Advances in Theoretical and Applied Mathematics.
ISSN 0973-4554 Vol.1 No.1 (2006), pp. 27-48

http://www.ripublication.com/atam.htm

Evaluation of the Convolution Sums $\sum_{l+12m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+4m=n} \sigma(l)\sigma(m)$

Ayşe Alaca, Şaban Alaca and Kenneth S. Williams[1]

Centre for Research in Algebra and Number Theory,
School of Mathematics and Statistics
Carleton University, Ottawa, Ontario, Canada K1S 5B6
E-mail: aalaca@math.carleton.ca, salaca@math.carleton.ca,
williams@math.carleton.ca

Abstract

The convolution sums $\sum_{l+12m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+4m=n} \sigma(l)\sigma(m)$ are evaluated for all $n \in \mathbb{N}$, and their evaluations used to determine the number of representations of a positive integer n by the form $x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + 4(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2)$.

AMS Mathematics Subject Classification: 11A25, 11E20, 11E25
Keywords: sum of divisors function, convolution sums, Eisenstein series.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of natural numbers, integers, real numbers, complex numbers respectively. For $k, n \in \mathbb{N}$ we set

$$\sigma_k(n) = \sum_{d \mid n} d^k, \tag{1.1}$$

where d runs through the positive divisors of n. If $n \notin \mathbb{N}$ we set $\sigma_k(n) = 0$. We write $\sigma(n)$ for $\sigma_1(n)$. For $a, b \in \mathbb{N}$ with $a \le b$ we define the convolution sum $W_{a,b}(n)$ by

$$W_{a,b}(n) := \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ al+bm=n}} \sigma(l)\sigma(m). \tag{1.2}$$

[1]Research of the third author was supported by Natural Sciences and Engineering Research Council of Canada grant A-7233.

Set $g = \gcd(a, b)$. Clearly

$$W_{a,b}(n) = \begin{cases} W_{a/g,\, b/g}(n/g), & \text{if } g \mid n, \\ 0, & \text{if } g \nmid n. \end{cases} \tag{1.3}$$

Hence we may suppose that $\gcd(a, b) = 1$. When $a = 1$ we have

$$W_{1,b}(n) = \sum_{\substack{m \in \mathbb{N} \\ m < n/b}} \sigma(m)\sigma(n - bm) \tag{1.4}$$

and we write $W_b(n)$ for $W_{1,b}(n)$.

The sum $W_k(n)$ has been evaluated for $k = 1$ [7], [10], [11], [12], $k = 2$ [12], [16], [17], $k = 3$ [12], [16], [17], [21], $k = 4$ [12], [16], [17], $k = 5$ [14], [16], [17], $k = 6$ [3], $k = 7$ [14], $k = 8$ [23], $k = 9$ [16], [17], [21], [22] and $k = 16$ [1]. The sum $W_{2,3}(n)$ was evaluated in [3]. In this paper we determine $W_{12}(n)$ and $W_{3,4}(n)$. These determinations are given in Theorem 2.1 in Section 2. The proof of Theorem 2.1 is given in Section 3. Some related convolution sums are evaluated in [8], [9].

For $k, n \in \mathbb{N}$ we let

$$\begin{aligned} N_k(n) := \operatorname{card}\Big\{&(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8 \mid n = x_1^2 + x_1x_2 + x_2^2 \\ &+ x_3^2 + x_3x_4 + x_4^2 + k(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2)\Big\}. \end{aligned} \tag{1.5}$$

The value of $N_1(n)$ has been given by Lomadze [15], the value of $N_2(n)$ by Alaca and Williams [3] and the value of $N_3(n)$ by Williams [22]. In Section 4 the evaluations of $W_{12}(n)$ and $W_{3,4}(n)$ are used to determine $N_4(n)$, see Theorem 2.2 in Section 2.

2. Statements of Theorems 2.1 and 2.2

Let $q \in \mathbb{C}$ be such that $|q| < 1$. Ramanujan's discriminant function $\Delta(q)$ is defined by

$$\Delta(q) := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n, \tag{2.1}$$

where $\tau(n)$ is Ramanujan's tau function [18, eqn. (92)], [20, p. 151].

From (2.1) we deduce

$$\begin{aligned} &\Delta(q)^{-1/24}\Delta(q^2)^{1/12}\Delta(q^3)^{1/8}\Delta(q^4)^{1/8}\Delta(q^6)^{1/12}\Delta(q^{12})^{-1/24} \\ &= q \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2(1 - q^{3n})^3(1 - q^{4n})^3(1 - q^{6n})^2}{(1 - q^n)(1 - q^{12n})} \\ &= q \prod_{n=1}^{\infty} (1 + q^n)(1 - q^{2n})(1 - q^{3n})^3(1 - q^{4n})^3(1 - q^{6n})(1 - q^{12n-6}) \\ &= \sum_{n=1}^{\infty} t_n q^n, \end{aligned} \tag{2.2}$$

where

$$t_n \in \mathbb{Z} \ \ (n \in \mathbb{N}), \ \ t_1 = 1. \tag{2.3}$$

Also

$$\begin{aligned}
&\Delta(q)^{-1/12}\Delta(q^2)^{1/3}\Delta(q^3)^{-1/12}\Delta(q^4)^{-1/12}\Delta(q^6)^{1/3}\Delta(q^{12})^{-1/12} \\
&= q\prod_{n=1}^{\infty}\frac{(1-q^{2n})^8(1-q^{6n})^8}{(1-q^n)^2(1-q^{3n})^2(1-q^{4n})^2(1-q^{12n})^2} \\
&= q\prod_{n=1}^{\infty}(1+q^n)^2(1-q^{2n})^6(1+q^{3n})^2(1-q^{6n})^6(1+q^{4n}+q^{8n})^2 \\
&= \sum_{n=1}^{\infty}u_nq^n,
\end{aligned} \tag{2.4}$$

where

$$u_n \in \mathbb{Z} \ \ (n \in \mathbb{N}), \ \ u_1 = 1. \tag{2.5}$$

Thus

$$\begin{aligned}
&\frac{10}{11}\Delta(q)^{-1/24}\Delta(q^2)^{1/12}\Delta(q^3)^{1/8}\Delta(q^4)^{1/8}\Delta(q^6)^{1/12}\Delta(q^{12})^{-1/24} \\
&+\frac{1}{11}\Delta(q)^{-1/12}\Delta(q^2)^{1/3}\Delta(q^3)^{-1/12}\Delta(q^4)^{-1/12}\Delta(q^6)^{1/3}\Delta(q^{12})^{-1/12} \\
&= \sum_{n=1}^{\infty}c_{1,\,12}(n)q^n,
\end{aligned} \tag{2.6}$$

where

$$c_{1,12}(n) = 10t_n + u_n \in \mathbb{Z} \ \ (n \in \mathbb{N}) \tag{2.7}$$

and

$$c_{1,12}(1) = 1. \tag{2.8}$$

The first 36 values of $c_{1,12}(n)$ are given in the following table.

n	$c_{1,12}(n)$	n	$c_{1,12}(n)$	n	$c_{1,12}(n)$
1	1	13	178/11	25	461/11
2	12/11	14	−192/11	26	456/11
3	−3/11	15	−378/11	27	−27/11
4	−24/11	16	−96/11	28	384/11
5	−54/11	17	−666/11	29	−90
6	−36/11	18	108/11	30	−216/11
7	−56/11	19	−380/11	31	−608/11
8	48/11	20	−144/11	32	192/11
9	9	21	408/11	33	324/11
10	72/11	22	144/11	34	−1512/11
11	252/11	23	1368/11	35	−1296/11
12	72/11	24	−144/11	36	−216/11

Also from (2.1) we have

$$\begin{aligned}&\Delta(q)^{1/8}\Delta(q^2)^{1/12}\Delta(q^3)^{-1/24}\Delta(q^4)^{-1/24}\Delta(q^6)^{1/12}\Delta(q^{12})^{1/8} \\ &= q^2\prod_{n=1}^{\infty}\frac{(1-q^n)^3(1-q^{2n})^2(1-q^{6n})^2(1-q^{12n})^3}{(1-q^{3n})(1-q^{4n})} \\ &= q^2\prod_{n=1}^{\infty}(1-q^n)^3(1-q^{2n})^2(1+q^{3n})(1-q^{6n})(1+q^{4n}+q^{8n})(1-q^{12n})^2 \\ &= \sum_{n=1}^{\infty} v_n q^n,\end{aligned} \tag{2.9}$$

where

$$v_n \in \mathbb{Z} \ \ (n \in \mathbb{N}), \quad v_1 = 0, \quad v_2 = 1. \tag{2.10}$$

Thus

$$\begin{aligned}&10\Delta(q)^{1/8}\Delta(q^2)^{1/12}\Delta(q^3)^{-1/24}\Delta(q^4)^{-1/24}\Delta(q^6)^{1/12}\Delta(q^{12})^{1/8} \\ &\quad+\Delta(q)^{-1/12}\Delta(q^2)^{1/3}\Delta(q^3)^{-1/12}\Delta(q^4)^{-1/12}\Delta(q^6)^{1/3}\Delta(q^{12})^{-1/12} \\ &\quad= \sum_{n=1}^{\infty} c_{3,4}(n)q^n,\end{aligned} \tag{2.11}$$

where

$$11c_{3,4}(n) = 10v_n + u_n \in \mathbb{Z} \ \ (n \in \mathbb{N}) \tag{2.12}$$

and

$$c_{3,4}(1) = 1. \tag{2.13}$$

The first 36 values of $c_{3,4}(n)$ are given in the following table.

n	$c_{3,4}(n)$	n	$c_{3,4}(n)$	n	$c_{3,4}(n)$
1	1	13	278	25	−1529
2	12	14	−192	26	456
3	−33	15	162	27	−297
4	−24	16	−96	28	384
5	126	17	−846	29	1350
6	−36	18	108	30	−216
7	−136	19	620	31	−448
8	48	20	−144	32	192
9	9	21	168	33	−756
10	72	22	144	34	−1512
11	−108	23	648	35	144
12	72	24	−144	36	−216

In Section 3 we prove the following theorem.

Theorem 2.1: Let $n \in \mathbb{N}$. Then

$$\sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+12m=n}} \sigma(l)\sigma(m) = \frac{1}{480}\sigma_3(n) + \frac{1}{160}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{160}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{30}\sigma_3\left(\frac{n}{4}\right) + \frac{9}{160}\sigma_3\left(\frac{n}{6}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{12}\right) + \left(\frac{1}{24} - \frac{n}{48}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{12}\right) - \frac{11}{480}c_{1,12}(n)$$

and

$$\sum_{\substack{(l,m) \in \mathbb{N}^2 \\ 3l+4m=n}} \sigma(l)\sigma(m) = \frac{1}{480}\sigma_3(n) + \frac{1}{160}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{160}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{30}\sigma_3\left(\frac{n}{4}\right) + \frac{9}{160}\sigma_3\left(\frac{n}{6}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{12}\right) + \left(\frac{1}{24} - \frac{n}{16}\right)\sigma\left(\frac{n}{3}\right) + \left(\frac{1}{24} - \frac{n}{12}\right)\sigma\left(\frac{n}{4}\right) - \frac{1}{480}c_{3,4}(n),$$

where $c_{1,12}(n)$ and $c_{3,4}(n)$ are defined in (2.6) and (2.12) respectively.

As an application of Theorem 2.1, we prove the following result in Section 4.

Theorem 2.2: Let $n \in \mathbb{N}$. The number $N_4(n)$ of $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8$ such that

$$n = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + 4(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2)$$

is given by

$$\begin{aligned} N_4(n) &= \frac{6}{5}\sigma_3(n) + \frac{18}{5}\sigma_3\left(\frac{n}{2}\right) + \frac{54}{5}\sigma_3\left(\frac{n}{3}\right) \\ &\quad + \frac{96}{5}\sigma_3\left(\frac{n}{4}\right) + \frac{162}{5}\sigma_3\left(\frac{n}{6}\right) + \frac{864}{5}\sigma_3\left(\frac{n}{12}\right) \\ &\quad + \frac{99}{10}c_{1,12}(n) + \frac{9}{10}c_{3,4}(n). \end{aligned}$$

3. Proof of Theorem 2.1

We define the Eisenstein series $L(q)$, $M(q)$ and $N(q)$ by

$$L(q) := 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n, \ q \in \mathbb{C}, \ |q| < 1, \tag{3.1}$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \ q \in \mathbb{C}, \ |q| < 1, \tag{3.2}$$

$$N(q) := 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \ q \in \mathbb{C}, \ |q| < 1, \tag{3.3}$$

see for example [5, p. 105], [18, eqn. (25)], [20, p. 140]. The discriminant function $\Delta(q)$ is given by

$$\Delta(q) = \frac{1}{1728}\left(M(q)^3 - N(q)^2\right), \tag{3.4}$$

see for example [13, p. 111], [18, eqn. (44)], [20, p. 144]. The Jacobi theta function $\varphi(q)$ is defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \ \ q \in \mathbb{C}, \ \ |q| < 1, \tag{3.5}$$

see for example [5, p. 92]. Set

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)} \tag{3.6}$$

and

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{3.7}$$

It is shown in [3, eqns. (3.84) and (3.87)] that

$$L(q) - 2L(q^2) = -(1 + 14p + 24p^2 + 14p^3 + p^4)k^2 \tag{3.8}$$

and

$$L(q) - 3L(q^3) = -(2 + 16p + 36p^2 + 16p^3 + 2p^4)k^2. \tag{3.9}$$

The following two results are proved in [2]:

Duplication principle.

$$p(q^2) = \frac{1 + p - p^2 - ((1-p)(1+p)(1+2p))^{1/2}}{p^2},$$

$$k(q^2) = \frac{(1 + p - p^2 + ((1-p)(1+p)(1+2p))^{1/2})k}{2}.$$

Triplication principle.

$$\begin{aligned} p(q^3) = 3^{-1} \Big(& (-4 - 3p + 6p^2 + 4p^3) \\ & + 2^{2/3}(1 - 2p - 2p^2)((1-p)(1+2p)(2+p))^{1/3} \\ & + 2^{1/3}(1+2p)((1-p)(1+2p)(2+p))^{2/3} \Big), \end{aligned}$$

$$\begin{aligned} k(q^3) = 3^{-2} \Big(& 3 + 2^{2/3}(1+2p)((1-p)(1+2p)(2+p))^{1/3} \\ & + 2^{4/3}((1-p)(1+2p)(2+p))^{2/3} \Big) k. \end{aligned}$$

Applying the triplication principle to (3.8), we obtain using MAPLE (here and throughout)

$$L(q^3) - 2L(q^6) = -(1 + 2p + 2p^3 + p^4)k^2, \tag{3.10}$$

in agreement with [3, eqn. (3.85)]. Applying the duplication principle to (3.10), we deduce

$$L(q^6) - 2L(q^{12}) = -\left(1 + 2p - p^3 - \frac{1}{2}p^4\right)k^2. \tag{3.11}$$

Then, from (3.9), (3.10), (3.11) and the relation

$$\begin{aligned} & L(q) - 12L(q^{12}) \\ & \quad = (L(q) - 3L(q^3)) + 3(L(q^3) - 2L(q^6)) + 6(L(q^6) - 2L(q^{12})), \end{aligned}$$

we obtain

$$L(q) - 12L(q^{12}) = -(11 + 34p + 36p^2 + 16p^3 + 2p^4)k^2. \tag{3.12}$$

Squaring (3.12) we obtain

$$\begin{aligned}(L(q) - 12L(q^{12}))^2 &= \Big(121 + 748p + 1948p^2 + 2800p^3 \\ &\quad +2428p^4 + 1288p^5 + 400p^6 + 64p^7 + 4p^8\Big) k^4.\end{aligned} \tag{3.13}$$

It is also shown in [3, eqn. (3.69)] that

$$\begin{aligned}M(q) &= (1 + 124p + 964p^2 + 2788p^3 + 3910p^4 \\ &\quad +2788p^5 + 964p^6 + 124p^7 + p^8)k^4.\end{aligned} \tag{3.14}$$

Applying the duplication and triplication principles successively to (3.14), we obtain

$$\begin{aligned}M(q^2) &= (1 + 4p + 64p^2 + 178p^3 + 235p^4 \\ &\quad +178p^5 + 64p^6 + 4p^7 + p^8)k^4,\end{aligned} \tag{3.15}$$

$$\begin{aligned}M(q^3) &= (1 + 4p + 4p^2 + 28p^3 + 70p^4 \\ &\quad +28p^5 + 4p^6 + 4p^7 + p^8)k^4,\end{aligned} \tag{3.16}$$

$$\begin{aligned}M(q^4) &= \Big(1 + 4p + 4p^2 - 2p^3 + 10p^4 + 28p^5 \\ &\quad +\frac{31}{4}p^6 - \frac{29}{4}p^7 + \frac{1}{16}p^8\Big) k^4,\end{aligned} \tag{3.17}$$

$$\begin{aligned}M(q^6) &= (1 + 4p + 4p^2 - 2p^3 - 5p^4 \\ &\quad -2p^5 + 4p^6 + 4p^7 + p^8)k^4,\end{aligned} \tag{3.18}$$

$$\begin{aligned}M(q^{12}) &= \Big(1 + 4p + 4p^2 - 2p^3 - 5p^4 - 2p^5 \\ &\quad +\frac{1}{4}p^6 + \frac{1}{4}p^7 + \frac{1}{16}p^8\Big) k^4.\end{aligned} \tag{3.19}$$

Equations (3.15), (3.16) and (3.18) are equations (3.70), (3.71) and (3.72) in [3] respectively. Then, from (3.14)-(3.19), we obtain

$$\begin{aligned}&22M(q) - 9M(q^2) - 27M(q^3) - 48M(q^4) - 81M(q^6) + 3168M(q^{12}) \\ &\quad = (3025 + 14740p + 32680p^2 + 52900p^3 + 66100p^4 \\ &\qquad +51460p^5 + 20620p^6 + 3400p^7 + 100p^8)k^4.\end{aligned} \tag{3.20}$$

Next, from (3.13) and (3.20), we deduce

$$
\begin{aligned}
&(L(q) - 12L(q^{12}))^2 \\
&-\frac{1}{25}\Big(22M(q) - 9M(q^2) - 27M(q^3) - 48M(q^4) - 81M(q^6) + 3168M(q^{12})\Big) \\
&= \frac{36}{5}(5p^2 + 17p + 11)p(1+p)(1-p)(1+2p)(2+p)k^4.
\end{aligned}
$$

Since

$$5p^2 + 17p + 11 = 5(1+p)(2+p) + (1+2p)$$

we obtain

$$
\begin{aligned}
(L(q) - 12L(q^{12}))^2 = &\frac{1}{25}\Big(22M(q) - 9M(q^2) - 27M(q^3) - 48M(q^4) \\
&-81M(q^6) + 3168M(q^{12})\Big) \\
&+36p(1+p)^2(1-p)(1+2p)(2+p)^2k^4 \\
&+\frac{36}{5}p(1+p)(1-p)(1+2p)^2(2+p)k^4. \qquad (3.21)
\end{aligned}
$$

From [3, eqn. (3.73)] we have

$$
\begin{aligned}
N(q) \;=\; &(1 - 246p - 5532p^2 - 38614p^3 - 135369p^4 \qquad (3.22)\\
&-276084p^5 - 348024p^6 - 276084p^7 - 135369p^8 \\
&-38614p^9 - 5532p^{10} - 246p^{11} + p^{12})k^6.
\end{aligned}
$$

Applying the duplication and triplication principles successively to (3.22), we obtain

$$
\begin{aligned}
N(q^2) \;=\; &\Big(1 + 6p - 114p^2 - 625p^3 - \frac{4059}{2}p^4 \qquad (3.23)\\
&-4302p^5 - 5556p^6 - 4302p^7 - \frac{4059}{2}p^8 \\
&- 625p^9 - 114p^{10} + 6p^{11} + p^{12}\Big)k^6,
\end{aligned}
$$

$$N(q^3) = (1+6p+12p^2-58p^3-297p^4-396p^5-264p^6 -396p^7-297p^8-58p^9+12p^{10}+6p^{11}+p^{12})k^6, \tag{3.24}$$

$$N(q^4) = \left(1+6p+12p^2+5p^3-45p^4-144p^5 -\frac{1167}{8}p^6+\frac{171}{8}p^7+\frac{2151}{32}p^8-\frac{739}{16}p^9 -\frac{345}{8}p^{10}+\frac{129}{32}p^{11}+\frac{1}{64}p^{12}\right)k^6, \tag{3.25}$$

$$N(q^6) = (1+6p+12p^2+5p^3-\frac{27}{2}p^4-18p^5-12p^6 -18p^7-\frac{27}{2}p^8+5p^9+12p^{10}+6p^{11}+p^{12})k^6, \tag{3.26}$$

$$N(q^{12}) = \left(1+6p+12p^2+5p^3-\frac{27}{2}p^4-18p^5 -\frac{33}{8}p^6+\frac{45}{8}p^7+\frac{135}{32}p^8+\frac{17}{16}p^9 +\frac{3}{16}p^{10}+\frac{3}{32}p^{11}+\frac{1}{64}p^{12}\right)k^6. \tag{3.27}$$

Equations (3.23), (3.24) and (3.26) are formulae (3.74), (3.75) and (3.76) in [3] respectively.

Next, from (3.4), (3.14)-(3.19) and (3.22)-(3.27), we obtain

$$\Delta(q) = \frac{1}{16}p(1+p)^4(1-p)^{12}(1+2p)^3(2+p)^3k^{12}, \tag{3.28}$$

$$\Delta(q^2) = \frac{1}{256}p^2(1+p)^2(1-p)^6(1+2p)^6(2+p)^6k^{12}, \tag{3.29}$$

$$\Delta(q^3) = \frac{1}{16}p^3(1+p)^{12}(1-p)^4(1+2p)(2+p)k^{12}, \tag{3.30}$$

$$\Delta(q^4) = \frac{1}{65536}p^4(1+p)(1-p)^3(1+2p)^3(2+p)^{12}k^{12}, \tag{3.31}$$

$$\Delta(q^6) = \frac{1}{256}p^6(1+p)^6(1-p)^2(1+2p)^2(2+p)^2k^{12}, \tag{3.32}$$

$$\Delta(q^{12}) = \frac{1}{65536}p^{12}(1+p)^3(1-p)(1+2p)(2+p)^4k^{12}. \tag{3.33}$$

Equations (3.28), (3.29), (3.30) and (3.32) are equations (3.78), (3.79), (3.80) and (3.81) in [3] respectively. Hence, from (3.28)-(3.33), we obtain

$$\begin{aligned}&\Delta(q)^{-1/24}\Delta(q^2)^{1/12}\Delta(q^3)^{1/8}\Delta(q^4)^{1/8}\Delta(q^6)^{1/12}\Delta(q^{12})^{-1/24}\\&= 2^{-3}p(1+p)^2(1-p)(1+2p)(2+p)^2k^4\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}&\Delta(q)^{-1/12}\Delta(q^2)^{1/3}\Delta(q^3)^{-1/12}\Delta(q^4)^{-1/12}\Delta(q^6)^{1/3}\Delta(q^{12})^{-1/12}\\&= 2^{-2}p(1+p)(1-p)(1+2p)^2(2+p)k^4.\end{aligned} \tag{3.35}$$

Thus, from (3.21), (3.34) and (3.35), we have

$$\begin{aligned}(L(q)-12L(q^{12}))^2 &= \frac{1}{25}\Big(22M(q)-9M(q^2)-27M(q^3)-48M(q^4)\\&\quad -81M(q^6)+3168M(q^{12})\Big)\\&\quad +288\Delta(q)^{-1/24}\Delta(q^2)^{1/12}\Delta(q^3)^{1/8}\Delta(q^4)^{1/8}\Delta(q^6)^{1/12}\Delta(q^{12})^{-1/24}\\&\quad +\frac{144}{5}\Delta(q)^{-1/12}\Delta(q^2)^{1/3}\Delta(q^3)^{-1/12}\Delta(q^4)^{-1/12}\Delta(q^6)^{1/3}\Delta(q^{12})^{-1/12},\end{aligned}$$

that is (by (2.6))

$$\begin{aligned}&(L(q)-12L(q^{12}))^2\\&= \frac{1}{25}\Big(22M(q)-9M(q^2)-27M(q^3)-48M(q^4)-81M(q^6)+3168M(q^{12})\Big)\\&+\frac{1584}{5}\sum_{n=1}^{\infty}c_{1,12}(n)q^n.\end{aligned} \tag{3.36}$$

Then, from (3.2) and (3.36), we have

$$\begin{aligned}(L(q)-12L(q^{12}))^2 &= 121+\frac{48}{5}\sum_{n=1}^{\infty}\Big(22\sigma_3(n)-9\sigma_3\left(\frac{n}{2}\right)-27\sigma_3\left(\frac{n}{3}\right)\\&\quad -48\sigma_3\left(\frac{n}{4}\right)-81\sigma_3\left(\frac{n}{6}\right)+3168\sigma_3\left(\frac{n}{12}\right)+33c_{1,12}(n)\Big)q^n.\end{aligned} \tag{3.37}$$

Recalling that (see for example [10], [11])

$$L(q)^2 = 1+\sum_{n=1}^{\infty}(240\sigma_3(n)-288n\sigma(n))q^n, \tag{3.38}$$

so that

$$L(q^{12})^2 = 1 + \sum_{n=1}^{\infty} \left(240\sigma_3\left(\frac{n}{12}\right) - 24n\sigma\left(\frac{n}{12}\right)\right) q^n, \tag{3.39}$$

we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{(l,\, m)\, \in\, \mathbb{N}^2 \\ l+12m=n}} \sigma(l)\sigma(m)\Bigg) q^n \\
&= \sum_{l,m=1}^{\infty} \sigma(l)\sigma(m) q^{l+12m} \\
&= \left(\sum_{l=1}^{\infty} \sigma(l) q^l\right)\left(\sum_{m=1}^{\infty} \sigma(m) q^{12m}\right) \\
&= \left(\frac{1-L(q)}{24}\right)\left(\frac{1-L(q^{12})}{24}\right) \\
&= \frac{1}{576} - \frac{1}{576}L(q) - \frac{1}{576}L(q^{12}) + \frac{1}{576}L(q)L(q^{12}) \\
&= \frac{1}{576} - \frac{1}{576}L(q) - \frac{1}{576}L(q^{12}) + \frac{1}{13824}L(q)^2 + \frac{1}{96}L(q^{12})^2 \\
&\quad - \frac{1}{13824}\left(L(q) - 12L(q^{12})\right)^2 \\
&= \frac{1}{576} - \frac{1}{576}\left(1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n\right) - \frac{1}{576}\left(1 - 24\sum_{n=1}^{\infty}\sigma\left(\frac{n}{12}\right)q^n\right) \\
&\quad + \frac{1}{13824}\left(1 + \sum_{n=1}^{\infty}(240\sigma_3(n) - 288n\sigma(n))q^n\right) \\
&\quad + \frac{1}{96}\left(1 + \sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{12}\right) - 24n\sigma\left(\frac{n}{12}\right)\right)q^n\right) \\
&\quad - \frac{1}{13824}\Bigg(121 + \frac{48}{5}\sum_{n=1}^{\infty}\Bigg(22\sigma_3(n) - 9\sigma_3\left(\frac{n}{2}\right) - 27\sigma_3\left(\frac{n}{3}\right) \\
&\qquad\qquad - 48\sigma_3\left(\frac{n}{4}\right) - 81\sigma_3\left(\frac{n}{6}\right) + 3168\sigma_3\left(\frac{n}{12}\right) + 33c_{1,\,12}(n)\Bigg) q^n\Bigg)
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{480}\sigma_3(n) + \frac{1}{160}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{160}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{30}\sigma_3\left(\frac{n}{4}\right) + \frac{9}{160}\sigma_3\left(\frac{n}{6}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{12}\right) + \left(\frac{1}{24} - \frac{n}{48}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{12}\right) - \frac{11}{480}c_{1,12}(n)\right) q^n.$$

Equating coefficients of q^n ($n \in \mathbb{N}$), we obtain the first assertion of Theorem 2.1.

Next we turn to the proof of the second assertion of Theorem 2.1. Applying duplication to (3.8) we obtain

$$L(q^2) - 2L(q^4) = -(1 + 2p + 6p^2 + 5p^3 - \frac{1}{2}p^4)k^2. \tag{3.40}$$

Then, from (3.8), (3.9), (3.40) and the relation

$$3L(q^3) - 4L(q^4) = (L(q) - 2L(q^2)) - (L(q) - 3L(q^3)) + 2(L(q^2) - 2L(q^4)),$$

we obtain

$$3L(q^3) - 4L(q^4) = -(1 + 2p + 8p^3 - 2p^4)k^2. \tag{3.41}$$

Squaring (3.41) we obtain

$$(3L(q^3) - 4L(q^4))^2 = (1 + 4p + 4p^2 + 16p^3 + 28p^4 - 8p^5 + 64p^6 - 32p^7 + 4p^8)k^4. \tag{3.42}$$

Next, from (3.14)-(3.19), we have

$$-\frac{3}{25}M(q) - \frac{9}{25}M(q^2) + \frac{198}{25}M(q^3) + \frac{352}{25}M(q^4) - \frac{81}{25}M(q^6) - \frac{432}{25}M(q^{12}) = \left(1 - \frac{52}{5}p - \frac{664}{5}p^2 - 164p^3 + 244p^4 + \frac{1292}{5}p^5 - \frac{76}{5}p^6 - 104p^7 + 4p^8\right)k^4. \tag{3.43}$$

Then, from (3.42) and (3.43), we deduce

$$(3L(q^3) - 4L(q^4))^2 = -\frac{3}{25}M(q) - \frac{9}{25}M(q^2) + \frac{198}{25}M(q^3) + \frac{352}{25}M(q^4) - \frac{81}{25}M(q^6) - \frac{432}{25}M(q^{12}) + \frac{36}{5}(1 + 7p - 5p^2)p(1 + p)(1 - p)(1 + 2p)(2 + p)k^4. \tag{3.44}$$

As

$$1 + 7p - 5p^2 = 5p(1-p) + (1+2p),$$

we obtain from (3.44)

$$\begin{aligned}(3L(q^3) - 4L(q^4))^2 &= -\frac{3}{25}M(q) - \frac{9}{25}M(q^2) + \frac{198}{25}M(q^3) \\ &+\frac{352}{25}M(q^4) - \frac{81}{25}M(q^6) - \frac{432}{25}M(q^{12}) \\ &+36p^2(1+p)(1-p)^2(1+2p)(2+p)k^4 \\ &+\frac{36}{5}p(1+p)(1-p)(1+2p)^2(2+p)k^4.\end{aligned} \tag{3.45}$$

From (3.28)-(3.33) we have

$$\begin{aligned}&\Delta(q)^{1/8}\Delta(q^2)^{1/12}\Delta(q^3)^{-1/24}\Delta(q^4)^{-1/24}\Delta(q^6)^{1/12}\Delta(q^{12})^{1/8} \\ &= 2^{-3}p^2(1+p)(1-p)^2(1+2p)(2+p)k^4\end{aligned} \tag{3.46}$$

and recalling (3.35) we have

$$\begin{aligned}&\Delta(q)^{-1/12}\Delta(q^2)^{1/3}\Delta(q^3)^{-1/12}\Delta(q^4)^{-1/12}\Delta(q^6)^{1/3}\Delta(q^{12})^{-1/12} \\ &= 2^{-2}p(1+p)(1-p)(1+2p)^2(2+p)k^4.\end{aligned} \tag{3.47}$$

Hence, from (3.45)-(3.47), we obtain

$$\begin{aligned}(3L(q^3) - 4L(q^4))^2 &= -\frac{3}{25}M(q) - \frac{9}{25}M(q^2) + \frac{198}{25}M(q^3) \\ &+\frac{352}{25}M(q^4) - \frac{81}{25}M(q^6) - \frac{432}{25}M(q^{12}) \\ &+288\Delta(q)^{1/8}\Delta(q^2)^{1/12}\Delta(q^3)^{-1/24}\Delta(q^4)^{-1/24}\Delta(q^6)^{1/12}\Delta(q^{12})^{1/8} \\ &+\frac{144}{5}\Delta(q)^{-1/12}\Delta(q^2)^{1/3}\Delta(q^3)^{-1/12}\Delta(q^4)^{-1/12}\Delta(q^6)^{1/3}\Delta(q^{12})^{-1/12}.\end{aligned} \tag{3.48}$$

Appealing to (3.2), (2.11) and (3.48), we obtain

$$\begin{aligned}&(3L(q^3) - 4L(q^4))^2 \\ &= 1 + \frac{48}{5}\sum_{n=1}^{\infty}\left(-3\sigma_3(n) - 9\sigma_3\left(\frac{n}{2}\right) + 198\sigma_3\left(\frac{n}{3}\right)\right. \\ &\left.+352\sigma_3\left(\frac{n}{4}\right) - 81\sigma_3\left(\frac{n}{6}\right) - 432\sigma_3\left(\frac{n}{12}\right)\right)q^n \\ &+\frac{144}{5}\sum_{n=1}^{\infty}c_{3,4}(n)q^n.\end{aligned} \tag{3.49}$$

Finally, appealing to (3.1), (3.38) and (3.49), we obtain

$$
\begin{aligned}
\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{(l,\,m)\,\in\,\mathbb{N}^2\\ 3l+4m\,=\,n}}\sigma(l)\sigma(m)\Bigg)q^n &= \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sigma(l)\sigma(m)q^{3l+4m}\\
&= \left(\sum_{l=1}^{\infty}\sigma(l)q^{3l}\right)\left(\sum_{m=1}^{\infty}\sigma(m)q^{4m}\right) = \left(\frac{1-L(q^3)}{24}\right)\left(\frac{1-L(q^4)}{24}\right)\\
&= \frac{1}{576}-\frac{1}{576}L(q^3)-\frac{1}{576}L(q^4)+\frac{1}{576}L(q^3)L(q^4)\\
&= \frac{1}{576}-\frac{1}{576}L(q^3)-\frac{1}{576}L(q^4)+\frac{1}{1536}L(q^3)^2+\frac{1}{864}L(q^4)^2\\
&\quad -\frac{1}{13824}\Big(3L(q^3)-4L(q^4)\Big)^2\\
&= \frac{1}{576}-\frac{1}{576}\left(1-24\sum_{n=1}^{\infty}\sigma\left(\frac{n}{3}\right)q^n\right)-\frac{1}{576}\left(1-24\sum_{n=1}^{\infty}\sigma\left(\frac{n}{4}\right)q^n\right)\\
&\quad +\frac{1}{1536}\left(1+\sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{3}\right)-96n\sigma\left(\frac{n}{3}\right)\right)q^n\right)\\
&\quad +\frac{1}{864}\left(1+\sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{4}\right)-72n\sigma\left(\frac{n}{4}\right)\right)q^n\right)\\
&\quad -\frac{1}{13824}\left(1+\frac{48}{5}\sum_{n=1}^{\infty}\left(-3\sigma_3(n)-9\sigma_3\left(\frac{n}{2}\right)+198\sigma_3\left(\frac{n}{3}\right)\right.\right.\\
&\qquad\qquad \left.+352\sigma_3\left(\frac{n}{4}\right)-81\sigma_3\left(\frac{n}{6}\right)-432\sigma_3\left(\frac{n}{12}\right)\right)q^n\\
&\qquad\qquad \left.+\frac{144}{5}\sum_{n=1}^{\infty}c_{3,\,4}(n)q^n\right)\\
&= \sum_{n=1}^{\infty}\left(\frac{1}{480}\sigma_3(n)+\frac{1}{160}\sigma_3\left(\frac{n}{2}\right)+\frac{3}{160}\sigma_3\left(\frac{n}{3}\right)+\frac{1}{30}\sigma_3\left(\frac{n}{4}\right)\right.\\
&\qquad +\frac{9}{160}\sigma_3\left(\frac{n}{6}\right)+\frac{3}{10}\sigma_3\left(\frac{n}{12}\right)+\left(\frac{1}{24}-\frac{n}{16}\right)\sigma\left(\frac{n}{3}\right)\\
&\qquad \left.+\left(\frac{1}{24}-\frac{n}{12}\right)\sigma\left(\frac{n}{4}\right)-\frac{1}{480}c_{3,\,4}(n)\right)q^n.
\end{aligned}
$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain the second assertion of Theorem 2.1. ■

4. Proof of Theorem 2.2

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $l \in \mathbb{N}_0$ we set

$$r(l) = \text{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid l = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2\} \quad (4.1)$$

so that $r(0) = 1$. It is known that [12, Theorem 13], [15]

$$r(l) = 12\sigma(l) - 36\sigma\left(\frac{l}{3}\right), \quad l \in \mathbb{N}. \quad (4.2)$$

By (1.5) and (4.1) we have

$$N_4(n) = \sum_{\substack{(l,m) \in \mathbb{N}_0^2 \\ l+4m=n}} r(l)r(m) \quad (4.3)$$

$$= r(0)r\left(\frac{n}{4}\right) + r(n)r(0) + \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+4m=n}} r(l)r(m). \quad (4.4)$$

Thus

$$N_4(n) - \left(12\sigma\left(\frac{n}{4}\right) - 36\sigma\left(\frac{n}{12}\right) + 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right)\right) \quad (4.5)$$

$$= \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+4m=n}} \left(12\sigma(l) - 36\sigma\left(\frac{l}{3}\right)\right)\left(12\sigma(m) - 36\sigma\left(\frac{m}{3}\right)\right) \quad (4.6)$$

$$= 144 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+4m=n}} \sigma(l)\sigma(m) - 432 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+4m=n}} \sigma\left(\frac{l}{3}\right)\sigma(m) \quad (4.7)$$

$$- 432 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+4m=n}} \sigma(l)\sigma\left(\frac{m}{3}\right) + 1296 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+4m=n}} \sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right). \quad (4.8)$$

The first sum is

$$\begin{aligned}\sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+4m=n}} \sigma(l)\sigma(m) &= \sum_{\substack{m \in \mathbb{N} \\ m < n/4}} \sigma(m)\sigma(n-4m) \\ &= \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{4}\right) \\ &\quad + \left(\frac{1}{24} - \frac{n}{16}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{4}\right),\end{aligned}$$

see for example [12, Theorem 4].

The second sum is

$$\begin{aligned}\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+4m=n}} \sigma\left(\frac{l}{3}\right)\sigma(m) &= \sum_{\substack{(l,m)\in\mathbb{N}^2\\ 3l+4m=n}} \sigma(l)\sigma(m)\\ &= \frac{1}{480}\sigma_3(n)+\frac{1}{160}\sigma_3\left(\frac{n}{2}\right)+\frac{3}{160}\sigma_3\left(\frac{n}{3}\right)+\frac{1}{30}\sigma_3\left(\frac{n}{4}\right)\\ &\quad+\frac{9}{160}\sigma_3\left(\frac{n}{6}\right)+\frac{3}{10}\sigma_3\left(\frac{n}{12}\right)+\left(\frac{1}{24}-\frac{n}{16}\right)\sigma\left(\frac{n}{3}\right)\\ &\quad+\left(\frac{1}{24}-\frac{n}{12}\right)\sigma\left(\frac{n}{4}\right)-\frac{1}{480}c_{3,4}(n)\end{aligned}$$

by Theorem 2.1.

The third sum is

$$\begin{aligned}\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+4m=n}} \sigma(l)\sigma\left(\frac{m}{3}\right) &= \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+12m=n}} \sigma(l)\sigma(m)\\ &= \frac{1}{480}\sigma_3(n)+\frac{1}{160}\sigma_3\left(\frac{n}{2}\right)+\frac{3}{160}\sigma_3\left(\frac{n}{3}\right)+\frac{1}{30}\sigma_3\left(\frac{n}{4}\right)\\ &\quad+\frac{9}{160}\sigma_3\left(\frac{n}{6}\right)+\frac{3}{10}\sigma_3\left(\frac{n}{12}\right)+\left(\frac{1}{24}-\frac{n}{48}\right)\sigma(n)\\ &\quad+\left(\frac{1}{24}-\frac{n}{4}\right)\sigma\left(\frac{n}{12}\right)-\frac{11}{480}c_{1,12}(n)\end{aligned}$$

by Theorem 2.1.

The fourth sum is

$$\begin{aligned}\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+4m=n}} \sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right) &= \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+4m=n/3}} \sigma(l)\sigma(m)\\ &= \frac{1}{48}\sigma_3\left(\frac{n}{3}\right)+\frac{1}{16}\sigma_3\left(\frac{n}{6}\right)+\frac{1}{3}\sigma_3\left(\frac{n}{12}\right)\\ &\quad+\left(\frac{1}{24}-\frac{n}{48}\right)\sigma\left(\frac{n}{3}\right)+\left(\frac{1}{24}-\frac{n}{12}\right)\sigma\left(\frac{n}{12}\right)\end{aligned}$$

by [12, Theorem 4].

Finally, putting these results together, we obtain

$$\begin{aligned}N_4(n) &= \frac{6}{5}\sigma_3(n)+\frac{18}{5}\sigma_3\left(\frac{n}{2}\right)+\frac{54}{5}\sigma_3\left(\frac{n}{3}\right)+\frac{96}{5}\sigma_3\left(\frac{n}{4}\right)\\ &\quad+\frac{162}{5}\sigma_3\left(\frac{n}{6}\right)+\frac{864}{5}\sigma_3\left(\frac{n}{12}\right)+\frac{99}{10}c_{1,12}(n)+\frac{9}{10}c_{3,4}(n),\end{aligned}$$

as asserted.

Denoting the right hand side of this equation by $E(n)$, we close this section by giving a short table of values of $N_4(n)$ and $E(n)$.

n	$N_4(n)$	$\sigma_3(n)$	$c_{1,12}(n)$	$c_{3,4}(n)$	$E(n)$
1	12	1	1	1	12
2	36	9	12/11	12	36
3	12	28	−3/11	−33	12
4	96	73	−24/11	−24	96
5	216	126	−54/11	126	216
6	468	252	−36/11	−36	468
7	240	344	−56/11	−136	240
8	1224	585	48/11	48	1224
9	1308	757	9	9	1308
10	1944	1134	72/11	72	1944
11	1728	1332	252/11	−108	1728
12	5280	2044	72/11	72	5280

5. Some properties of $c_{1,12}(n)$ and $c_{3,4}(n)$.

Let $n \in \mathbb{N}$. It was shown in [3, Theorem 2.1] that

$$\begin{aligned} W_6(n) &= \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ l+6m=n}} \sigma(l)\sigma(m) \qquad (5.1) \\ &= \frac{1}{120}\sigma_3(n) + \frac{1}{30}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{40}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{6}\right) \\ &\quad + \left(\frac{1}{24} - \frac{n}{24}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{6}\right) - \frac{1}{120}c_{1,6}(n) \end{aligned}$$

and

$$\begin{aligned} W_{2,3}(n) &= \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ 2l+3m=n}} \sigma(l)\sigma(m) \qquad (5.2) \\ &= \frac{1}{120}\sigma_3(n) + \frac{1}{30}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{40}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{6}\right) \\ &\quad + \left(\frac{1}{24} - \frac{n}{12}\right)\sigma\left(\frac{n}{2}\right) + \left(\frac{1}{24} - \frac{n}{8}\right)\sigma\left(\frac{n}{3}\right) - \frac{1}{120}c_{1,6}(n), \end{aligned}$$

where

$$q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})^2(1-q^{3n})^2(1-q^{6n})^2 = \sum_{n=1}^{\infty} c_{1,6}(n)q^n. \qquad (5.3)$$

We show that

$$c_{1,12}(2n) = \frac{12}{11} c_{1,6}(n), \quad c_{3,4}(2n) = 12 c_{1,6}(n) \tag{5.4}$$

and

$$c_{1,12}(3n) = -\frac{3}{11} c_{3,4}(n), \quad c_{3,4}(3n) = -33 c_{1,12}(n). \tag{5.5}$$

Using the elementary identity

$$\sigma(2k) = 3\sigma(k) - 2\sigma\left(\frac{k}{2}\right) \quad (k \in \mathbb{N}) \tag{5.6}$$

we obtain

$$\begin{aligned} W_{12}(2n) &= \sum_{m<2n/12} \sigma(m)\sigma(2n-12m) \\ &= 3\sum_{m<n/6} \sigma(m)\sigma(n-6m) - 2\sum_{m<n/6} \sigma(m)\sigma\left(\frac{n-6m}{2}\right). \end{aligned}$$

Hence we have

$$W_{12}(2n) - 3W_6(n) + 2W_3\left(\frac{n}{2}\right) = 0. \tag{5.7}$$

Appealing to the result

$$\begin{aligned} W_3(n) &= \frac{1}{24}\sigma_3(n) + \frac{3}{8}\sigma_3\left(\frac{n}{3}\right) \\ &\quad + \left(\frac{1}{24} - \frac{n}{12}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{3}\right), \end{aligned} \tag{5.8}$$

see for example [12, Theorem 3, p. 248], [21, Theorem 1.9, p. 528] and to (5.1), we obtain, using (5.6) and the elementary identity

$$\sigma_3(2k) = 9\sigma_3(k) - 8\sigma_3\left(\frac{k}{2}\right) \quad (k \in \mathbb{N}), \tag{5.9}$$

the first relation in (5.4).

Similarly we can show that

$$W_{3,4}(2n) - 3W_{2,3}(n) + 2W_3\left(\frac{n}{2}\right) = 0 \tag{5.10}$$

from which we obtain the second relation in (5.4).

Using the elementary identity

$$\sigma(3k) = 4\sigma(k) - 3\sigma\left(\frac{k}{3}\right) \quad (k \in \mathbb{N}) \tag{5.11}$$

we obtain

$$\begin{aligned} W_{12}(3n) &= \sum_{m<3n/12} \sigma(m)\sigma(3n-12m) \\ &= 4\sum_{m<n/4} \sigma(m)\sigma(n-4m) - 3\sum_{m<n/4} \sigma(m)\sigma\left(\frac{n-4m}{3}\right). \end{aligned}$$

Hence we have

$$W_{12}(3n) - 4W_4(n) + 3W_{3,4}(n) = 0. \tag{5.12}$$

Appealing to the result

$$\begin{aligned} W_4(n) &= \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{4}\right) \\ &\quad + \left(\frac{1}{24} - \frac{n}{16}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{4}\right), \end{aligned} \tag{5.13}$$

see for example [12, Theorem 4], [8, Theorem 5.2] and to Theorem 2.1, we obtain, using (5.11) and the elementary identity

$$\sigma_3(3k) = 28\sigma_3(k) - 27\sigma_3\left(\frac{k}{3}\right) \quad (k \in \mathbb{N}), \tag{5.14}$$

the first relation in (5.5).

Similarly we can show that

$$W_{3,4}(3n) - 4W_4(n) + 3W_{12}(n) = 0 \tag{5.15}$$

from which we obtain the second relation in (5.5).

As

$$c_{1,6}(2^r3^s) = (-1)^{r+s}2^r3^s \quad (r, s \in \mathbb{N}_0), \tag{5.16}$$

see [3, §5], we obtain from (5.4) and (5.5)

$$c_{1,12}(2^r3^s) = \begin{cases} 3^s, & \text{if } r = 0,\ s \text{ (even) } \geq 0, \\ -\dfrac{3^s}{11}, & \text{if } r = 0,\ s \text{ (odd) } \geq 1, \\ \dfrac{(-1)^{r+s+1}2^{r+1}3^{s+1}}{11}, & \text{if } r \geq 1,\ s \geq 0, \end{cases} \tag{5.17}$$

and

$$c_{3,4}(2^r 3^s) = \begin{cases} 3^s, & \text{if } r = 0,\ s \text{ (even) } \geq 0, \\ -3^s \cdot 11, & \text{if } r = 0,\ s \text{ (odd) } \geq 1, \\ (-1)^{r+s+1} 2^{r+1} 3^{s+1}, & \text{if } r \geq 1,\ s \geq 0. \end{cases} \quad (5.18)$$

Hence the sums $W_{12}(n)$ and $W_{3,4}(n)$ have elementary evaluations when $n = 2^r 3^s$.

References

[1] Alaca, A., Alaca, Ş., and Williams, K. S., "The convolution sum $\sum_{m<n/16} \sigma(m)\sigma(n - 16m)$," Canad. Math. Bull., (to appear).

[2] Alaca, A., Alaca, Ş., and Williams, K. S., "On the two-dimensional theta functions of the Borweins," Acta Arith., to appear.

[3] Alaca, Ş., and Williams, K. S., "Evaluation of the convolution sums $\sum_{l+6m=n} \sigma(l)\sigma(m)$ and $\sum_{2l+3m=n} \sigma(l)\sigma(m)$," submitted for publication.

[4] Berndt, B. C., (1991), "Ramanujan's Notebooks," Part III, Springer-Verlag, New York.

[5] Berndt, B. C., (1998), "Ramanujan's Notebooks," Part V, Springer-Verlag, New York.

[6] Berndt, B. C., Bhargava, S., and Garvan, F. G., (1995), "Ramanujan's theories of elliptic functions to alternative bases," Trans. Amer. Math. Soc., 347, pp. 4163–4244.

[7] Besge, M., (1862), "Extrait d'une lettre de M. Besge à M. Liouville," J. Math. Pures Appl., 7, p. 256.

[8] Cheng, N., and Williams, K. S., (2004), "Convolution sums involving the divisor function," Proc. Edinburgh Math. Soc., 47, pp. 561–572.

[9] Cheng, N., and Williams, K. S., (2005), "Evaluation of some convolution sums involving the sum of divisors functions," Yokohama Math. J., 52, pp. 39–57.

[10] Glaisher, J. W. L., (1885), "On the square of the series in which the coefficients are the sums of the divisors of the exponents," Mess. Math., 14, pp. 156–163.

[11] Glaisher, J. W. L., (1885), "Mathematical Papers," 1883–1885, W. Metcalfe and Son, Cambridge.

[12] Huard, J. G., Ou, Z. M., Spearman, B. K., and Williams, K. S., (2002), "Elementary evaluation of certain convolution sums involving divisor functions," Number

Theory for the Millenium II, edited by M. A. Bennet, B. C. Berndt, N. Boston, H. G. Diamond, A. J. H. Hildebrand, and W. Philipp, A. K. Peters, Natick, Massachusetts, pp. 229–274.

[13] Lehmer, D. H., (1959), "Some functions of Ramanujan," Math. Student, 27, pp. 105–116.

[14] Lemire, M., and Williams, K. S., (2006), "Evaluation of two convolution sums involving the sum of divisors function," Bull. Austral. Math. Soc., 73, pp. 107–115.

[15] Lomadze, G. A., (1989), "Representation of numbers by sums of the quadratic forms $x_1^2 + x_1x_2 + x_2^2$," Acta Arith., 54, pp. 9–36.

[16] Melfi, G., (1998), "Some Problems in Elementary Number Theory and Modular Forms," Ph.D. thesis, University of Pisa.

[17] Melfi, G., (1998), "On some modular identities," Number Theory (K. Györy, A. Pethö, and V. Sos, eds), de Gruyter, Berlin, pp. 371–382.

[18] Ramanujan, S., (1916), "On certain arithmetic functions," Trans. Cambridge Phil. Soc., 22, pp. 159–184.

[19] Ramanujan, S., (1957), "Notebooks," 2 vols, Tata Institute of Fundamental Research, Bombay.

[20] Ramanujan, S., (2000), "Collected Papers," AMS Chelsea Publishing, Providence, Rhode Island.

[21] Williams, K. S., (2004), "A cubic transformation formula for ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1, z\right)$ and some arithmetic convolution formulae," Math. Proc. Cambridge Philos. Soc., 137, pp. 519–539.

[22] Williams, K. S., (2005), "The convolution sum $\sum_{m<n/9} \sigma(m)\sigma(n-9m)$," Internat. J. Number Theory, 1, pp. 193–205.

[23] Williams, K. S., "The convolution sum $\sum_{m<n/8} \sigma(m)\sigma(n-8m)$," Pacific J. Math. (to appear).