D-B^2=\square \\ B\equiv 0 \pmod 2 \\ B\equiv 0 \pmod 3 \\ B\equiv 1-A \pmod 4}} 1 = \frac{1}{4} \sum_{\substack{B,C \\ D=C^2+36B^2 \\ 6B\equiv 1-A \pmod 4}} 1$$

$$= \begin{cases} \frac{1}{4} \sum\limits_{\substack{B,C \\ D=C^2+144B^2}} 1, & A\equiv 1 \pmod 4, \\ \frac{1}{4} \sum\limits_{\substack{B,C \\ D=C^2+36B^2 \\ B\equiv 1 \pmod 2}} 1, & A\equiv 3 \pmod 4. \end{cases}$$

For the sum S_1 we have $D\equiv 1 \pmod 2$ so that $C\equiv 1 \pmod 2$. Thus, in the case $A\equiv 3 \pmod 4$, we have $B\equiv C\equiv 1 \pmod 2$, so that there exists an integer X such that $C = B + 2X$. Then

$$\sum_{\substack{B,C \\ D=C^2+36B^2 \\ B\equiv 1 \pmod 2}} 1 = \sum_{\substack{B,X \\ D=4X^2+4XB+37B^2}} 1 = N_{(4,4,37)}(D).$$

Hence

$$\sum_{\substack{B>0 \\ D-B^2=\square \\ B\equiv 0 \pmod 2 \\ B\equiv 0 \pmod 3 \\ B\equiv 1-A \pmod 4}} 1 = \begin{cases} \frac{1}{4}N_{(1,0,144)}(D), & A\equiv 1 \pmod 4, \\ \frac{1}{4}N_{(4,4,37)}(D), & A\equiv 3 \pmod 4. \end{cases}$$

Similarly for the inner sum of S_2 we find that

$$\sum_{\substack{B>0 \\ D-B^2=\square\equiv 0 \pmod 9 \\ B\equiv 0 \pmod 2 \\ B\equiv 1-A \pmod 4}} 1 = \begin{cases} \frac{1}{4}N_{(9,0,16)}(D), & A\equiv 1 \pmod 4, \\ \frac{1}{4}N_{(13,10,13)}(D), & A\equiv 3 \pmod 4, \end{cases}$$

for the inner sum of S_3

$$\sum_{\substack{B>0 \\ D-B^2=\square \\ B\equiv 0 \pmod 2 \\ B\equiv 0 \pmod 3 \\ B\equiv 3-A \pmod 4}} 1 = \begin{cases} \frac{1}{4}N_{(4,4,37)}(D), & A\equiv 1 \pmod 4, \\ \frac{1}{4}N_{(1,0,144)}(D), & A\equiv 3 \pmod 4, \end{cases}$$

for the inner sum of S_4

$$\sum_{\substack{B>0\\ D-B^2=\square\equiv 0\,(\mathrm{mod}\,9)\\ B\equiv 0\,(\mathrm{mod}\,2)\\ B\equiv 3-A\,(\mathrm{mod}\,4)}} 1 = \begin{cases} \frac{1}{4}N_{(13,\,10,\,13)}(D), & A\equiv 1\,(\mathrm{mod}\,4),\\ \frac{1}{4}N_{(9,\,0,\,16)}(D), & A\equiv 3\,(\mathrm{mod}\,4),\end{cases}$$

for the inner sum of S_5

$$\sum_{\substack{B>0\\ D-B^2=\square\equiv 0\,(\mathrm{mod}\,4)\\ B\equiv 1\,(\mathrm{mod}\,2)\\ B\equiv 0\,(\mathrm{mod}\,3)}} 1 = \frac{1}{4}N_{(4,\,0,\,9)}(D),$$

for the inner sum of S_6

$$\sum_{\substack{B>0\\ D-B^2=\square\equiv 0\,(\mathrm{mod}\,36)\\ B\equiv 1\,(\mathrm{mod}\,2)}} 1 = \frac{1}{4}N_{(1,\,0,\,36)}(D),$$

for the inner sum of S_7

$$\sum_{\substack{B>0\\ 2D-B^2=\square\\ B\equiv 0\,(\mathrm{mod}\,3)}} 1 = \frac{1}{4}N_{(1,\,0,\,9)}(2D),$$

and for the inner sum of S_8

$$\sum_{\substack{B>0\\ 2D-B^2=\square\equiv 0\,(\mathrm{mod}\,9)}} 1 = \frac{1}{4}N_{(1,\,0,\,9)}(2D).$$

Thus

$$S_1 = \frac{1}{4}\sum_{\substack{D\leqslant x^{\frac{1}{3}}\\ D\in\wp\\ D\equiv 1\,(\mathrm{mod}\,3)}} \left(N_{(1,\,0,\,144)}(D) \sum_{\substack{1\leqslant A\leqslant x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\ \mathrm{sqf}\\ (A,6D)=1\\ A\equiv\ \pm 1\,(\mathrm{mod}\,12)}} 1 + N_{(4,\,4,\,37)}(D) \sum_{\substack{1\leqslant A\leqslant x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\ \mathrm{sqf}\\ (A,6D)=1\\ A\equiv\ \pm 5\,(\mathrm{mod}\,12)}} 1 \right),$$

$$S_2 = \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\mathrm{mod}\, 3)}} \left(N_{(9,0,16)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1 \\ A \equiv \pm 5 \,(\mathrm{mod}\, 12)}} 1 + N_{(13,10,13)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1 \\ A \equiv \pm 1 \,(\mathrm{mod}\, 12)}} 1 \right),$$

$$S_3 = \frac{1}{4} \sum_{\substack{D \leqslant 2^{-\frac{4}{3}} x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\mathrm{mod}\, 3)}} \left(N_{(4,4,37)}(D) \sum_{\substack{1 \leqslant A \leqslant 2^{-2} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1 \\ A \equiv \pm 1 \,(\mathrm{mod}\, 12)}} 1 + N_{(1,0,144)}(D) \sum_{\substack{1 \leqslant A \leqslant 2^{-2} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1 \\ A \equiv \pm 5 \,(\mathrm{mod}\, 12)}} 1 \right),$$

$$S_4 = \frac{1}{4} \sum_{\substack{D \leqslant 2^{-\frac{4}{3}} x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\mathrm{mod}\, 3)}} \left(N_{(13,10,13)}(D) \sum_{\substack{1 \leqslant A \leqslant 2^{-2} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1 \\ A \equiv \pm 5 \,(\mathrm{mod}\, 12)}} 1 + N_{(9,0,16)}(D) \sum_{\substack{1 \leqslant A \leqslant 2^{-2} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1 \\ A \equiv \pm 1 \,(\mathrm{mod}\, 12)}} 1 \right),$$

$$S_5 = \frac{1}{4} \sum_{\substack{D \leqslant 2^{-2} x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\mathrm{mod}\, 3)}} N_{(4,0,9)}(D) \sum_{\substack{1 \leqslant A \leqslant 2^{-3} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1,$$

$$S_6 = \frac{1}{4} \sum_{\substack{D \leqslant 2^{-2} x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\mathrm{mod}\, 3)}} N_{(1,0,36)}(D) \sum_{\substack{1 \leqslant A \leqslant 2^{-3} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1,$$

$$S_7 = \frac{1}{4} \sum_{\substack{D \leqslant 2^{-\frac{11}{3}} x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 2 \,(\mathrm{mod}\, 3)}} N_{(1,0,9)}(2D) \sum_{\substack{1 \leqslant A \leqslant 2^{-11/2} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1,$$

$$S_8 = \frac{1}{4} \sum_{\substack{D \leqslant 2^{-\frac{11}{3}} x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 2 \,(\mathrm{mod}\, 3)}} N_{(1,0,9)}(2D) \sum_{\substack{1 \leqslant A \leqslant 2^{-11/2} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1.$$

Now we determine $S_1 + S_2$. We have for $(A, 6) = 1$

$$A \equiv \pm 1 (\text{mod } 12) \Longleftrightarrow \left(\frac{12}{A}\right) = 1, \quad A \equiv \pm 5 (\text{mod } 12) \Longleftrightarrow \left(\frac{12}{A}\right) = -1,$$

so that

$$\begin{aligned} S_1 + S_2 &= \frac{1}{8} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\text{mod}\, 3)}} (N_{(1,0,144)}(D) + N_{(13,10,13)}(D)) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} \left\{1 + \left(\frac{12}{A}\right)\right\} \\ &\quad + \frac{1}{8} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\text{mod}\, 3)}} (N_{(4,4,37)}(D) + N_{(9,0,16)}(D)) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} \left\{1 - \left(\frac{12}{A}\right)\right\} \\ &= \frac{1}{8} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\text{mod}\, 3)}} (N_{(1,0,144)}(D) + N_{(13,10,13)}(D) \\ &\qquad + N_{(4,4,37)}(D) + N_{(9,0,16)}(D)) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1 \\ &\quad + \frac{1}{8} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\text{mod}\, 3)}} (N_{(1,0,144)}(D) + N_{(13,10,13)}(D) \\ &\qquad - N_{(4,4,37)}(D) - N_{(9,0,16)}(D)) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} \left(\frac{12}{A}\right). \end{aligned}$$

Appealing to Lemma 2.4.3, we obtain

$$\begin{aligned} S_1 + S_2 &= \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\text{mod}\, 3)}} d(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1 + E_{12}, \\ &= \frac{1}{8} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp}} d(D) \left\{1 + \left(\frac{-3}{D}\right)\right\} \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{sqf} \\ (A,6D)=1}} 1 + E_{12}, \end{aligned}$$

where

$$E_{12} = O\left(\sum_{D \leqslant x^{1/3}} d(D) \left| \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} \left(\frac{12}{A} \right) \right| \right).$$

Then, appealing to Lemma 2.3.6 (b) (with $r = s = 0$ and $\Delta = -3$) and Lemma 2.3.7 (with $r = s = 0$, $m = 6$ and $\Delta = 12$), we deduce that

$$S_1 + S_2 = \frac{3}{8\pi^2}(c(3/2) + c(3/2, -3))x^{\frac{1}{2}} + O(x^{\frac{1}{3}} \log^3 x).$$

Next we determine $S_3 + S_4$. We have

$$\begin{aligned} S_3 + S_4 = &\frac{1}{8} \sum_{\substack{D \leqslant 2^{-\frac{4}{3}} x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\mathrm{mod}\, 3)}} (N_{(4,4,37)}(D) + N_{(9,0,16)}(D)) \\ &\times \sum_{\substack{1 \leqslant A \leqslant 2^{-2} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} \left\{ 1 + \left(\frac{12}{A} \right) \right\} \\ &+ \frac{1}{8} \sum_{\substack{D \leqslant 2^{-\frac{4}{3}} x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\mathrm{mod}\, 3)}} (N_{(1,0,144)}(D) + N_{(13,10,13)}(D)) \\ &\times \sum_{\substack{1 \leqslant A \leqslant 2^{-2} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} \left\{ 1 - \left(\frac{12}{A} \right) \right\} \\ = &\frac{1}{8} \sum_{\substack{D \leqslant 2^{-\frac{4}{3}} x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\mathrm{mod}\, 3)}} (N_{(1,0,144)}(D) + N_{(13,10,13)}(D) \\ &+ N_{(4,4,37)}(D) + N_{(9,0,16)}(D)) \sum_{\substack{1 \leqslant A \leqslant 2^{-2} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1 \end{aligned}$$

$$-\frac{1}{8}\sum_{\substack{D\leqslant 2^{-\frac{4}{3}}x^{\frac{1}{3}}\\ D\in\wp\\ D\equiv 1\,(\mathrm{mod}\,3)}}(N_{(1,0,144)}(D)+N_{(13,10,13)}(D)$$

$$-N_{(4,4,37)}(D)-N_{(9,0,16)}(D))\sum_{\substack{1\leqslant A\leqslant 2^{-2}x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\text{ sqf}\\ (A,6D)=1}}\left(\frac{12}{A}\right).$$

Appealing to Lemma 2.4.3, we obtain

$$S_3+S_4=\frac{1}{4}\sum_{\substack{D\leqslant 2^{-\frac{4}{3}}x^{\frac{1}{3}}\\ D\in\wp\\ D\equiv 1\,(\mathrm{mod}\,3)}}d(D)\sum_{\substack{1\leqslant A\leqslant 2^{-2}x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\text{ sqf}\\ (A,6D)=1}}1+E_{3\,4},$$

$$=\frac{1}{8}\sum_{\substack{D\leqslant 2^{-\frac{4}{3}}x^{\frac{1}{3}}\\ D\in\wp}}d(D)\left\{1+\left(\frac{-3}{D}\right)\right\}\sum_{\substack{1\leqslant A\leqslant 2^{-2}x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\text{ sqf}\\ (A,6D)=1}}1+E_{3\,4},$$

where

$$E_{3\,4}=O\left(\sum_{D\leqslant 2^{-\frac{4}{3}}x^{1/3}}d(D)\left|\sum_{\substack{1\leqslant A\leqslant 2^{-2}x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\text{ sqf}\\ (A,6D)=1}}\left(\frac{12}{A}\right)\right|\right).$$

Then, appealing to Lemma 2.3.6 (b) (with $r=4/3$, $s=2$ and $\Delta=-3$) and Lemma 2.3.7 (with $r=4/3$, $s=2$, $m=6$ and $\Delta=12$), we deduce that

$$S_3+S_4=\frac{3}{32\pi^2}(c(3/2)+c(3/2,-3))x^{\frac{1}{2}}+O(x^{\frac{1}{3}}\log^3 x).$$

Now we turn to S_5+S_6. We have

$$S_5+S_6=\frac{1}{4}\sum_{\substack{D\leqslant 2^{-2}x^{\frac{1}{3}}\\ D\in\wp\\ D\equiv 1\,(\mathrm{mod}\,3)}}(N_{(1,0,36)}(D)+N_{(4,0,9)}(D))\sum_{\substack{1\leqslant A\leqslant 2^{-3}x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\text{ sqf}\\ (A,6D)=1}}1.$$

Appealing to Lemma 2.4.4, we obtain

$$S_5+S_6=\frac{1}{2}\sum_{\substack{D\leqslant 2^{-2}x^{\frac{1}{3}}\\ D\in\wp\\ D\equiv 1\,(\mathrm{mod}\,3)}}d(D)\sum_{\substack{1\leqslant A\leqslant 2^{-3}x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\text{ sqf}\\ (A,6D)=1}}1$$

$$= \frac{1}{4} \sum_{\substack{D \leqslant 2^{-2} x^{\frac{1}{3}} \\ D \in \wp}} d(D) \left\{ 1 + \left(\frac{-3}{D} \right) \right\} \sum_{\substack{1 \leqslant A \leqslant 2^{-3} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1.$$

Then, appealing to Lemma 2.3.6 (b) (with $r = 2$, $s = 3$ and $\Delta = -3$), we deduce that

$$S_5 + S_6 = \frac{3}{32\pi^2} (c(3/2) + c(3/2, -3)) x^{\frac{1}{2}} + O(x^{\frac{1}{3}} \log^3 x).$$

Finally we consider $S_7 + S_8$. We have

$$S_7 + S_8 = \frac{1}{2} \sum_{\substack{D \leqslant 2^{-\frac{11}{3}} x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 2 \pmod 3}} N_{(1,0,9)}(2D) \sum_{\substack{1 \leqslant A \leqslant 2^{-\frac{11}{2}} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1.$$

Appealing to Lemma 2.4.5, we obtain

$$\begin{aligned} S_7 + S_8 &= \sum_{\substack{D \leqslant 2^{-\frac{11}{3}} x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 2 \pmod 3}} d(D) \sum_{\substack{1 \leqslant A \leqslant 2^{-\frac{11}{3}} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1 \\ &= \frac{1}{2} \sum_{\substack{D \leqslant 2^{-\frac{11}{3}} x^{\frac{1}{3}} \\ D \in \wp}} d(D) \left\{ 1 - \left(\frac{-3}{D} \right) \right\} \sum_{\substack{1 \leqslant A \leqslant 2^{-\frac{11}{3}} x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1. \end{aligned}$$

Appealing to Lemma 2.3.6 (b) (with $r = 11/3$, $s = 11/2$ and $\Delta = -3$), we obtain

$$S_7 + S_8 = \frac{3}{2^{\frac{13}{2}} \pi^2} (c(3/2) - c(3/2, -3)) x^{\frac{1}{2}} + O(x^{\frac{1}{3}} \log^3 x).$$

Finally

$$\begin{aligned} N(x, 0, 3) &= (S_1 + S_2) + (S_3 + S_4) + (S_5 + S_6) + (S_7 + S_8) \\ &= x^{\frac{1}{2}} \left\{ \frac{3}{8\pi^2} (c(3/2) + c(3/2, -3)) + \frac{3}{32\pi^2} (c(3/2) + c(3/2, -3)) \right. \\ &\quad \left. + \frac{3}{32\pi^2} (c(3/2) + c(3/2, -3)) + \frac{3}{64\sqrt{2}\pi^2} (c(3/2) - c(3/2, -3)) \right\} \\ &\quad + O(x^{\frac{1}{3}} \log^3 x) \\ &= x^{\frac{1}{2}} \left\{ \frac{3(24 + \sqrt{2})}{128\pi^2} c(3/2) + \frac{3(24 - \sqrt{2})}{128\pi^2} c(3/2, -3) \right\} + O(x^{\frac{1}{3}} \log^3 x), \end{aligned}$$

as asserted. □

Our next goal is to determine an asymptotic formula for $N(x, 0, 4)$.

Theorem 2.5.3.

$$N(x,0,4) = \frac{1}{2\pi^2}(c(3/2)+c(3/2,8))x^{\frac{1}{2}} + O(x^{\frac{1}{3}}\log^3 x).$$

Proof. By (1.2.1)–(1.2.6) and Theorem 1.3.10 we have

$$N(x,0,4) = \text{number of } (A,\ B,\ D) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N} \text{ such that}$$

$$\begin{aligned}
&A^2D^3 \leqslant x,\\
&A \equiv 1 \pmod 8,\ B \equiv 0 \pmod 8 \text{or}\\
&A \equiv 5 \pmod 8,\ B \equiv 4 \pmod 8,\\
&A \text{ sqf},\\
&B \geqslant 1,\ D \geqslant 2,\ D \text{ sqf},\\
&D - B^2 = \square,\\
&(A,D) = 1,
\end{aligned}$$

so that

$$N(x,0,4) = \sum_{\substack{2\leqslant D\leqslant x^{\frac{1}{3}}\\ D\text{ sqf}\\ D\equiv 1 \pmod 8}} \sum_{\substack{|A|\leqslant x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\text{ sqf}\\ A\equiv 1 \pmod 8\\ (A,D)=1}} \sum_{\substack{B>0\\ D-B^2=\square\\ B\equiv 0 \pmod 8}} 1$$
$$+ \sum_{\substack{2\leqslant D\leqslant x^{\frac{1}{3}}\\ D\text{ sqf}\\ D\equiv 1 \pmod 8}} \sum_{\substack{|A|\leqslant x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\text{ sqf}\\ A\equiv 5 \pmod 8\\ (A,D)=1}} \sum_{\substack{B>0\\ D-B^2=\square\\ B\equiv 4 \pmod 8}} 1.$$

First we examine the inner sums over B. We have

$$\sum_{\substack{B>0\\ D-B^2=\square\\ B\equiv 0 \pmod 8}} 1 = \sum_{\substack{B>0\\ D-64B^2=\square}} 1 = \frac{1}{2}\sum_{\substack{B\\ D-64B^2=\square}} 1$$
$$= \frac{1}{4}\sum_{\substack{B,C\\ D-64B^2=C^2}} 1 = \frac{1}{4}N_{(1,0,64)}(D),$$

and (remembering that D is squarefree and odd)

$$\sum_{\substack{B>0\\ B\equiv 4 \pmod 8\\ D-B^2=\square}} 1 = \frac{1}{2}\sum_{\substack{B\\ B\equiv 4 \pmod 8\\ D-B^2=\square}} 1 = \frac{1}{4}\sum_{\substack{B,C\\ B\equiv 4 \pmod 8\\ D=B^2+C^2}} 1$$
$$= \frac{1}{4}\sum_{\substack{B,C\\ B\equiv 1 \pmod 2\\ D=16B^2+C^2}} 1 = \frac{1}{4}\sum_{\substack{B,C\\ B\equiv C\equiv 1 \pmod 2\\ D=16B^2+C^2}} 1$$

$$= \frac{1}{4} \sum_{\substack{B,X \\ B \equiv 1 \pmod 2 \\ D=16B^2+(B+2X)^2}} 1 \quad = \frac{1}{4} \sum_{\substack{B,X \\ B \equiv 1 \pmod 2 \\ D=4X^2+4XB+17B^2}} 1$$

$$= \frac{1}{4} \sum_{\substack{B,X \\ D=4X^2+4XB+17B^2}} 1 \quad = \frac{1}{4} N_{(4,4,17)}(D).$$

Hence

$$N(x,0,4) = \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod 8}} N_{(1,0,64)}(D) \sum_{\substack{|A| \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv 1 \pmod 8 \\ (A,D)=1}} 1$$

$$+ \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod 8}} N_{(4,4,17)}(D) \sum_{\substack{|A| \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv 5 \pmod 8 \\ (A,D)=1}} 1$$

$$= \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod 8}} N_{(1,0,64)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv \pm 1 \pmod 8 \\ (A,D)=1}} 1$$

$$+ \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod 8}} N_{(4,4,17)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv \pm 3 \pmod 8 \\ (A,D)=1}} 1$$

$$= \frac{1}{8} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod 8}} N_{(1,0,64)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,2D)=1}} \left\{ 1 + \left(\frac{8}{A} \right) \right\}$$

$$+ \frac{1}{8} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod 8}} N_{(4,4,17)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,2D)=1}} \left\{ 1 - \left(\frac{8}{A} \right) \right\}$$

$$= \frac{1}{8} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod 8}} \left(N_{(1,0,64)}(D) + N_{(4,4,17)}(D) \right) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,2D)=1}} 1$$

$$+\frac{1}{8}\sum_{\substack{D\leqslant x^{\frac{1}{3}}\\ D\in\wp\\ D\equiv 1\,(\mathrm{mod}\,8)}}(N_{(1,0,64)}(D)-N_{(4,4,17)}(D))\sum_{\substack{1\leqslant A\leqslant x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\ \mathrm{sqf}\\ (A,2D)=1}}\left(\frac{8}{A}\right).$$

Appealing to Lemma 2.4.6, we obtain

$$\begin{aligned}N(x,0,4)&=\frac{1}{4}\sum_{\substack{D\leqslant x^{\frac{1}{3}}\\ D\in\wp\\ D\equiv 1\,(\mathrm{mod}\,8)}}d(D)\sum_{\substack{1\leqslant A\leqslant x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\ \mathrm{sqf}\\ (A,2D)=1}}1\\ &\quad+O\left(\sum_{D\leqslant x^{\frac{1}{3}}}d(D)\left|\sum_{\substack{1\leqslant A\leqslant x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\ \mathrm{sqf}\\ (A,2D)=1}}\left(\frac{8}{A}\right)\right|\right)\\ &=\frac{1}{8}\sum_{\substack{D\leqslant x^{\frac{1}{3}}\\ D\in\wp}}d(D)\left\{1+\left(\frac{8}{D}\right)\right\}\sum_{\substack{1\leqslant A\leqslant x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\ \mathrm{sqf}\\ (A,2D)=1}}1+O(x^{\frac{1}{3}}\log^3 x),\end{aligned}$$

by Lemma 2.3.7 (with $r=0$, $s=0$, $m=2$ and $\Delta=8$). Then, appealing to Lemma 2.3.6 (a) (with $r=s=0$ and $\Delta=8$), we obtain

$$N(x,0,4)=\frac{1}{2\pi^2}(c(3/2)+c(3/2,8))x^{\frac{1}{2}}+O(x^{\frac{1}{3}}\log^3 x),$$

as asserted. □

Next we turn to the evaluation of $N(x,0,6)$.

Theorem 2.5.4.

$$\begin{aligned}N(x,0,6)=\frac{3}{16\pi^2}&(c(3/2)+c(3/2,-3)+c(3/2,8)+c(3/2,-24))x^{\frac{1}{2}}\\ &+O(x^{\frac{1}{3}}\log^3 x).\end{aligned}$$

Proof. By (1.2.1)–(1.2.6), Theorem 1.3.2 and Theorem 1.3.9, we have

$$N(x,0,6)=\text{number of}(A,\,B,\,D)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}$$

such that

$$\begin{aligned}&A^2D^3\leqslant x,\\ &A\equiv 1\,(\mathrm{mod}\,4),\ B\equiv 0\,(\mathrm{mod}\,4),\\ &A\equiv 1\,(\mathrm{mod}\,3),\ B\equiv 0\,(\mathrm{mod}\,3)\text{or}\\ &A\equiv 2\,(\mathrm{mod}\,3),\ D-B^2\equiv 0\,(\mathrm{mod}\,9),\end{aligned}$$

$$\begin{aligned}&A \text{ sqf},\\&B \geqslant 1,\ D \geqslant 2,\ D \text{ sqf},\\&D - B^2 = \square,\\&(A, D) = 1,\end{aligned}$$

so that

$$N(x,0,6) = \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod 8 \\ D \equiv 1 \pmod 3}} \ \sum_{\substack{|A| \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv 1 \pmod 4 \\ A \equiv 1 \pmod 3 \\ (A,D)=1}} \ \sum_{\substack{B>0 \\ B \equiv 0 \pmod 4 \\ B \equiv 0 \pmod 3 \\ D-B^2=\square}} 1$$

$$+ \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod 8 \\ D \equiv 1 \pmod 3}} \ \sum_{\substack{|A| \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv 1 \pmod 4 \\ A \equiv 2 \pmod 3 \\ (A,D)=1}} \ \sum_{\substack{B>0 \\ B \equiv 0 \pmod 4 \\ D-B^2=\square \equiv 0 \pmod 9}} 1.$$

Examining the inner sums over B, we obtain

$$\sum_{\substack{B>0 \\ B \equiv 0 \pmod{12} \\ D-B^2=\square}} 1 = \sum_{\substack{B>0 \\ D-144B^2=\square}} 1 = \tfrac{1}{2} \sum_{\substack{B \\ D-144B^2=\square}} 1$$

$$= \tfrac{1}{4} \sum_{\substack{B,C \\ D-144B^2=C^2}} 1 = \tfrac{1}{4} N_{(1,0,144)}(D)$$

and

$$\sum_{\substack{B>0 \\ B \equiv 0 \pmod 4 \\ D-B^2=\square \equiv 0 \pmod 9}} 1 = \sum_{\substack{B>0 \\ D-16B^2=9\square}} 1 = \tfrac{1}{2} \sum_{\substack{B \\ D-16B^2=9\square}} 1$$

$$= \tfrac{1}{4} \sum_{\substack{B,C \\ D-16B^2=9C^2}} 1 = \tfrac{1}{4} N_{(9,0,16)}(D).$$

Hence

$$N(x,0,6) = \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} N_{(1,0,144)}(D) \sum_{\substack{|A| \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv 1 \pmod{12} \\ (A,D)=1}} 1$$

$$+ \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} N_{(9,0,16)}(D) \sum_{\substack{|A| \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv 5 \pmod{12} \\ (A,D)=1}} 1$$

$$= \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} N_{(1,0,144)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv \pm 1 \pmod{12} \\ (A,D)=1}} 1$$

$$+ \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} N_{(9,0,16)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv \pm 5 \pmod{12} \\ (A,D)=1}} 1$$

$$= \frac{1}{8} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} N_{(1,0,144)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} \left\{ 1 + \left(\frac{12}{A} \right) \right\}$$

$$+ \frac{1}{8} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} N_{(9,0,16)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} \left\{ 1 - \left(\frac{12}{A} \right) \right\}$$

$$= \frac{1}{8} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} (N_{(1,0,144)}(D) + N_{(9,0,16)}(D)) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1$$

$$+ \frac{1}{8} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} (N_{(1,0,144)}(D) - N_{(9,0,16)}(D)) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} \left(\frac{12}{A} \right).$$

Appealing to Lemma 2.4.3, we deduce

$$N(x,0,6) = \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} d(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1$$

$$+ O\left(\sum_{D \leqslant x^{\frac{1}{3}}} d(D) \left| \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} \left(\frac{12}{A} \right) \right| \right).$$

Then, appealing to Lemma 2.3.7 (with $r = 0$, $s = 0$, $m = 6$ and $\Delta = 12$), we obtain

$$N(x,0,6) = \frac{1}{16} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp}} d(D) \left\{ 1 + \left(\frac{-3}{D} \right) \right\} \left\{ 1 + \left(\frac{8}{D} \right) \right\} \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1 + O(x^{\frac{1}{3}} \log^3 x).$$

Finally, appealing to Lemma 2.3.6 (c) (with $\Delta = -3$ and $\Delta' = 8$), we obtain

$$N(x,0,6) = \frac{3}{16\pi^2}(c(3/2) + c(3/2,-3) + c(3/2,8) + c(3/2,-24))x^{\frac{1}{2}} + O(x^{\frac{1}{3}} \log^3 x),$$

as asserted. □

Our final theorem of this section is an asymptotic formula for $N(x, 0, 12)$.

Theorem 2.5.5.

$$N(x,0,12) = \frac{3}{32\pi^2}(c(3/2) + c(3/2,-3) + c(3/2,8) + c(3/2,-24))x^{\frac{1}{2}} + O(x^{\frac{1}{3}} \log^3 x).$$

Proof. By (1.2.1)–(1.2.6) and Theorems 1.3.9 and 1.3.10, we have

$$N(x,0,12) = \text{number of } (A,\, B,\, D) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$$

such that

$$\begin{aligned}
&A^2D^3 \leqslant x,\\
&A \equiv 1 \pmod{24},\ B \equiv 0 \pmod{24}, \text{ or}\\
&A \equiv 5 \pmod{24},\ B \equiv \pm 4 \pmod{24},\ D \equiv 1 \pmod 3, \text{ or}\\
&A \equiv 13 \pmod{24},\ B \equiv 12 \pmod{24}, \text{ or}\\
&A \equiv 17 \pmod{24},\ B \equiv \pm 8 \pmod{24},\ D \equiv 1 \pmod 3,\\
&A \text{ sqf},\\
&B \geqslant 1,\ D \geqslant 2,\ D \text{ sqf},\\
&D - B^2 = \square,\\
&(A,D) = 1,
\end{aligned}$$

so that

$$\begin{aligned}
N(x,0,12) = &\sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} \sum_{\substack{|A| \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv 1 \pmod{24} \\ (A,D)=1}} \sum_{\substack{B>0 \\ B \equiv 0 \pmod{24} \\ D-B^2=\square}} 1\\
&+ \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} \sum_{\substack{|A| \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv 5 \pmod{24} \\ (A,D)=1}} \sum_{\substack{B>0 \\ B \equiv \pm 4 \pmod{24} \\ D-B^2=\square}} 1\\
&+ \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} \sum_{\substack{|A| \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv 13 \pmod{24} \\ (A,D)=1}} \sum_{\substack{B>0 \\ B \equiv 12 \pmod{24} \\ D-B^2=\square}} 1
\end{aligned}$$

$$+ \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} \sum_{\substack{|A| \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv 17 \pmod{24} \\ (A,D)=1}} \sum_{\substack{B>0 \\ B \equiv \pm 8 \pmod{24} \\ D-B^2=\square}} 1.$$

Straightforward calculations show that

$$\sum_{\substack{B>0 \\ B \equiv 0 \pmod{24} \\ D-B^2=\square}} 1 = \frac{1}{4} N_{(1,0,576)}(D),$$

and

$$\begin{aligned}
\sum_{\substack{B>0 \\ B \equiv \pm 4 \pmod{24} \\ D-B^2=\square}} 1 &= \frac{1}{4} \sum_{\substack{B,C \\ B \equiv 4 \pmod 8 \\ C \equiv 3 \pmod 6 \\ D=B^2+C^2}} 1 \\
&= \frac{1}{4} \sum_{\substack{U,V \\ U \equiv V \equiv 1 \pmod 2 \\ D=16U^2+9V^2}} 1 \\
&= \frac{1}{4} \sum_{\substack{X,Y \\ D=16(X+Y)^2+9(X-Y)^2}} 1 \\
&= \frac{1}{4} \sum_{\substack{X,Y \\ D=25X^2+14XY+25Y^2}} 1 \\
&= \frac{1}{4} N_{(25,14,25)}(D),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\substack{B>0 \\ B \equiv 12 \pmod{24} \\ D-B^2=\square}} 1 &= \frac{1}{4} \sum_{\substack{B,C \\ B \equiv 12 \pmod{24} \\ D=B^2+C^2}} 1 \\
&= \frac{1}{4} \sum_{\substack{C,E \\ E \equiv 1 \pmod 2 \\ D=C^2+144E^2}} 1 \\
&= \frac{1}{4} \sum_{\substack{E,F \\ E \equiv 1 \pmod 2 \\ D=(E+2F)^2+144E^2}} 1
\end{aligned}$$

$$= \frac{1}{4} \sum_{\substack{E,F \\ D=4F^2+4FE+145E^2}} 1$$

$$= \frac{1}{4} N_{(4,4,145)}(D),$$

and

$$\sum_{\substack{B>0 \\ B \equiv \pm 8 \pmod{24} \\ D-B^2=\square}} 1 = \frac{1}{4} \sum_{\substack{B,C \\ B \equiv \pm 8 \pmod{24} \\ D=B^2+C^2}} 1$$

$$= \frac{1}{4} \sum_{\substack{C,E \\ E \equiv \pm 1 \pmod{3} \\ D=C^2+64E^2}} 1$$

$$= \frac{1}{4} \sum_{\substack{C,E \\ C \equiv 0 \pmod{3} \\ D=C^2+64E^2}} 1$$

$$= \frac{1}{4} N_{(9,0,64)}(D).$$

Hence

$$N(x,0,12) = \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} N_{(1,0,576)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv \pm 1 \pmod{24} \\ (A,D)=1}} 1$$

$$+ \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} N_{(25,14,25)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv \pm 5 \pmod{24} \\ (A,D)=1}} 1$$

$$+ \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} N_{(4,4,145)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv \pm 11 \pmod{12} \\ (A,D)=1}} 1$$

$$+ \frac{1}{4} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \pmod{24}}} N_{(9,0,64)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ A \equiv \pm 7 \pmod{24} \\ (A,D)=1}} 1.$$

Thus

$$N(x,0,12)$$

$$= \frac{1}{16} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\mathrm{mod}\, 24)}} N_{(1,0,576)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} \left\{1 + \left(\frac{8}{A}\right)\right\}\left\{1 + \left(\frac{12}{A}\right)\right\}$$

$$+ \frac{1}{16} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\mathrm{mod}\, 24)}} N_{(25,14,25)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} \left\{1 - \left(\frac{8}{A}\right)\right\}\left\{1 - \left(\frac{12}{A}\right)\right\}$$

$$+ \frac{1}{16} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\mathrm{mod}\, 24)}} N_{(4,4,145)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} \left\{1 - \left(\frac{8}{A}\right)\right\}\left\{1 + \left(\frac{12}{A}\right)\right\}$$

$$+ \frac{1}{16} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\mathrm{mod}\, 24)}} N_{(9,0,64)}(D) \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} \left\{1 + \left(\frac{8}{A}\right)\right\}\left\{1 - \left(\frac{12}{A}\right)\right\}.$$

Therefore

$$N(x,0,12) = \frac{1}{16} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\mathrm{mod}\, 24)}} \{N_{(1,0,576)}(D) + N_{(25,14,25)}(D)$$

$$+ N_{(4,4,145)}(D) + N_{(9,0,64)}(D)\} \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1$$

$$+ \frac{1}{16} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp \\ D \equiv 1 \,(\mathrm{mod}\, 24)}} \{N_{(1,0,576)}(D) - N_{(25,14,25)}(D)$$

$$- N_{(4,4,145)}(D) + N_{(9,0,64)}(D)\} \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} \left(\frac{8}{A}\right)$$

$$+\frac{1}{16}\sum_{\substack{D\leqslant x^{\frac{1}{3}}\\ D\in\wp\\ D\equiv 1\ (\mathrm{mod}\ 24)}}\{N_{(1,0,576)}(D)-N_{(25,14,25)}(D)$$

$$+N_{(4,4,145)}(D)-N_{(9,0,64)}(D)\}\sum_{\substack{1\leqslant A\leqslant x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\ \mathrm{sqf}\\ (A,6D)=1}}\left(\frac{12}{A}\right)$$

$$+\frac{1}{16}\sum_{\substack{D\leqslant x^{\frac{1}{3}}\\ D\in\wp\\ D\equiv 1\ (\mathrm{mod}\ 24)}}\{N_{(1,0,576)}(D)+N_{(25,14,25)}(D)$$

$$-N_{(4,4,145)}(D)-N_{(9,0,64)}(D)\}\sum_{\substack{1\leqslant A\leqslant x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\ \mathrm{sqf}\\ (A,6D)=1}}\left(\frac{24}{A}\right).$$

By Lemma 2.4.7 we have

$$N(x,0,12)=\frac{1}{8}\sum_{\substack{D\leqslant x^{\frac{1}{3}}\\ D\in\wp\\ D\equiv 1\ (\mathrm{mod}\ 24)}}d(D)\sum_{\substack{1\leqslant A\leqslant x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\ \mathrm{sqf}\\ (A,6D)=1}}1$$

$$+O\left(\sum_{D\leqslant x^{\frac{1}{3}}}d(D)\left|\sum_{\substack{1\leqslant A\leqslant x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\ \mathrm{sqf}\\ (A,6D)=1}}\left(\frac{8}{A}\right)\right|\right)$$

$$+O\left(\sum_{D\leqslant x^{\frac{1}{3}}}d(D)\left|\sum_{\substack{1\leqslant A\leqslant x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\ \mathrm{sqf}\\ (A,6D)=1}}\left(\frac{12}{A}\right)\right|\right)$$

$$+O\left(\sum_{D\leqslant x^{\frac{1}{3}}}d(D)\left|\sum_{\substack{1\leqslant A\leqslant x^{\frac{1}{2}}D^{-\frac{3}{2}}\\ A\ \mathrm{sqf}\\ (A,6D)=1}}\left(\frac{24}{A}\right)\right|\right).$$

Appealing to Lemma 2.3.7, we obtain

$$N(x,0,12) = \frac{1}{32} \sum_{\substack{D \leqslant x^{\frac{1}{3}} \\ D \in \wp}} d(D) \left\{1 + \left(\frac{-3}{D}\right)\right\} \left\{1 + \left(\frac{8}{D}\right)\right\} \sum_{\substack{1 \leqslant A \leqslant x^{\frac{1}{2}} D^{-\frac{3}{2}} \\ A \text{ sqf} \\ (A,6D)=1}} 1 + O(x^{\frac{1}{3}} \log^3 x).$$

Finally, appealing to Lemma 2.3.6 (c), we obtain

$$N(x,0,12) = \frac{3}{32\pi^2} (c(3/2) + c(3/2,-3) + c(3/2,8) + c(3/2,-24)) x^{\frac{1}{2}} + O(x^{\frac{1}{3}} \log^3 x).$$

This completes the proof of Theorem 2.5.5. □

2.6. Number of cyclic quartic fields with discriminant $\leqslant x$ and given index: Proof of Theorem B. By a theorem of Ou and Williams [14] the number $N(x)$ of cyclic quartic fields K with $d(K) \leqslant x$ satisfies

$$N(x) = \frac{3}{\pi^2} \left\{ \frac{24 + \sqrt{2}}{24} \prod_{p \equiv 1 \pmod 4} \left(1 + \frac{2}{(p+1)\sqrt{p}}\right) - 1 \right\} x^{\frac{1}{2}} + O(x^{\frac{1}{3}} \log^3 x),$$

so that

$$N(x) = \frac{1}{8\pi^2} \{(24 + \sqrt{2}) c(3/2) + \sqrt{2}\} x^{\frac{1}{2}} + O(x^{\frac{1}{3}} \log^3 x). \tag{2.6.1}$$

As

$$\begin{aligned}
N(x;1) &= N(x) - N(x,0,2) - N(x,0,3) + N(x,0,6), \\
N(x;2) &= N(x,0,2) - N(x,0,4) - N(x,0,6) + N(x,0,12), \\
N(x;3) &= N(x,0,3) - N(x,0,6), \\
N(x;4) &= N(x,0,4) - N(x,0,12), \\
N(x;6) &= N(x,0,6) - N(x,0,12), \\
N(x;12) &= N(x,0,12),
\end{aligned}$$

appealing to (2.6.1) and Theorems 2.5.1–2.5.5, we obtain Theorem B. □

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