International Journal of Number Theory
Vol. 4, No. 2 (2008) 219–239

THETA FUNCTION IDENTITIES AND REPRESENTATIONS BY CERTAIN QUATERNARY QUADRATIC FORMS

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Received 7 December 2006
Accepted 19 January 2007

Some new theta function identities are proved and used to determine the number of representations of a positive integer n by certain quaternary quadratic forms.

Keywords: Quaternary quadratic forms; theta functions.

Mathematics Subject Classification 2000: 11E20, 11E25

1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers, rational numbers, real numbers and complex numbers respectively. For $q \in \mathbb{C}$ with $|q| < 1$ Jacobi's one-dimensional theta function $\varphi(q)$ is defined by

$$\varphi(q) := \sum_{n \in \mathbb{Z}} q^{n^2}, \tag{1.1}$$

and the Borweins' two-dimensional theta function $a(q)$ is defined by

$$a(q) := \sum_{(m,n) \in \mathbb{Z}^2} q^{m^2+mn+n^2}. \tag{1.2}$$

For $a, b, c, d \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we define

$$N(a, b, c, d; n) = \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = ax^2 + by^2 + cz^2 + dt^2\} \tag{1.3}$$

and

$$M(a, b; n) = \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = a(x^2 + y^2) + b(z^2 + zt + t^2)\}. \tag{1.4}$$

Clearly

$$N(a,b,c,d;0) = M(a,b;0) = 1. \tag{1.5}$$

Also

$$\sum_{n=0}^{\infty} N(a,b,c,d;n)q^n = \varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d) \tag{1.6}$$

and

$$\sum_{n=0}^{\infty} M(a,b;n)q^n = \varphi^2(q^a)a(q^b). \tag{1.7}$$

In this paper, using some recent results of Alaca, Alaca and Williams [1, 2], we prove some new theta function identities, see Theorems 2.6 and 2.7. For example we show that

$$\varphi(q)\varphi(q^2)\varphi(q^6) = \frac{3}{4}\varphi^3(q^3) + \frac{1}{4}\varphi^2(q)\varphi(q^3) - \frac{1}{2}\varphi(q)\varphi(-q)\varphi(-q^3) - \frac{1}{4}\varphi(q^3)\varphi^2(-q) + \frac{3}{4}\varphi(q^3)\varphi^2(-q^3),$$

see Theorem 2.6(a). Then, using these identities in conjunction with some results of Petr [17], we determine the number of representations of $n \in \mathbb{N}$ by each of the following quaternary quadratic forms:

$x^2 + y^2 + z^2 + 3t^2$	(Sec. 4)
$x^2 + y^2 + 2z^2 + 6t^2$	(Sec. 5)
$x^2 + 2y^2 + 2z^2 + 3t^2$	(Sec. 6)
$x^2 + 2y^2 + 4z^2 + 6t^2$	(Sec. 7)
$x^2 + 3y^2 + 3z^2 + 3t^2$	(Sec. 8)
$x^2 + 3y^2 + 6z^2 + 6t^2$	(Sec. 9)
$2x^2 + 3y^2 + 3z^2 + 6t^2$	(Sec. 10)
$x^2 + y^2 + z^2 + zt + t^2$	(Sec. 11)
$x^2 + y^2 + 2(z^2 + zt + t^2)$	(Sec. 12)
$x^2 + y^2 + 4(z^2 + zt + t^2)$	(Sec. 13)
$3(x^2 + y^2) + (z^2 + zt + t^2)$	(Sec. 14)
$3(x^2 + y^2) + 2(z^2 + zt + t^2)$	(Sec. 15)
$3(x^2 + y^2) + 4(z^2 + zt + t^2)$	(Sec. 16)

2. Theta Function Identities

As in [2, pp. 32–33] we set

$$p = p(q) = \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)} \tag{2.1}$$

and

$$k = k(q) = \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{2.2}$$

Next we recall the duplication, triplication and change of sign principles for p and k, which were proved in [1, Theorems 9–11].

Theorem 2.1 (Duplication Principle).

$$p(q^2) = \frac{1 + p - p^2 - ((1-p)(1+p)(1+2p))^{1/2}}{p^2},$$

$$k(q^2) = \frac{(1 + p - p^2 + ((1-p)(1+p)(1+2p))^{1/2})k}{2}.$$

Theorem 2.2 (Triplication Principle).

$$\begin{aligned} p(q^3) &= 3^{-1}((-4 - 3p + 6p^2 + 4p^3) \\ &\quad + 2^{2/3}(1 - 2p - 2p^2)((1-p)(1+2p)(2+p))^{1/3} \\ &\quad + 2^{1/3}(1+2p)((1-p)(1+2p)(2+p))^{2/3}), \\ k(q^3) &= 3^{-2}(3 + 2^{2/3}(1+2p)((1-p)(1+2p)(2+p))^{1/3} \\ &\quad + 2^{4/3}((1-p)(1+2p)(2+p))^{2/3})k. \end{aligned}$$

Theorem 2.3 (Change of Sign Principle).

$$p(-q) = \frac{-p}{1+p}, \quad k(-q) = (1+p)^2 k.$$

From (2.1) and (2.2) we obtain

$$\varphi(q) = (1+2p)^{3/4}k^{1/2}, \quad \varphi(q^3) = (1+2p)^{1/4}k^{1/2}. \tag{2.3}$$

Applying the change of sign principle (Theorem 2.3) to (2.3), we obtain

$$\varphi(-q) = (1-p)^{3/4}(1+p)^{1/4}k^{1/2}, \quad \varphi(-q^3) = (1-p)^{1/4}(1+p)^{3/4}k^{1/2}. \tag{2.4}$$

Now

$$\varphi(q) + \varphi(-q) = \sum_{n=-\infty}^{\infty} (1 + (-1)^{n^2})q^{n^2} = 2 \sum_{n=-\infty}^{\infty} q^{4n^2} = 2\varphi(q^4), \tag{2.5}$$

so appealing to (2.3)–(2.5), we obtain

$$\varphi(q^4) = \frac{1}{2}((1+2p)^{3/4} + (1-p)^{3/4}(1+p)^{1/4})k^{1/2}. \tag{2.6}$$

Replacing q by q^3 in (2.5), we have

$$\varphi(q^3) + \varphi(-q^3) = 2\varphi(q^{12}). \tag{2.7}$$

Then, appealing to (2.3), (2.4) and (2.7), we deduce

$$\varphi(q^{12}) = \frac{1}{2}((1+2p)^{1/4} + (1-p)^{1/4}(1+p)^{3/4})k^{1/2}. \tag{2.8}$$

Next let

$$r(n) = \mathrm{card}\{(x,y) \in \mathbb{Z}^2 \mid n = x^2 + y^2\}, \quad n \in \mathbb{N}_0. \tag{2.9}$$

Then

$$\varphi^2(q) = \sum_{n=0}^{\infty} r(n)q^n = \sum_{\substack{n=0 \\ n\,\text{even}}}^{\infty} r(n)q^n + \sum_{\substack{n=0 \\ n\,\text{odd}}}^{\infty} r(n)q^n \tag{2.10}$$

and

$$\varphi^2(-q) = \sum_{n=0}^{\infty} r(n)(-q)^n = \sum_{\substack{n=0 \\ n\,\text{even}}}^{\infty} r(n)q^n - \sum_{\substack{n=0 \\ n\,\text{odd}}}^{\infty} r(n)q^n. \tag{2.11}$$

Adding (2.10) and (2.11), and recalling that $r(2n) = r(n)$ $(n \in \mathbb{N}_0)$, we obtain

$$\varphi^2(q) + \varphi^2(-q) = 2\sum_{\substack{n=0 \\ n\,\text{even}}}^{\infty} r(n)q^n = 2\sum_{n=0}^{\infty} r(2n)q^{2n} = 2\sum_{n=0}^{\infty} r(n)q^{2n},$$

that is

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2). \tag{2.12}$$

Appealing to (2.3), (2.4) and (2.12), we obtain

$$\varphi(q^2) = \frac{1}{\sqrt{2}}((1+2p)^{3/2} + (1-p)^{3/2}(1+p)^{1/2})^{1/2}k^{1/2}. \tag{2.13}$$

Replacing q by q^3 in (2.12), we have

$$\varphi^2(q^3) + \varphi^2(-q^3) = 2\varphi^2(q^6). \tag{2.14}$$

Then, appealing to (2.3), (2.4) and (2.14), we deduce

$$\varphi(q^6) = \frac{1}{\sqrt{2}}((1+2p)^{1/2} + (1-p)^{1/2}(1+p)^{3/2})^{1/2}k^{1/2}. \tag{2.15}$$

We have proved the following result.

Theorem 2.4. *Define* p *and* k *by* (2.1) *and* (2.2), *respectively. Then*

(a) $\varphi(q) = (1+2p)^{3/4}k^{1/2}$,

(b) $\varphi(q^2) = \dfrac{1}{\sqrt{2}}((1+2p)^{3/2} + (1-p)^{3/2}(1+p)^{1/2})^{1/2}k^{1/2}$,

(c) $\varphi(q^3) = (1+2p)^{1/4}k^{1/2}$,

(d) $\varphi(q^4) = \dfrac{1}{2}((1+2p)^{3/4} + (1-p)^{3/4}(1+p)^{1/4})k^{1/2}$,

(e) $\varphi(q^6) = \dfrac{1}{\sqrt{2}}((1+2p)^{1/2} + (1-p)^{1/2}(1+p)^{3/2})^{1/2}k^{1/2}$,

(f) $\varphi(q^{12}) = \dfrac{1}{2}((1+2p)^{1/4} + (1-p)^{1/4}(1+p)^{3/4})k^{1/2}$,

(g) $\varphi(-q) = (1-p)^{3/4}(1+p)^{1/4}k^{1/2}$,

(h) $\varphi(-q^3) = (1-p)^{1/4}(1+p)^{3/4}k^{1/2}$.

We remark that parts (b)–(f) can also be obtained from part (a) by applying Theorems 2.1 and 2.2.

From Theorem 2.4(b), (e), we obtain the following result.

Theorem 2.5.

$$\varphi(q^2)\varphi(q^6) = \frac{1}{2}(1+p+p^2+(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2})k.$$

Proof. From parts (b) and (e) of Theorem 2.4, we have

$$\begin{aligned}\varphi(q^2)\varphi(q^6) &= \frac{1}{2}(((1+2p)^{3/2}+(1-p)^{3/2}(1+p)^{1/2})\\ &\quad\times((1+2p)^{1/2}+(1-p)^{1/2}(1+p)^{3/2}))^{1/2}k\\ &= \frac{1}{2}(2+4p+2p^2+p^4\\ &\quad+(2+2p+2p^2)(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2})^{1/2}k\\ &= \frac{1}{2}(1+p+p^2+(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2})k. \qquad \square\end{aligned}$$

The theta function identities given in the next theorem follow from Theorems 2.4 and 2.5.

Theorem 2.6.

(a) $$\begin{aligned}\varphi(q)\varphi(q^2)\varphi(q^6) &= \frac{3}{4}\varphi^3(q^3)+\frac{1}{4}\varphi^2(q)\varphi(q^3)-\frac{1}{2}\varphi(q)\varphi(-q)\varphi(-q^3)\\ &\quad-\frac{1}{4}\varphi(q^3)\varphi^2(-q)+\frac{3}{4}\varphi(q^3)\varphi^2(-q^3).\end{aligned}$$

(b) $$\begin{aligned}\varphi(q^2)\varphi(q^3)\varphi(q^6) &= -\frac{1}{4}\varphi(q)\varphi^2(q^3)+\frac{1}{4}\varphi^3(q)+\frac{1}{2}\varphi(q^3)\varphi(-q)\varphi(-q^3)\\ &\quad+\frac{1}{4}\varphi^2(-q)\varphi(q)+\frac{1}{4}\varphi^2(-q^3)\varphi(q).\end{aligned}$$

(c) $$\begin{aligned}\varphi(q)\varphi(q^2)\varphi(q^4)\varphi(q^6) &= \frac{3}{8}\varphi(q)\varphi^3(q^3)+\frac{1}{8}\varphi^3(q)\varphi(q^3)-\frac{1}{8}\varphi^2(q)\varphi(-q)\varphi(-q^3)\\ &\quad-\frac{3}{8}\varphi^2(q^3)\varphi(-q)\varphi(-q^3)-\frac{1}{2}\varphi^2(q^4)\varphi(-q^4)\varphi(-q^{12})\\ &\quad+\frac{3}{2}\varphi^2(q^{12})\varphi(-q^4)\varphi(-q^{12}).\end{aligned}$$

Proof. We just give the proof of (a) as the remaining identities can be proved in a similar manner.

(a) By Theorem 2.4(a), (c), (g), (h) and Theorem 2.5, we have

$$
\begin{aligned}
&3\varphi^3(q^3)+\varphi^2(q)\varphi(q^3)-2\varphi(q)\varphi(-q)\varphi(-q^3)-\varphi(q^3)\varphi^2(-q)+3\varphi(q^3)\varphi^2(-q^3)\\
&\quad= 3(1+2p)^{3/4}k^{3/2}+(1+2p)^{7/4}k^{3/2}-2(1+2p)^{3/4}(1-p)(1+p)k^{3/2}\\
&\qquad -(1+2p)^{1/4}(1-p)^{3/2}(1+p)^{1/2}k^{3/2}\\
&\qquad +3(1+2p)^{1/4}(1-p)^{1/2}(1+p)^{3/2}k^{3/2}\\
&\quad= (1+2p)^{1/4}k^{3/2}((2+2p+2p^2)(1+2p)^{1/2}+(2+4p)(1-p)^{1/2}(1+p)^{1/2})\\
&\quad= 2(1+2p)^{3/4}k^{3/2}(1+p+p^2+(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2})\\
&\quad= 4\varphi(q)\varphi(q^2)\varphi(q^6).
\end{aligned}
$$

□

In [1, p. 178] Alaca, Alaca and Williams showed that

$$a(q)=(1+4p+p^2)k, \tag{2.16}$$

$$a(q^2)=(1+p+p^2)k, \tag{2.17}$$

$$a(q^4)=\left(1+p-\frac{1}{2}p^2\right)k. \tag{2.18}$$

Making use of Theorem 2.4 and (2.16)–(2.18), we obtain the theta function identities stated in the next theorem.

Theorem 2.7.

(a) $a(q)=2\varphi(q)\varphi(q^3)-\varphi(-q)\varphi(-q^3)$.

(b) $a(q^4)=\frac{1}{2}\varphi(q)\varphi(q^3)+\frac{1}{2}\varphi(-q)\varphi(-q^3)$.

(c) $\varphi(q)a(q^2)=\frac{3}{2}\varphi^3(q^3)+\frac{1}{2}\varphi^2(q)\varphi(q^3)-\varphi(q)\varphi(-q)\varphi(-q^3)$.

(d) $\varphi(q^3)a(q^2)=-\frac{1}{2}\varphi(q)\varphi^2(q^3)+\frac{1}{2}\varphi^3(q)+\varphi(q^3)\varphi(-q)\varphi(-q^3)$.

Proof. (a) By Theorem 2.4(a), (c), (g), (h) and (2.16), we have

$$2\varphi(q)\varphi(q^3)-\varphi(-q)\varphi(-q^3)=(1+4p+p^2)k=a(q).$$

(b) By Theorem 2.4(a), (c), (g), (h) and (2.18), we have

$$\frac{1}{2}\varphi(q)\varphi(q^3)+\frac{1}{2}\varphi(-q)\varphi(-q^3)=\left(1+p-\frac{1}{2}p^2\right)k=a(q^4).$$

(c) By Theorem 2.4(a), (c), (g), (h) and (2.17), we have

$$
\begin{aligned}
&\frac{3}{2}\varphi^3(q^3)+\frac{1}{2}\varphi^2(q)\varphi(q^3)-\varphi(q)\varphi(-q)\varphi(-q^3)\\
&\qquad= \frac{3}{2}(1+2p)^{3/4}k^{3/2}+\frac{1}{2}(1+2p)^{7/4}k^{3/2}-(1-p^2)(1+2p)^{3/4}k^{3/2}
\end{aligned}
$$

$$= (1+2p)^{3/4}(1+p+p^2)k^{3/2}$$
$$= \varphi(q)a(q^2).$$

(d) By Theorem 2.4(a), (c), (g), (h) and (2.17), we have

$$-\frac{1}{2}\varphi(q)\varphi^2(q^3) + \frac{1}{2}\varphi^3(q) + \varphi(q^3)\varphi(-q)\varphi(-q^3)$$
$$= -\frac{1}{2}(1+2p)^{5/4}k^{3/2} + \frac{1}{2}(1+2p)^{9/4}k^{3/2} + (1-p^2)(1+2p)^{1/4}k^{3/2}$$
$$= (1+2p)^{1/4}(1+p+p^2)k^{3/2}$$
$$= \varphi(q^3)a(q^2). \qquad \square$$

Parts (a) and (b) are due to Borwein, Borwein and Garvan [4, Lemma 2.1(i), (a), (b), p. 36].

3. Some Results of Petr

A nonsquare integer Δ satisfying $\Delta \equiv 0$ or $1 \pmod 4$ is called a discriminant. If Δ is a discriminant the Legendre–Jacobi–Kronecker symbol corresponding to the discriminant Δ is denoted by $\left(\frac{\Delta}{*}\right)$. We begin with a definition.

Definition 3.1. For $n \in \mathbb{N}$, we set

(a) $A(n) := \sum_{d|n} d\left(\frac{12}{n/d}\right) = \sum_{d|n} \frac{n}{d}\left(\frac{12}{d}\right),$

(b) $B(n) := \sum_{d|n} d\left(\frac{-3}{d}\right)\left(\frac{-4}{n/d}\right) = \sum_{d|n} \frac{n}{d}\left(\frac{-3}{n/d}\right)\left(\frac{-4}{d}\right),$

(c) $C(n) := \sum_{d|n} d\left(\frac{-3}{n/d}\right)\left(\frac{-4}{d}\right) = \sum_{d|n} \frac{n}{d}\left(\frac{-3}{d}\right)\left(\frac{-4}{n/d}\right),$

(d) $D(n) := \sum_{d|n} d\left(\frac{12}{d}\right) = \sum_{d|n} \frac{n}{d}\left(\frac{12}{n/d}\right).$

The following result is easily proved so the proof is omitted.

Theorem 3.1. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and* $\gcd(N,6) = 1$. *Then*

(a) $A(n) = 2^\alpha 3^\beta A(N),$

(b) $B(n) = (-1)^{\alpha+\beta} 2^\alpha \left(\frac{N}{3}\right) A(N),$

(c) $C(n) = (-1)^{\alpha+\beta+(N-1)/2} 3^{\beta} A(N)$,

(d) $D(n) = (-1)^{(N-1)/2} \left(\frac{N}{3}\right) A(N) = \left(\frac{3}{N}\right) A(N)$.

Here $\left(\frac{N}{3}\right)$ and $\left(\frac{3}{N}\right)$ are Legendre–Jacobi symbols, which are related by $\left(\frac{3}{N}\right) = (-1)^{(N-1)/2}\left(\frac{N}{3}\right)$ by the Law of Quadratic Reciprocity.

We now turn to the results of Petr [17] that we need. Petr's notation is as follows:

$$\Theta_1 = 2\sum_{n=0}^{\infty} q^{(2n+1)^2/4}, \quad \Theta_1(0,3\tau) = 2\sum_{n=0}^{\infty} q^{3(2n+1)^2/4}, \tag{3.1}$$

$$\Theta_2 = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2}, \quad \Theta_2(0,3\tau) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{3n^2}, \tag{3.2}$$

$$\Theta_3 = 1 + 2\sum_{n=1}^{\infty} q^{n^2}, \quad \Theta_3(0,3\tau) = 1 + 2\sum_{n=1}^{\infty} q^{3n^2}. \tag{3.3}$$

Petr [17, Eq. (23), p. 12] notes the following classical relation of Jacobi

$$\Theta_3\Theta_3(0,3\tau) = \Theta_1\Theta_1(0,3\tau) + \Theta_2\Theta_2(0,3\tau) \tag{3.4}$$

and proves the identities

$$(\Theta_3^2 + 3\Theta_3^2(0,3\tau))\Theta_2\Theta_2(0,3\tau) = 4 - 4\sum_{n=1}^{\infty}\left(\sum_{d\mid n} d\left(\frac{12}{d}\right)\right) q^n, \tag{3.5}$$

$$(\Theta_3^2 - \Theta_3^2(0,3\tau))\Theta_2\Theta_2(0,3\tau) = 4\sum_{n=1}^{\infty}\left(\sum_{d\mid n} d\left(\frac{-3}{n/d}\right)\left(\frac{-4}{d}\right)\right) q^n, \tag{3.6}$$

$$(3\Theta_3^2(0,3\tau) - \Theta_3^2)\Theta_1\Theta_1(0,3\tau) = 8\sum_{n=1}^{\infty}\left(\sum_{d\mid n} d\left(\frac{-3}{d}\right)\left(\frac{-4}{n/d}\right)\right) q^n, \tag{3.7}$$

$$(\Theta_3^2(0,3\tau) + \Theta_3^2)\Theta_1\Theta_1(0,3\tau) = 8\sum_{n=1}^{\infty}\left(\sum_{d\mid n} d\left(\frac{12}{n/d}\right)\right) q^n, \tag{3.8}$$

see [17, p. 15, 5th Eq. (30)], [17, p. 15, 6th Eq. (30)], [17, p. 15, 1st Eq. (30)], [17, p. 15, 2nd Eq. (30)], respectively. From (1.1), (3.2) and (3.3), we obtain

$$\Theta_2 = \varphi(-q), \quad \Theta_2(0,3\tau) = \varphi(-q^3), \tag{3.9}$$

$$\Theta_3 = \varphi(q), \quad \Theta_3(0,3\tau) = \varphi(q^3). \tag{3.10}$$

From (3.4), (3.9) and (3.10), we deduce

$$\Theta_1\Theta_1(0,3\tau) = \varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3). \tag{3.11}$$

Using (3.9)–(3.11) in (3.5)–(3.8), and recalling Definition 3.1, we have the following result.

Theorem 3.2. *For* $|q| < 1$,

$$\sum_{n=1}^{\infty} A(n)q^n = \frac{1}{8}\varphi(q)\varphi^3(q^3) + \frac{1}{8}\varphi^3(q)\varphi(q^3)$$
$$- \frac{1}{8}\varphi^2(q)\varphi(-q)\varphi(-q^3) - \frac{1}{8}\varphi^2(q^3)\varphi(-q)\varphi(-q^3),$$
$$\sum_{n=1}^{\infty} B(n)q^n = \frac{3}{8}\varphi(q)\varphi^3(q^3) - \frac{1}{8}\varphi^3(q)\varphi(q^3)$$
$$+ \frac{1}{8}\varphi^2(q)\varphi(-q)\varphi(-q^3) - \frac{3}{8}\varphi^2(q^3)\varphi(-q)\varphi(-q^3),$$
$$\sum_{n=1}^{\infty} C(n)q^n = \frac{1}{4}\varphi^2(q)\varphi(-q)\varphi(-q^3) - \frac{1}{4}\varphi^2(q^3)\varphi(-q)\varphi(-q^3),$$
$$\sum_{n=1}^{\infty} D(n)q^n = 1 - \frac{1}{4}\varphi^2(q)\varphi(-q)\varphi(-q^3) - \frac{3}{4}\varphi^2(q^3)\varphi(-q)\varphi(-q^3).$$

Solving for the quantities $\varphi(q)\varphi^3(q^3)$, $\varphi^3(q)\varphi(q^3)$, $\varphi^2(q)\varphi(-q)\varphi(-q^3)$ and $\varphi^2(q^3)\varphi(-q)\varphi(-q^3)$ in Theorem 3.2, we obtain the following theorem.

Theorem 3.3. *For* $|q| < 1$,

(a) $\varphi(q)\varphi^3(q^3) = 1 + \sum_{n=1}^{\infty}(2A(n) + 2B(n) - C(n) - D(n))q^n$,

(b) $\varphi^3(q)\varphi(q^3) = 1 + \sum_{n=1}^{\infty}(6A(n) - 2B(n) + 3C(n) - D(n))q^n$,

(c) $\varphi^2(q)\varphi(-q)\varphi(-q^3) = 1 + \sum_{n=1}^{\infty}(3C(n) - D(n))q^n$,

(d) $\varphi^2(q^3)\varphi(-q)\varphi(-q^3) = 1 - \sum_{n=1}^{\infty}(C(n) + D(n))q^n$.

Replacing q by $-q$ in (a)–(d) of Theorem 3.2, we obtain the expansions of $\varphi(-q)\varphi^3(-q^3)$, $\varphi^3(-q)\varphi(-q^3)$, $\varphi(q)\varphi(q^3)\varphi^2(-q)$ and $\varphi(q)\varphi(q^3)\varphi^2(-q^3)$ respectively in powers of q.

4. The Form $x^2 + y^2 + z^2 + 3t^2$

By (1.3), (1.5), (1.6) and Theorem 3.3(b), we have

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} N(1,1,1,3;n)q^n &= \sum_{n=0}^{\infty} N(1,1,1,3;n)q^n \\ &= \varphi^3(q)\varphi(q^3) \\ &= 1 + \sum_{n=1}^{\infty}(6A(n) - 2B(n) + 3C(n) - D(n))q^n. \end{aligned}$$

Equating the coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$N(1,1,1,3;n) = 6A(n) - 2B(n) + 3C(n) - D(n).$$

Appealing to Theorem 3.1, we have the following result.

Theorem 4.1. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N, 6) = 1$. *Then*

$$N(1,1,1,3;n) = (2^{\alpha+1} + (-1)^{\alpha+\beta+\frac{N-1}{2}})\left(3^{\beta+1} - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right)A(N).$$

This theorem was first stated but not proved by Liouville [6]. Proofs have been given by Benz [3, pp. 175–192], Demuth [5, pp. 243–245] and Petr [17, p. 16] and [18, p. 186].

5. The Form $x^2 + y^2 + 2z^2 + 6t^2$

By (1.3), (1.5), (1.6), Theorem 2.6(a) and Theorem 3.3, we have

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} N(1,1,2,6;n)q^n &= \varphi^2(q)\varphi(q^2)\varphi(q^6) \\ &= \frac{3}{4}\varphi(q)\varphi^3(q^3) + \frac{1}{4}\varphi^3(q)\varphi(q^3) - \frac{1}{2}\varphi^2(q)\varphi(-q)\varphi(-q^3) \\ &\quad - \frac{1}{4}\varphi(q)\varphi(q^3)\varphi^2(-q) + \frac{3}{4}\varphi(q)\varphi(q^3)\varphi^2(-q^3) \\ &= 1 + \sum_{n=1}^{\infty}(3A(n) + B(n) \\ &\quad - \frac{3}{2}(1 + (-1)^n)C(n) - \frac{1}{2}(1 + (-1)^n)D(n))q^n. \end{aligned}$$

Equating the coefficients of q^n $(n \in \mathbb{N})$, we have

$$N(1,1,2,6;n) = 3A(n) + B(n) - \frac{3}{2}(1 + (-1)^n)C(n) - \frac{1}{2}(1 + (-1)^n)D(n).$$

Appealing to Theorem 3.1, we deduce the following result.

Theorem 5.1. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N, 6) = 1$. *Then*

$$N(1,1,2,6;n) = \left(2^\alpha - \frac{1}{2}(1+(-1)^n)(-1)^{\alpha+\beta+\frac{N-1}{2}}\right) \times \left(3^{\beta+1} + (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right)A(N).$$

This theorem was first stated but not proved by Liouville [9]. The only proof of this result that the authors have found in the literature is due to Benz [3, pp. 175–192]. Our proof is considerably simpler and more direct than that of Benz.

6. The Form $x^2 + 2y^2 + 2z^2 + 3t^2$

By (1.3), (1.5), (1.6), (2.12) and Theorem 3.3(b), (c), we have

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} N(1,2,2,3;n)q^n &= \varphi(q)\varphi^2(q^2)\varphi(q^3) \\ &= \frac{1}{2}\varphi(q)(\varphi^2(q) + \varphi^2(-q))\varphi(q^3) \\ &= \frac{1}{2}\varphi^3(q)\varphi(q^3) + \frac{1}{2}\varphi(q)\varphi(q^3)\varphi^2(-q) \\ &= 1 + \sum_{n=1}^{\infty}(3A(n) - B(n) + \frac{3}{2}(1+(-1)^n)C(n) \\ &\quad - \frac{1}{2}(1+(-1)^n)D(n))q^n. \end{aligned}$$

Equating coefficients of q^n ($n \in \mathbb{N}$), we obtain

$$N(1,2,2,3;n) = 3A(n) - B(n) + \frac{3}{2}(1+(-1)^n)C(n) - \frac{1}{2}(1+(-1)^n)D(n).$$

Appealing to Theorem 3.1, we deduce the following result.

Theorem 6.1. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N, 6) = 1$. *Then*

$$N(1,2,2,3;n) = \left(2^\alpha + \frac{1}{2}(1+(-1)^n)(-1)^{\alpha+\beta+\frac{N-1}{2}}\right) \times \left(3^{\beta+1} - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right)A(N).$$

This result was stated but not proved by Liouville [10]. The authors have not found a proof of Theorem 6.1 in the literature.

7. The Form $x^2 + 2y^2 + 4z^2 + 6t^2$

Making use of (1.3), (1.5), (1.6), Theorem 2.6(c) and Theorem 3.3, we obtain

$$\begin{aligned}
1 + \sum_{n=1}^{\infty} N(1,2,4,6;n)q^n &= \varphi(q)\varphi(q^2)\varphi(q^4)\varphi(q^6) \\
&= \frac{3}{8}\varphi(q)\varphi^3(q^3) + \frac{1}{8}\varphi^3(q)\varphi(q^3) - \frac{1}{8}\varphi^2(q)\varphi(-q)\varphi(-q^3) \\
&\quad - \frac{3}{8}\varphi^2(q^3)\varphi(-q)\varphi(-q^3) - \frac{1}{2}\varphi^2(q^4)\varphi(-q^4)\varphi(-q^{12}) \\
&\quad + \frac{3}{2}\varphi^2(q^{12})\varphi(-q^4)\varphi(-q^{12}) \\
&= \frac{3}{8}\left(1 + \sum_{n=1}^{\infty}(2A(n) + 2B(n) - C(n) - D(n))q^n\right) \\
&\quad + \frac{1}{8}\left(1 + \sum_{n=1}^{\infty}(6A(n) - 2B(n) + 3C(n) - D(n))q^n\right) \\
&\quad - \frac{1}{8}\left(1 + \sum_{n=1}^{\infty}(3C(n) - D(n))q^n\right) \\
&\quad - \frac{3}{8}\left(1 - \sum_{n=1}^{\infty}(C(n) + D(n))q^n\right) \\
&\quad - \frac{1}{2}\left(1 + \sum_{n=1}^{\infty}(3C(n) - D(n))q^{4n}\right) \\
&\quad + \frac{3}{2}\left(1 - \sum_{n=1}^{\infty}(C(n) + D(n))q^{4n}\right) \\
&= 1 + \sum_{n=1}^{\infty}\left(\frac{3}{2}A(n) + \frac{1}{2}B(n)\right)q^n \\
&\quad + \sum_{n=1}^{\infty}(-3C(n) - D(n))q^{4n}.
\end{aligned}$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$N(1,2,4,6;n) = \begin{cases} \frac{3}{2}A(n) + \frac{1}{2}B(n), & \text{if } n \not\equiv 0 \pmod 4, \\ \frac{3}{2}A(n) + \frac{1}{2}B(n) - 3C\left(\frac{n}{4}\right) - D\left(\frac{n}{4}\right), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Appealing to Theorem 3.1, we deduce the following result.

Theorem 7.1. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(1,2,4,6;n) = \begin{cases} \frac{1}{2}\left(3^{\beta+1} + (-1)^\beta \left(\frac{N}{3}\right)\right) A(N), & \text{if } n \equiv 1 \pmod 2, \\ \left(3^{\beta+1} - (-1)^\beta \left(\frac{N}{3}\right)\right) A(N), & \text{if } n \equiv 2 \pmod 4, \\ \left(2^{\alpha-1} - (-1)^{\alpha+\beta+\frac{N-1}{2}}\right) & \\ \qquad \times \left(3^{\beta+1} + (-1)^{\alpha+\beta} \left(\frac{N}{3}\right)\right) A(N), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

This result was stated but not proved by Liouville [11]. The authors have not found a proof in the literature.

8. The Form $x^2 + 3y^2 + 3z^2 + 3t^2$

By (1.3), (1.5), (1.6) and Theorem 3.3(a), we have

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} N(1,3,3,3;n)q^n &= \varphi(q)\varphi^3(q^3) \\ &= 1 + \sum_{n=1}^{\infty} (2A(n) + 2B(n) - C(n) - D(n))q^n. \end{aligned}$$

Equating the coefficients of q^n $(n \in \mathbb{N})$, we have

$$N(1,3,3,3;n) = 2A(n) + 2B(n) - C(n) - D(n). \tag{8.1}$$

Appealing to Theorem 3.1, we deduce the following result.

Theorem 8.1. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(1,3,3,3;n) = \left(2^{\alpha+1} - (-1)^{\alpha+\beta+\frac{N-1}{2}}\right)\left(3^\beta + (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right) A(N).$$

A less compact form of Theorem 8.1 was given without proof by Liouville [14]. A proof of Theorem 8.1 have been given by Benz [3, pp. 175–192].

9. The Form $x^2 + 3y^2 + 6z^2 + 6t^2$

By (1.3), (1.5), (1.6), (2.14) and Theorem 3.3(a)(d), we have

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} N(1,3,6,6;n)q^n &= \varphi(q)\varphi(q^3)\varphi^2(q^6) \\ &= \frac{1}{2}\varphi(q)\varphi(q^3)(\varphi^2(q^3) + \varphi^2(-q^3)) \end{aligned}$$

$$= \frac{1}{2}\varphi(q)\varphi^3(q^3) + \frac{1}{2}\varphi(q)\varphi(q^3)\varphi^2(-q^3)$$

$$= 1 + \sum_{n=1}^{\infty} \Big(A(n) + B(n) - \frac{1}{2}(1+(-1)^n)C(n) - \frac{1}{2}(1+(-1)^n)D(n) \Big) q^n.$$

Equating the coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$N(1,3,6,6;n) = A(n) + B(n) - \frac{1}{2}(1+(-1)^n)C(n) - \frac{1}{2}(1+(-1)^n)D(n). \quad (9.1)$$

Appealing to Theorem 3.1, we deduce the following result.

Theorem 9.1. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(1,3,6,6;n) = \Big(2^\alpha - \frac{1}{2}(1+(-1)^n)(-1)^{\alpha+\beta+\frac{N-1}{2}} \Big) \times \left(3^\beta + (-1)^{\alpha+\beta} \left(\frac{N}{3}\right) \right) A(N).$$

A complicated result equivalent to Theorem 9.1 was stated but not proved by Liouville [12]. The authors have not found a proof of Theorem 9.1 in the literature.

10. The Form $2x^2 + 3y^2 + 3z^2 + 6t^2$

By (1.3), (1.5), (1.6), Theorem 2.6(b) and Theorem 3.3, we have

$$1 + \sum_{n=1}^{\infty} N(2,3,3,6;n)q^n = \varphi(q^2)\varphi^2(q^3)\varphi(q^6)$$

$$= -\frac{1}{4}\varphi(q)\varphi^3(q^3) + \frac{1}{4}\varphi^3(q)\varphi(q^3) + \frac{1}{2}\varphi^2(q^3)\varphi(-q)\varphi(-q^3) + \frac{1}{4}\varphi^2(-q)\varphi(q)\varphi(q^3) + \frac{1}{4}\varphi^2(-q^3)\varphi(q)\varphi(q^3)$$

$$= 1 + \sum_{n=1}^{\infty} \Big(A(n) - B(n) + \frac{1}{2}(1+(-1)^n)C(n) - \frac{1}{2}(1+(-1)^n)D(n) \Big) q^n.$$

Equating the coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$N(2,3,3,6;n) = A(n) - B(n) + \frac{1}{2}(1+(-1)^n)C(n) - \frac{1}{2}(1+(-1)^n)D(n).$$

Appealing to Theorem 3.1, we deduce the following result.

Theorem 10.1. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and* $\gcd(N,6) = 1$. *Then*

$$N(2,3,3,6;n) = \left(2^\alpha + \frac{1}{2}(1+(-1)^n)(-1)^{\alpha+\beta+\frac{N-1}{2}}\right) \times \left(3^\beta - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right) A(N).$$

This result was stated by Liouville in [13] without proof. A proof has been given by Benz [3, pp. 175–192].

11. The Form $x^2 + y^2 + z^2 + zt + t^2$

By (1.4), (1.5), (1.7), Theorem 2.7(a) and Theorem 3.3(b), (c), we have

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} M(1,1;n)q^n &= \sum_{n=0}^{\infty} M(1,1;n)q^n \\ &= \varphi^2(q)a(q) \\ &= 2\varphi^3(q)\varphi(q^3) - \varphi^2(q)\varphi(-q)\varphi(-q^3) \\ &= 2\left(1 + \sum_{n=1}^{\infty}(6A(n) - 2B(n) + 3C(n) - D(n))q^n\right) \\ &\quad - \left(1 + \sum_{n=1}^{\infty}(3C(n) - D(n))q^n\right) \\ &= 1 + \sum_{n=1}^{\infty}(12A(n) - 4B(n) + 3C(n) - D(n))q^n. \end{aligned}$$

Equating the coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$M(1,1;n) = 12A(n) - 4B(n) + 3C(n) - D(n).$$

Appealing to Theorem 3.1, we have the following result.

Theorem 11.1. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and* $\gcd(N,6) = 1$. *Then*

$$M(1,1;n) = \left(2^{\alpha+2} + (-1)^{\alpha+\beta+\frac{N-1}{2}}\right)\left(3^{\beta+1} - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right) A(N).$$

This result was stated without proof by Liouville in [8]. A proof was given by Benz in [3, pp. 192–194]. Our proof is considerably shorter than that of Benz.

12. The Form $x^2 + y^2 + 2(z^2 + zt + t^2)$

By (1.4), (1.5), (1.7), Theorem 2.7(c) and Theorem 3.3(a)–(c), we obtain

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} M(1,2;n)q^n &= \varphi^2(q)a(q^2) \\ &= \frac{3}{2}\varphi(q)\varphi^3(q^3) + \frac{1}{2}\varphi^3(q)\varphi(q^3) - \varphi^2(q)\varphi(-q)\varphi(-q^3) \\ &= 1 + \sum_{n=1}^{\infty}(6A(n) + 2B(n) - 3C(n) - D(n))q^n. \end{aligned}$$

Equating the coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$M(1,2;n) = 6A(n) + 2B(n) - 3C(n) - D(n).$$

Appealing to Theorem 3.1, we deduce the following result.

Theorem 12.1. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and* $\gcd(N,6) = 1$. *Then*

$$M(1,2;n) = \left(2^{\alpha+1} - (-1)^{\alpha+\beta+\frac{N-1}{2}}\right)\left(3^{\beta+1} + (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right)A(N).$$

This theorem was stated but not proved by Liouville in [7]. Benz considered the evaluation of $M(1,2;n)$ in [3, pp. 194–198].

13. The Form $x^2 + y^2 + 4(z^2 + zt + t^2)$

By (1.4), (1.5), (1.7), Theorem 2.7(b) and Theorem 3.3(b), (c), we obtain

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} M(1,4;n)q^n &= \varphi^2(q)a(q^4) \\ &= \frac{1}{2}\varphi^3(q)\varphi(q^3) + \frac{1}{2}\varphi^2(q)\varphi(-q)\varphi(-q^3) \\ &= 1 + \sum_{n=1}^{\infty}(3A(n) - B(n) + 3C(n) - D(n))q^n. \end{aligned}$$

Equating the coefficients of q^n $(n \in \mathbb{N})$, we deduce

$$M(1,4;n) = 3A(n) - B(n) + 3C(n) - D(n). \tag{13.1}$$

Appealing to Theorem 3.1, we obtain the following result.

Theorem 13.1. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and* $\gcd(N,6) = 1$. *Then*

$$M(1,4;n) = \left(2^{\alpha} + (-1)^{\alpha+\beta+\frac{N-1}{2}}\right)\left(3^{\beta+1} - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right)A(N).$$

This result was not stated by Liouville and appears to be new. We observe that

$$M(1,4;n) = \text{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + 3t^2,\ z \equiv t \pmod 2\}. \tag{13.2}$$

Hence, by Theorems 4.1 and 13.1, we have the following result.

Theorem 13.2. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$\begin{aligned} &\text{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + 3t^2,\ z \not\equiv t (\text{mod}\, 2)\} \\ &= 2^\alpha \left(3^\beta - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right) A(N). \end{aligned}$$

14. The Form $3(x^2 + y^2) + z^2 + zt + t^2$

By (1.4), (1.5), (1.7), Theorem 2.7(a) and Theorem 3.3(a), (d), we have

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} M(3,1;n)q^n &= \varphi^2(q^3)a(q) \\ &= \varphi^2(q^3)(2\varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3)) \\ &= 2\varphi(q)\varphi^3(q^3) - \varphi^2(q^3)\varphi(-q)\varphi(-q^3) \\ &= 1 + \sum_{n=1}^{\infty}(4A(n) + 4B(n) - C(n) - D(n))q^n. \end{aligned}$$

Equating the coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$M(3,1;n) = 4A(n) + 4B(n) - C(n) - D(n).$$

Appealing to Theorem 3.1, we have the following result.

Theorem 14.1. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$M(3,1;n) = \left(2^{\alpha+2} - (-1)^{\alpha+\beta+\frac{N-1}{2}}\right)\left(3^\beta + (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right) A(N).$$

This result was stated without proof by Liouville in [16]. No proof seems to exist in the literature.

15. The Form $3(x^2+y^2)+2(z^2+zt+t^2)$

By (1.4), (1.5), (1.7), Theorem 2.7(d) and Theorem 3.3(a), (b), (d), we obtain

$$\begin{aligned}1+\sum_{n=1}^{\infty} M(3,2;n)q^n &= \varphi^2(q^3)a(q^2)\\ &= \varphi(q^3)\left(-\frac{1}{2}\varphi(q)\varphi^2(q^3)+\frac{1}{2}\varphi^3(q)+\varphi(q^3)\varphi(-q)\varphi(-q^3)\right)\\ &= 1+\sum_{n=1}^{\infty}(2A(n)-2B(n)+C(n)-D(n))q^n.\end{aligned}$$

Equating the coefficients of q^n ($n \in \mathbb{N}$), we deduce

$$M(3,2;n) = 2A(n)-2B(n)+C(n)-D(n).$$

Appealing to Theorem 3.3, we have the following result.

Theorem 15.1. *Let $n \in \mathbb{N}$. Set $n = 2^{\alpha}3^{\beta}N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and* $\gcd(N,6)=1$. *Then*

$$M(3,2;n) = \left(2^{\alpha+1}+(-1)^{\alpha+\beta+\frac{N-1}{2}}\right)\left(3^{\beta}-(-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right)A(N).$$

This result was stated without proof by Liouville in [15]. No proof seems to exist in the literature.

16. The Form $3(x^2+y^2)+4(z^2+zt+t^2)$

By (1.4), (1.5), (1.7), Theorem 2.7(b) and Theorem 3.3(a), (d), we obtain

$$\begin{aligned}1+\sum_{n=1}^{\infty} M(3,4;n)q^n &= \varphi^2(q^3)a(q^4)\\ &= \varphi^2(q^3)\left(\frac{1}{2}\varphi(q)\varphi(q^3)+\frac{1}{2}\varphi(-q)\varphi(-q^3)\right)\\ &= \frac{1}{2}\varphi(q)\varphi^3(q^3)+\frac{1}{2}\varphi^2(q^3)\varphi(-q)\varphi(-q^3)\\ &= \frac{1}{2}\left(1+\sum_{n=1}^{\infty}(2A(n)+2B(n)-C(n)-D(n))q^n\right)\\ &\quad+\frac{1}{2}\left(1-\sum_{n=1}^{\infty}(C(n)+D(n))q^n\right)\\ &= 1+\sum_{n=1}^{\infty}(A(n)+B(n)-C(n)-D(n))q^n.\end{aligned}$$

Equating the coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$M(3,4;n) = A(n) + B(n) - C(n) - D(n).$$

Appealing to Theorem 3.1, we deduce the following result.

Theorem 16.1. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$M(3,4;n) = \left(2^\alpha - (-1)^{\alpha+\beta+\frac{N-1}{2}}\right)\left(3^\beta + (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right) A(N).$$

This result was not stated by Liouville and appears to be new. We observe that

$$M(3,4;n) = \operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + 3y^2 + 3z^2 + 3t^2,\ z \equiv t \pmod 2\}.$$

From Theorems 8.1 and 16.1 we deduce our final result of this paper.

Theorem 16.2. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$\begin{aligned}&\operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + 3y^2 + 3z^2 + 3t^2,\ z \not\equiv t \pmod 2\}\\ &\quad = 2^\alpha \left(3^\beta + (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right) A(N).\end{aligned}$$

Note added in proof. The authors recently noticed that the methods of their paper also apply to the form $2x^2 + 3y^2 + 6z^2 + 12t^2$. From (2.12) with q replaced by q^2 and Theorem 2.4(b), (d), we obtain $\varphi(-q^2)$ in terms of p and k. Applying the duplication principle to $\varphi(-q^2)$, we obtain $\varphi(-q^4)$ in terms of p and k. Applying the triplication principle to $\varphi(-q^4)$, we obtain $\varphi(-q^{12})$ in terms of p and k. Then, appealing to Theorem 2.4, we can verify the identity

$$\begin{aligned}\varphi(q^2)\varphi(q^3)\varphi(q^6)\varphi(q^{12}) &= -\frac{1}{8}\varphi(q)\varphi^3(q^3) + \frac{1}{8}\varphi^3(q)\varphi(q^3)\\ &\quad + \frac{1}{8}\varphi(-q)\varphi^2(q^3)\varphi(-q^3) - \frac{1}{8}\varphi^2(q)\varphi(-q)\varphi(-q^3)\\ &\quad + \frac{1}{2}\varphi^2(q^4)\varphi(-q^4)\varphi(-q^{12}) + \frac{1}{2}\varphi^2(q^{12})\varphi(-q^4)\varphi(-q^{12}).\end{aligned}$$

From this identity and Theorem 3.3, we deduce

$$N(2,3,6,12;n) = \begin{cases} \frac{1}{2}A(n) - \frac{1}{2}B(n), & \text{if } n \not\equiv 0 \pmod 4,\\ \frac{1}{2}A(n) - \frac{1}{2}B(n) + C\left(\frac{n}{4}\right) - D\left(\frac{n}{4}\right), & \text{if } n \equiv 0 \pmod 4.\end{cases}$$

Finally, with $n = 2^\alpha 3^\beta N$, we deduce from Theorem 3.1

$$N(2,3,6,12;n) = \begin{cases} \frac{1}{2}\left(3^\beta - (-1)^\beta\left(\frac{N}{3}\right)\right)A(N), & \text{if } n \equiv 1 \pmod 2, \\ \left(3^\beta + (-1)^\beta\left(\frac{N}{3}\right)\right)A(N), & \text{if } n \equiv 2 \pmod 4, \\ \left(2^{\alpha-1} + (-1)^{\alpha+\beta+\frac{N-1}{2}}\right) & \\ \quad \times\left(3^\beta - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right)A(N), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Acknowledgment

The fourth author was supported by the research grant A-7233 from the Natural Sciences and Engineering Research Council of Canada.

References

[1] A. Alaca, Ş. Alaca and K. S. Williams, On the two-dimensional theta functions of the Borweins, *Acta Arith.* **124** (2006) 177–195.
[2] A. Alaca, Ş. Alaca and K. S. Williams, Evaluation of the convolution sums $\sum_{l+12m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+4m=n} \sigma(l)\sigma(m)$, *Adv. Theoret. Appl. Math.* **1** (2006) 27–48.
[3] E. Benz, Über die Anzahl Darstellungen einer Zahlen n durch gewisse quaternäre quadratische Formen: Beweise, welche auf Identitäten aus dem Gebiete der Thetafunktionen basieren, Dissertation, Zürich, 1964, in *Studien zur Theorie der quadratischen Formen*, eds. B. L. van der Waerden and H. Gross (Stuttgart, 1968), pp. 165–198.
[4] J. M. Borwein, P. B. Borwein and F. G. Garvan, Some cubic modular identities of Ramanujan, *Trans. Amer. Math. Soc.* **343** (1994) 35–47.
[5] P. Demuth, Die Zahl der Darstellungen einer natürlichen Zahl durch spezielle quaternäre quadratishe Formen aufgrund der Siegelschen Massformel, in *Studien zur Theorie der quadratischen Formen*, eds. B. L. van der Waerden and H. Gross (Stuttgart, 1968), pp. 224–254.
[6] J. Liouville, Sur la forme $x^2 + y^2 + z^2 + 3t^2$, *J. Math. Pures Appl.* **8** (1863) 105–114.
[7] J. Liouville, Sur la forme $x^2 + y^2 + 2z^2 + 2zt + 2t^2$, *J. Math. Pures Appl.* **8** (1863) 115–119.
[8] J. Liouville, Sur la forme $x^2+y^2+z^2+zt+t^2$, *J. Math. Pures Appl.* **8** (1863) 120–123.
[9] J. Liouville, Sur la forme $x^2 + y^2 + 2z^2 + 6t^2$, *J. Math. Pures Appl.* **8** (1863) 124–128.
[10] J. Liouville, Sur la forme $x^2+2y^2+2z^2+3t^2$, *J. Math. Pures Appl.* **8** (1863) 129–133.
[11] J. Liouville, Sur la forme $x^2+2y^2+4z^2+6t^2$, *J. Math. Pures Appl.* **8** (1863) 134–136.
[12] J. Liouville, Sur la forme $x^2+3y^2+6z^2+6t^2$, *J. Math. Pures Appl.* **8** (1863) 209–213.
[13] J. Liouville, Sur la forme $2x^2+3y^2+3z^2+6t^2$, *J. Math. Pures Appl.* **8** (1863) 214–218.
[14] J. Liouville, Sur la forme $x^2 + 3(y^2 + z^2 + t^2)$, *J. Math. Pures Appl.* **8** (1863) 219–224.
[15] J. Liouville, Sur la forme $2x^2 + 2xy + 2y^2 + 3(z^2 + t^2)$, *J. Math. Pures Appl.* **8** (1863) 225–226.

[16] J. Liouville, Sur la forme $x^2 + xy + y^2 + 3(z^2 + t^2)$, *J. Math. Pures Appl.* **8** (1863) 227–228.

[17] K. Petr, O počtu třid forem kvadratických záporného diskriminantu, *Rozpravy Ceské Akademie Císare Frantiska Josefa I* **10** (1901) 1–22.

[18] K. Petr, Über die Klassenzahl der quadratischen Formen mit negativer Discriminante, *Bull. Internat. de l'Acad. des Sciences de Bohème (Prague)* **7** (1903) 180–187.