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FOURTEEN OCTONARY QUADRATIC FORMS

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We use the recent evaluation of certain convolution sums involving the sum of divisors function to determine the number of representations of a positive integer by certain diagonal octonary quadratic forms whose coefficients are 1, 2 or 4.

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1. Introduction

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{Z}$ the set of all integers. Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $k \in \mathbb{N}$ we set

$$\sigma_k(n) := \begin{cases} \displaystyle\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d^k, & \text{if } n \in \mathbb{N}, \\ 0, & \text{if } n \notin \mathbb{N}. \end{cases} \tag{1.1}$$

We write $\sigma(n)$ for $\sigma_1(n)$.

In this paper, we use the evaluation of the convolution sums

$$\sum_{\substack{m \in \mathbb{N} \\ m < n/k}} \sigma(m)\sigma(n-km)$$

for $k \in \{1, 2, 4, 8, 16\}$ to determine the number of representations $(x_1, \ldots, x_8) \in \mathbb{Z}^8$ of the positive integer n by each of the 14 diagonal octonary quadratic forms

$$[800] \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2,$$

$$[620] \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + 2x_7^2 + 2x_8^2,$$

$$\begin{aligned}
&[521] \quad x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+2x_6^2+2x_7^2+4x_8^2,\\
&[440] \quad x_1^2+x_2^2+x_3^2+x_4^2+2x_5^2+2x_6^2+2x_7^2+2x_8^2,\\
&[422] \quad x_1^2+x_2^2+x_3^2+x_4^2+2x_5^2+2x_6^2+4x_7^2+4x_8^2,\\
&[404] \quad x_1^2+x_2^2+x_3^2+x_4^2+4x_5^2+4x_6^2+4x_7^2+4x_8^2,\\
&[341] \quad x_1^2+x_2^2+x_3^2+2x_4^2+2x_5^2+2x_6^2+2x_7^2+4x_8^2,\\
&[323] \quad x_1^2+x_2^2+x_3^2+2x_4^2+2x_5^2+4x_6^2+4x_7^2+4x_8^2,\\
&[260] \quad x_1^2+x_2^2+2x_3^2+2x_4^2+2x_5^2+2x_6^2+2x_7^2+2x_8^2,\\
&[242] \quad x_1^2+x_2^2+2x_3^2+2x_4^2+2x_5^2+2x_6^2+4x_7^2+4x_8^2,\\
&[224] \quad x_1^2+x_2^2+2x_3^2+2x_4^2+4x_5^2+4x_6^2+4x_7^2+4x_8^2,\\
&[161] \quad x_1^2+2x_2^2+2x_3^2+2x_4^2+2x_5^2+2x_6^2+2x_7^2+4x_8^2,\\
&[143] \quad x_1^2+2x_2^2+2x_3^2+2x_4^2+2x_5^2+4x_6^2+4x_7^2+4x_8^2,\\
&[125] \quad x_1^2+2x_2^2+2x_3^2+4x_4^2+4x_5^2+4x_6^2+4x_7^2+4x_8^2.
\end{aligned}$$

(We found the adjective "octonary" in the dictionary [6, Vol. 2, p. 914].) The notation $[abc]$ ($a,b,c \in \mathbb{N}_0,\ a+b+c=8$) identifies the diagonal octonary quadratic form $A_1x_1^2+\cdots+A_8x_8^2$ as having

$$A_1 = \cdots = A_a = 1,$$

$$A_{a+1} = \cdots = A_{a+b} = 2,$$

$$A_{a+b+1} = \cdots = A_8 = 4.$$

We write $N([abc];n)$ for the number of representations of n ($\in \mathbb{N}$) by the form $[abc]$, and determine $N([abc];n)$ for the 14 forms [800], [620], [521], [440], [422], [404], [341], [323], [260], [242], [224], [161], [143] and [125]. The number $N([800];n)$ is just the number of representations of n as the sum of eight squares. Jacobi implicitly gave a formula for $N([800];n)$ in his famous work on elliptic functions [9, §§40–42, pp. 159–170]. An arithmetic proof of Jacobi's eight squares formula has been given in [11]. Williams [12, Theorem 2, p. 388] has determined $N([440];n)$ and Alaca, Alaca and Williams [2, Theorem 1.2, p. 4] have determined $N([404];n)$. The evaluation of $N([abc];n)$ for the remaining 11 forms $[abc]$ is new.

Theorem 1.1. *Let $n \in \mathbb{N}$. Then*

(i) $N([800];n) = 16\sigma_3(n) - 32\sigma_3(n/2) + 256\sigma_3(n/4)$,

(ii) $N([620];n) = 8\sigma_3(n) - 8\sigma_3(n/2) - 16\sigma_3(n/4) + 256\sigma_3(n/8) + 4c_8(n)$,

(iii) $N([521];n) = 4\sigma_3(n) - 4\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) + (2+4(\frac{-4}{n}))c_8(n) + 12c_8(n/2)$,

(iv) $N([440];n) = 4\sigma_3(n) - 4\sigma_3(n/2) - 16\sigma_3(n/4) + 256\sigma_3(n/8) + 4c_8(n)$,

(v) $N([422];n) = 2\sigma_3(n) - 2\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) + (2+4(\frac{-4}{n}))c_8(n) + 12c_8(n/2)$,

$$\begin{aligned}
\text{(vi)}\quad N([404];n) &= \sigma_3(n) + 3\sigma_3(n/2) - 68\sigma_3(n/4) + 48\sigma_3(n/8)\\
&\quad + 256\sigma_3(n/16) + (3 + 4(\tfrac{-4}{n}))c_8(n) + 12c_8(n/2),\\
\text{(vii)}\quad N([341];n) &= 2\sigma_3(n) - 2\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16)\\
&\quad + (2 + 2(\tfrac{-4}{n}))c_8(n) + 4c_8(n/2),\\
\text{(viii)}\quad N([323];n) &= \sigma_3(n) - \sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16)\\
&\quad + (2 + 3(\tfrac{-4}{n}))c_8(n) + 8c_8(n/2),\\
\text{(ix)}\quad N([260];n) &= 2\sigma_3(n) - 2\sigma_3(n/2) - 16\sigma_3(n/4) + 256\sigma_3(n/8) + 2c_8(n),\\
\text{(x)}\quad N([242];n) &= \sigma_3(n) - \sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16)\\
&\quad + (1 + 2(\tfrac{-4}{n}))c_8(n) + 4c_8(n/2),\\
\text{(xi)}\quad N([224];n) &= \tfrac{1}{2}\sigma_3(n) - \tfrac{1}{2}\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16)\\
&\quad + (\tfrac{3}{2} + 2(\tfrac{-4}{n}))c_8(n) + 4c_8(n/2),\\
\text{(xii)}\quad N([161];n) &= \sigma_3(n) - \sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16)\\
&\quad + c_8(n) + 4c_8(n/2),\\
\text{(xiii)}\quad N([143];n) &= \tfrac{1}{2}\sigma_3(n) - \tfrac{1}{2}\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16)\\
&\quad + (\tfrac{1}{2} + (\tfrac{-4}{n}))c_8(n) + 4c_8(n/2),\\
\text{(xiv)}\quad N([125];n) &= \tfrac{1}{4}\sigma_3(n) - \tfrac{1}{4}\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16)\\
&\quad + (\tfrac{3}{4} + (\tfrac{-4}{n}))c_8(n) + 2c_8(n/2),
\end{aligned}$$

where the integers $c_8(n)$ $(n \in \mathbb{N})$ *are given by*

$$\sum_{n=1}^{\infty} c_8(n)q^n = q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4, \tag{1.2}$$

and $(\frac{-4}{n})$ $(n \in \mathbb{N})$ *is the Legendre–Jacobi–Kronecker symbol for discriminant* -4, *that is*

$$\left(\frac{-4}{n}\right) = \begin{cases} +1, & \text{if } n \equiv 1 \pmod 4,\\ -1, & \text{if } n \equiv 3 \pmod 4,\\ 0, & \text{if } n \equiv 0 \pmod 2.\end{cases}$$

In [3], the quantity $N([abc];n)$ is evaluated for $[abc] = [701]$, $[602]$, $[503]$, $[404]$, $[305]$, $[206]$ and $[107]$.

2. Some Convolution Sums

Let $n \in \mathbb{N}$. The formula

$$\sum_{\substack{m \in \mathbb{N}\\ m < n}} \sigma(m)\sigma(n-m) = \frac{5}{12}\sigma_3(n) + \left(\frac{1}{12} - \frac{1}{2}n\right)\sigma(n) \tag{2.1}$$

originally appeared in a letter from Besge to Liouville [5]. Lützen [10, p. 81] indicates that Besge is a pseudonym for Liouville. Many proofs of (2.1) have been given, see for example [7]. An arithmetic proof is given in [8, p. 236]. The following two evaluations are due to Huard, Ou, Spearman and Williams [8, Theorem 2, p. 247; Theorem 4, p. 249]

$$\sum_{\substack{m \in \mathbb{N} \\ m < n/2}} \sigma(m)\sigma(n-2m) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3(n/2) + \left(\frac{1}{24} - \frac{1}{8}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n/2) \tag{2.2}$$

and

$$\sum_{\substack{m \in \mathbb{N} \\ m < n/4}} \sigma(m)\sigma(n-4m) = \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3(n/2) + \frac{1}{3}\sigma_3(n/4) + \left(\frac{1}{24} - \frac{1}{16}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n/4). \tag{2.3}$$

Recently Williams [12] has shown that

$$\sum_{\substack{m \in \mathbb{N} \\ m < n/8}} \sigma(m)\sigma(n-8m) = \frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3(n/2) + \frac{1}{16}\sigma_3(n/4) + \frac{1}{3}\sigma_3(n/8) + \left(\frac{1}{24} - \frac{1}{32}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n/8) - \frac{1}{64}c_8(n), \tag{2.4}$$

where the integers $c_8(n)$ $(n \in \mathbb{N})$ are defined in (1.2). Clearly

$$c_8(n) = 0, \quad \text{if } n \equiv 0 \pmod 2). \tag{2.5}$$

Recently Alaca, Alaca and Williams [2] have proved that

$$\sum_{\substack{m \in \mathbb{N} \\ m < n/16}} \sigma(m)\sigma(n-16m) = \frac{1}{768}\sigma_3(n) + \frac{1}{256}\sigma_3(n/2) + \frac{1}{64}\sigma_3(n/4) + \frac{1}{16}\sigma_3(n/8) + \frac{1}{3}\sigma_3(n/16) + \left(\frac{1}{24} - \frac{1}{64}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n/16) - \frac{7}{256}c_{16}(n), \tag{2.6}$$

where the rational numbers $c_{16}(n)$ $(n \in \mathbb{N})$ are defined by

$$\sum_{n=1}^{\infty} c_{16}(n)q^n = \frac{1}{32}A_1(q) + \frac{3}{112}A_2(q) + \frac{1}{224}A_3(q) - \frac{1}{32}A_5(q) - \frac{3}{112}A_6(q) - \frac{1}{224}A_7(q), \tag{2.7}$$

where

$$A_k(q) := \prod_{n=1}^{\infty} (1+q^n)^{24-4k}(1-q^n)^8(1-q^{4n-2})^{16-2k}. \tag{2.8}$$

More recently the authors [3] have proved that

$$c_{16}(n) = \left(\frac{3}{7} + \frac{4}{7}\left(\frac{-4}{n}\right)\right) c_8(n) + \frac{12}{7} c_8(n/2). \tag{2.9}$$

Thus (2.6) becomes

$$\begin{aligned}\sum_{\substack{m \in \mathbb{N} \\ m < n/16}} \sigma(m)\sigma(n-16m) &= \frac{1}{768}\sigma_3(n) + \frac{1}{256}\sigma_3(n/2) + \frac{1}{64}\sigma_3(n/4) \\ &\quad + \frac{1}{16}\sigma_3(n/8) + \frac{1}{3}\sigma_3(n/16) + \left(\frac{1}{24} - \frac{1}{64}n\right)\sigma(n) \\ &\quad + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n/16) - \left(\frac{3}{256} + \frac{1}{64}\left(\frac{-4}{n}\right)\right) c_8(n) \\ &\quad - \frac{3}{64} c_8(n/2). \end{aligned} \tag{2.10}$$

We require the following result, which is a simple consequence of (2.1)–(2.4) and (2.10).

Theorem 2.1. *Let* $n \in \mathbb{N}$. *Let* $k \in \{1, 2, 4\}$. *Let* $\beta \in \{0, 1, 2, 3, 4\}$ *and*

$$\begin{aligned} &\alpha \in \{0,1,2,3,4\}, && \text{if } k = 1, \\ &\alpha \in \{0,1,2,3\}, && \text{if } k = 2, \\ &\alpha \in \{0,1,2\}, && \text{if } k = 4. \end{aligned}$$

Then there exist 18 *rational numbers*

$$A_0, A_1, A_2, A_3, A_4, B_0, B_1, B_2, B_3, B_4, C_0, C_1, C_2, C_3, C_4, D_0, D_1, E,$$

which depend upon k, α *and* β *but not on* n, *such that*

$$\begin{aligned}\sum_{\substack{m \in \mathbb{N} \\ m < n/k}} \sigma(m/2^\alpha)\sigma((n-km)/2^\beta) &= \sum_{r=0}^{4}(A_r\sigma_3(n/2^r) + (B_r + C_r n)\sigma(n/2^r)) \\ &\quad + \left(D_0 + D_1\left(\frac{-4}{n}\right)\right) c_8(n) + E c_8(n/2). \end{aligned}$$

Proof. It suffices to treat one case as the remaining cases can be treated in a similar manner.

Suppose $k = 1$ and $(\alpha, \beta) = (1, 4)$. Then

$$\sum_{\substack{m \in \mathbb{N} \\ m < n}} \sigma(m/2)\sigma((n-m)/16)$$

$$= \sum_{\substack{m \in \mathbb{N} \\ m < n/2}} \sigma(m)\sigma((n-2m)/16)$$

$$= \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ 8l+m=n/2}} \sigma(l)\sigma(m)$$

$$= \sum_{\substack{l \in \mathbb{N} \\ l < n/16}} \sigma(l)\sigma\left(\frac{n}{2} - 8l\right)$$

$$= \frac{1}{192}\sigma_3(n/2) + \frac{1}{64}\sigma_3(n/4) + \frac{1}{16}\sigma_3(n/8) + \frac{1}{3}\sigma_3(n/16)$$

$$+ \left(\frac{1}{24} - \frac{1}{64}n\right)\sigma(n/2) + \left(\frac{1}{24} - \frac{1}{8}n\right)\sigma(n/16) - \frac{1}{64}c_8(n/2),$$

by (2.4). This is of the asserted form. □

3. Three Quaternary Forms

We define three diagonal quaternary quadratic forms $f_1,\ f_2,\ f_3$ by

$$f_1(x_1, x_2, x_3, x_4) := x_1^2 + x_2^2 + x_3^2 + x_4^2, \tag{3.1}$$

$$f_2(x_1, x_2, x_3, x_4) := x_1^2 + x_2^2 + 2x_3^2 + 2x_4^2, \tag{3.2}$$

$$f_3(x_1, x_2, x_3, x_4) := x_1^2 + 2x_2^2 + 2x_3^2 + 4x_4^2. \tag{3.3}$$

It is known that

$$N_{f_1}(n) := \mathrm{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = f_1(x_1, x_2, x_3, x_4)\}$$
$$= 8\sigma(n) - 32\sigma(n/4), \quad n \in \mathbb{N}, \tag{3.4}$$

$$N_{f_2}(n) := \mathrm{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = f_2(x_1, x_2, x_3, x_4)\}$$
$$= 4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 32\sigma(n/8), \quad n \in \mathbb{N}, \tag{3.5}$$

$$N_{f_3}(n) := \mathrm{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = f_3(x_1, x_2, x_3, x_4)\}$$
$$= 2\sigma(n) - 2\sigma(n/2) + 8\sigma(n/8) - 32\sigma(n/16), \quad n \in \mathbb{N}, \tag{3.6}$$

see for example [1, pp. 296, 297, 300]. Hence, for $n \in \mathbb{N}$ and $i \in \{1, 2, 3\}$, we have

$$N_{f_i}(n) = \sum_{\alpha=0}^{i+1} c_i(\alpha)\sigma(n/2^\alpha) \tag{3.7}$$

for rational numbers $c_i(\alpha)$, which depend on i and α but not on n.

We observe that

$$f_1(x_1,x_2,x_3,x_4)+f_1(x_5,x_6,x_7,x_8)=[800],$$
$$f_1(x_1,x_2,x_3,x_4)+f_2(x_5,x_6,x_7,x_8)=[620],$$
$$f_1(x_1,x_2,x_3,x_4)+f_3(x_5,x_6,x_7,x_8)=[521],$$
$$f_2(x_1,x_2,x_3,x_4)+f_2(x_5,x_6,x_7,x_8)=[440],$$
$$f_2(x_1,x_2,x_3,x_4)+f_3(x_5,x_6,x_7,x_8)=[341],$$
$$f_3(x_1,x_2,x_3,x_4)+f_3(x_5,x_6,x_7,x_8)=[242].$$

Also

$$f_1(x_1,x_2,x_3,x_4)+2f_2(x_5,x_6,x_7,x_8)=[422],$$
$$f_2(x_1,x_2,x_3,x_4)+2f_1(x_5,x_6,x_7,x_8)=[260],$$
$$f_3(x_1,x_2,x_3,x_4)+2f_1(x_5,x_6,x_7,x_8)=[161],$$
$$f_3(x_1,x_2,x_3,x_4)+2f_2(x_5,x_6,x_7,x_8)=[143].$$

Finally

$$f_1(x_1,x_2,x_3,x_4)+4f_1(x_5,x_6,x_7,x_8)=[404],$$
$$f_2(x_1,x_2,x_3,x_4)+4f_1(x_5,x_6,x_7,x_8)=[224],$$
$$f_3(x_1,x_2,x_3,x_4)+4f_1(x_5,x_6,x_7,x_8)=[125].$$

We note that

$$[323]\neq f_i(x_1,x_2,x_3,x_4)+rf_j(x_5,x_6,x_7,x_8),\quad i,j\in\{1,2,3\},\quad r\in\{1,2,4\}.$$

For this reason the determination of $N([323];n)$ $(n\in\mathbb{N})$ is handled separately and differently from the others. Combinations $f_i(x_1,x_2,x_3,x_4)+rf_j(x_5,x_6,x_7,x_8)$ not listed above either lead to repetitions (for example $f_1(x_1,x_2,x_3,x_4)+2f_1(x_5,x_6,x_7,x_8)=[440]$) or to parameters outside the range of applicability of Theorem 2.1 (for example the proof in Sec. 5 applied to $f_1(x_1,x_2,x_3,x_4)+2f_3(x_5,x_6,x_7,x_8)$ allows $\alpha=4$, $\beta=2$ and Theorem 2.1 with $k=2$ is not applicable).

4. The Forms [800], [620], [521], [440], [341] and [242]

For $n\in\mathbb{N}$ and $i,j\in\{1,2,3\}$ with $i\leq j$, we have by (3.7)

$$\begin{aligned}N_{f_i+f_j}(n) &:= \mathrm{card}\{(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\in\mathbb{Z}^8 \mid \\ &\qquad n=f_i(x_1,x_2,x_3,x_4)+f_j(x_5,x_6,x_7,x_8)\}\\ &= N_{f_i}(n)+N_{f_j}(n)+\sum_{m=1}^{n-1}N_{f_i}(n-m)N_{f_j}(m)\end{aligned}$$

$$= \sum_{\alpha=0}^{i+1} c_i(\alpha)\sigma(n/2^\alpha) + \sum_{\alpha=0}^{j+1} c_j(\alpha)\sigma(n/2^\alpha)$$
$$+ \sum_{\alpha=0}^{j+1}\sum_{\beta=0}^{i+1} c_j(\alpha)c_i(\beta) \sum_{m=1}^{n-1} \sigma(m/2^\alpha)\sigma((n-m)/2^\beta).$$

As $i, j \in \{1, 2, 3\}$ we have $\alpha, \beta \in \{0, 1, 2, 3, 4\}$. Thus, by Theorem 2.1 with $k = 1$, we have

$$\begin{aligned} N_{f_i+f_j}(n) &= A_0\sigma_3(n) + A_1\sigma_3(n/2) + A_2\sigma_3(n/4) + A_3\sigma_3(n/8) + A_4\sigma_3(n/16) \\ &\quad + (B_0 + C_0 n)\sigma(n) + (B_1 + C_1 n)\sigma(n/2) + (B_2 + C_2 n)\sigma(n/4) \\ &\quad + (B_3 + C_3 n)\sigma(n/8) + (B_4 + C_4 n)\sigma(n/16) \\ &\quad + \left(D_0 + D_1\left(\frac{-4}{n}\right)\right) c_8(n) + Ec_8(n/2) \end{aligned}$$

for some rational numbers $A_0, A_1, \ldots, E$ depending on i and j but not on n. Next, determining $N_{f_i+f_j}(n)$ numerically for $n = 1, 2, \ldots, 18$ for each pair (i, j) with $i, j \in \{1, 2, 3\}$ and $i \le j$, we obtain 18 linearly independent linear equations for the 18 quantities $A_0, A_1, \ldots, E$. Using MAPLE to solve these equations, we obtain the following values of $A_0, A_1, \ldots, E$. For each of the six pairs (i, j) we find

$$B_0 = B_1 = B_2 = B_3 = B_4 = C_0 = C_1 = C_2 = C_3 = C_4 = 0.$$

For $(i, j) = (1, 1)$, so that $f_i + f_j = [800]$, we find

$$A_0 = 16, \quad A_1 = -32, \quad A_2 = 256, \quad A_3 = 0, \quad A_4 = 0,$$
$$D_0 = 0, \quad D_1 = 0, \quad E = 0.$$

For $(i, j) = (1, 2)$, so that $f_i + f_j = [620]$, we find

$$A_0 = 8, \quad A_1 = -8, \quad A_2 = -16, \quad A_3 = 256, \quad A_4 = 0,$$
$$D_0 = 4, \quad D_1 = 0, \quad E = 0.$$

For $(i, j) = (1, 3)$, so that $f_i + f_j = [521]$, we find

$$A_0 = 4, \quad A_1 = -4, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 256,$$
$$D_0 = 2, \quad D_1 = 4, \quad E = 12.$$

For $(i, j) = (2, 2)$, so that $f_i + f_j = [440]$, we find

$$A_0 = 4, \quad A_1 = -4, \quad A_2 = -16, \quad A_3 = 256, \quad A_4 = 0,$$
$$D_0 = 4, \quad D_1 = 0, \quad E = 0.$$

For $(i, j) = (2, 3)$, so that $f_i + f_j = [341]$, we find

$$A_0 = 2, \quad A_1 = -2, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 256,$$
$$D_0 = 2, \quad D_1 = 2, \quad E = 4.$$

For $(i,j)=(3,3)$, so that $f_i+f_j=[242]$, we find

$$A_0=1,\quad A_1=-1,\quad A_2=0,\quad A_3=-16,\quad A_4=256,$$
$$D_0=1,\quad D_1=2,\quad E=4.$$

This completes the proof of parts (i)–(iv), (vii) and (x) of Theorem 1.1. □

5. The Forms [422], [260], [161] and [143]

For $n\in\mathbb{N}$ and $(i,j)=(1,2),\ (2,1),\ (3,1)$ and $(3,2)$, we have by (3.7)

$$\begin{aligned}N_{f_i+2f_j}(n) &:= \text{card}\{(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\in\mathbb{Z}^8 \mid \\ &\qquad n=f_i(x_1,x_2,x_3,x_4)+2f_j(x_5,x_6,x_7,x_8)\}\\ &= N_{f_i}(n)+N_{f_j}(n/2)+\sum_{\substack{m\in\mathbb{N}\\ m<n/2}} N_{f_i}(n-2m)N_{f_j}(m)\\ &= \sum_{\alpha=0}^{i+1} c_i(\alpha)\sigma(n/2^\alpha)+\sum_{\alpha=0}^{j+1} c_j(\alpha)\sigma(n/2^{\alpha+1})\\ &\quad+\sum_{\alpha=0}^{j+1}\sum_{\beta=0}^{i+1} c_j(\alpha)c_i(\beta)\sum_{\substack{m\in\mathbb{N}\\ m<n/2}}\sigma(m/2^\alpha)\sigma((n-2m)/2^\beta).\end{aligned}$$

For the pairs (i,j) under consideration we have

$$0\le\alpha\le j+1\le 3$$

and

$$0\le\beta\le i+1\le 4$$

so, by Theorem 2.1 with $k=2$, we have

$$\begin{aligned}N_{f_i+2f_j}(n) &= A_0\sigma_3(n)+A_1\sigma_3(n/2)+A_2\sigma_3(n/4)+A_3\sigma_3(n/8)+A_4\sigma_3(n/16)\\ &\quad+(B_0+C_0n)\sigma(n)+(B_1+C_1n)\sigma(n/2)+(B_2+C_2n)\sigma(n/4)\\ &\quad+(B_3+C_3n)\sigma(n/8)+(B_4+C_4n)\sigma(n/16)\\ &\quad+\left(D_0+D_1\left(\frac{-4}{n}\right)\right)c_8(n)+Ec_8(n/2).\end{aligned}$$

Next, determining $N_{f_i+2f_j}(n)$ numerically for $n=1,2,\ldots,18$ for each of the four specified pairs (i,j), we obtain 18 linearly independent linear equations for the 18 quantities $A_0,A_1,\ldots,E$. Using MAPLE to solve these equations, we find that for all four pairs (i,j)

$$B_0=B_1=B_2=B_3=B_4=C_0=C_1=C_2=C_3=C_4=0.$$

The values of $A_0, \ldots, A_4, D_0, D_1$ and E are given below. For $(i,j) = (1,2)$, so that $f_i + 2f_j = [422]$, we find

$$A_0 = 2, \quad A_1 = -2, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 256,$$
$$D_0 = 2, \quad D_1 = 4, \quad E = 12.$$

For $(i,j) = (2,1)$, so that $f_i + 2f_j = [260]$, we find

$$A_0 = 2, \quad A_1 = -2, \quad A_2 = -16, \quad A_3 = 256, \quad A_4 = 0,$$
$$D_0 = 2, \quad D_1 = 0, \quad E = 0.$$

For $(i,j) = (3,1)$, so that $f_i + 2f_j = [161]$, we find

$$A_0 = 1, \quad A_1 = -1, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 256,$$
$$D_0 = 1, \quad D_1 = 0, \quad E = 4.$$

For $(i,j) = (3,2)$, so that $f_i + 2f_j = [143]$, we find

$$A_0 = \frac{1}{2}, \quad A_1 = -\frac{1}{2}, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 256,$$
$$D_0 = \frac{1}{2}, \quad D_1 = 1, \quad E = 4.$$

This completes the proof of parts (v), (ix), (xii) and (xiii). □

6. The Forms [404], [224] and [125]

For $n \in \mathbb{N}$ and $(i,j) = (1,1),\ (2,1)$ and $(3,1)$, we have by (3.7)

$$\begin{aligned}
N_{f_i+4f_j}(n) &:= \operatorname{card}\{(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) \in \mathbb{Z}^8 \mid \\
&\qquad n = f_i(x_1,x_2,x_3,x_4) + 4f_j(x_5,x_6,x_7,x_8)\} \\
&= N_{f_i}(n) + N_{f_j}(n/4) + \sum_{\substack{m \in \mathbb{N} \\ m < n/4}} N_{f_i}(n-4m) N_{f_j}(m) \\
&= \sum_{\alpha=0}^{i+1} c_i(\alpha)\sigma(n/2^\alpha) + \sum_{\alpha=0}^{j+1} c_j(\alpha)\sigma(n/2^{\alpha+2}) \\
&\quad + \sum_{\alpha=0}^{j+1}\sum_{\beta=0}^{i+1} c_j(\alpha)c_i(\beta) \sum_{\substack{m \in \mathbb{N} \\ m < n/4}} \sigma(m/2^\alpha)\sigma((n-4m)/2^\beta).
\end{aligned}$$

For the pairs (i,j) under consideration we have

$$0 \le \alpha \le j+1 = 2$$

and

$$0 \le \beta \le i+1 \le 4$$

so, by Theorem 2.1 with $k = 4$, we have

$$\begin{aligned} N_{f_i+4f_j}(n) = & A_0\sigma_3(n) + A_1\sigma_3(n/2) + A_2\sigma_3(n/4) + A_3\sigma_3(n/8) + A_4\sigma_3(n/16) \\ & + (B_0 + C_0 n)\sigma(n) + (B_1 + C_1 n)\sigma(n/2) + (B_2 + C_2 n)\sigma(n/4) \\ & + (B_3 + C_3 n)\sigma(n/8) + (B_4 + C_4 n)\sigma(n/16) \\ & + \left(D_0 + D_1\left(\frac{-4}{n}\right)\right) c_8(n) + E c_8(n/2). \end{aligned}$$

Next, determining $N_{f_i+4f_j}(n)$ numerically for $n = 1, 2, \ldots, 18$ for each of the three specified pairs (i, j), we obtain 18 linearly independent linear equations for the 18 quantities $A_0, A_1, \ldots, E$. Using MAPLE to solve these equations, we find that for all three pairs (i, j)

$$B_0 = B_1 = B_2 = B_3 = B_4 = C_0 = C_1 = C_2 = C_3 = C_4 = 0.$$

The values of $A_0, \ldots, A_4, D_0, D_1$ and E are given below. For $(i, j) = (1, 1)$, so that $f_i + 4f_j = [404]$, we find

$$A_0 = 1, \quad A_1 = 3, \quad A_2 = -68, \quad A_3 = 48, \quad A_4 = 256,$$
$$D_0 = 3, \quad D_1 = 4, \quad E = 12.$$

For $(i, j) = (2, 1)$, so that $f_i + 4f_j = [224]$, we find

$$A_0 = \frac{1}{2}, \quad A_1 = -\frac{1}{2}, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 256,$$
$$D_0 = \frac{3}{2}, \quad D_1 = 2, \quad E = 4.$$

For $(i, j) = (3, 1)$, so that $f_i + 4f_j = [125]$, we find

$$A_0 = \frac{1}{4}, \quad A_1 = -\frac{1}{4}, \quad A_2 = 0, \quad A_3 = -16, \quad A_4 = 256,$$
$$D_0 = \frac{3}{4}, \quad D_1 = 1, \quad E = 2.$$

This completes the proof of parts (vi), (xi) and (xiv). □

7. The Form [323]

Let

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q \in \mathbb{C}, \quad |q| < 1.$$

It is well-known that [4, p. 71]

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4)$$

and [4, p. 72]

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2).$$

Hence

$$\varphi^2(q) + \left(2\varphi(q^4) - \varphi(q)\right)^2 = 2\varphi^2(q^2)$$

so that

$$\varphi^2(q) - 2\varphi(q)\varphi(q^4) - \varphi^2(q^2) + 2\varphi^2(q^4) = 0.$$

Multiplying both sides by $\varphi^r(q)\varphi^s(q^2)\varphi^{6-r-s}(q^4)$ $(r, s \in \mathbb{N}_0,\ r + s \leq 6)$, we obtain

$$\begin{aligned}&\varphi^{r+2}(q)\varphi^s(q^2)\varphi^{6-r-s}(q^4) - 2\varphi^{r+1}(q)\varphi^s(q^2)\varphi^{7-r-s}(q^4)\\ &\quad - \varphi^r(q)\varphi^{s+2}(q^2)\varphi^{6-r-s}(q^4) + 2\varphi^r(q)\varphi^s(q^2)\varphi^{8-r-s}(q^4) = 0.\end{aligned}$$

As

$$\varphi^t(q)\varphi^u(q^2)\varphi^{8-t-u}(q^4) = \sum_{n=0}^{\infty} N([t\,u\,8-t-u]; n)q^n,$$

we deduce

$$\begin{aligned}&N([r+2\,s\,6-r-s]; n) - 2N([r+1\,s\,7-r-s]; n)\\ &\quad - N([r\,s+2\,6-r-s]; n) + 2N([r\,s\,8-r-s]; n) = 0,\ n \in \mathbb{N}. \qquad (7.1)\end{aligned}$$

Taking $r = s = 2$ in (7.1), we obtain

$$N([422]; n) - 2N([323]; n) - N([242]; n) + 2N([224]; n) = 0.$$

Appealing to Theorem 1.1(v), (x), (xi), we deduce

$$\begin{aligned}N([323]; n) &= \frac{1}{2}N([422]; n) - \frac{1}{2}N([242]; n) + N([224]; n)\\ &= \sigma_3(n) - \sigma_3(n/2) - 8\sigma_3(n/8) + 128\sigma_3(n/16)\\ &\quad + \left(1 + 2\left(\frac{-4}{n}\right)\right) c_8(n) + 6c_8(n/2)\\ &\quad - \frac{1}{2}\sigma_3(n) + \frac{1}{2}\sigma_3(n/2) + 8\sigma_3(n/8) - 128\sigma_3(n/16)\\ &\quad - \frac{1}{2}\left(1 + 2\left(\frac{-4}{n}\right)\right) c_8(n) - 2c_8(n/2)\\ &\quad + \frac{1}{2}\sigma_3(n) - \frac{1}{2}\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16)\\ &\quad + \left(\frac{3}{2} + 2\left(\frac{-4}{n}\right)\right) c_8(n) + 4c_8(n/2)\\ &= \sigma_3(n) - \sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16)\\ &\quad + \left(2 + 3\left(\frac{-4}{n}\right)\right) c_8(n) + 8c_8(n/2),\end{aligned}$$

which is part (viii) of Theorem 1.1. □

The eight choices $(r, s) = (0, 2),\ (0, 4),\ (0, 6),\ (1, 2),\ (1, 4),\ (2, 4),\ (3, 2),\ (4, 2)$ in (7.1) lead to the relationships (as $N([0ab]; n) = N([ab0]; n/2)$)

$$N([224]; n) - 2N([125]; n) - N([440]; n/2) + 2N([260]; n/2) = 0,$$
$$N([242]; n) - 2N([143]; n) - N([620]; n/2) + 2N([440]; n/2) = 0,$$
$$N([260]; n) - 2N([161]; n) - N([800]; n/2) + 2N([620]; n/2) = 0,$$
$$N([323]; n) - 2N([224]; n) - N([143]; n) + 2N([125]; n) = 0,$$
$$N([341]; n) - 2N([242]; n) - N([161]; n) + 2N([143]; n) = 0,$$
$$N([440]; n) - 2N([341]; n) - N([260]; n) + 2N([242]; n) = 0,$$
$$N([521]; n) - 2N([422]; n) - N([341]; n) + 2N([323]; n) = 0,$$
$$N([620]; n) - 2N([521]; n) - N([440]; n) + 2N([422]; n) = 0.$$

These relations serve as checks on various parts of Theorem 1.1. The remaining choices of (r, s) do not lead to new determinations of $N([abc]; n)$.

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References

[1] A. Alaca, Ş. Alaca, M. F. Lemire and K. S. Williams, Nineteen quaternary quadratic forms, *Acta Arith.* **130** (2007) 277–310.

[2] A. Alaca, Ş. Alaca and K. S. Williams, The convolution sum $\sum_{m<n/16} \sigma(m)\sigma(n-16m)$, *Canad. Math. Bull.* **51** (2008) 3–14.

[3] A. Alaca, Ş. Alaca and K. S. Williams, Seven octonary quadratic forms, *Acta Arith.* **135** (2008) 1427–1432.

[4] B. C. Berndt, *Number Theory in the Spirit of Ramanujan* (Amer. Math. Soc., Providence, RI, USA, 2006).

[5] M. Besge, Extrait d'une lettre de M. Besge à M. Liouville, *J. Math. Pures Appl.* **7** (1862) 256.

[6] C. E. Funk, *New Practical Standard Dictionary of the English Language,* Vols. 1, 2 (Funk and Wagnalls Co., New York, 1953).

[7] J. W. L. Glaisher, On the square of the series in which the coefficients are the sums of the divisors of the exponents, *Mess. Math.* **14** (1885) 156–163.

[8] J. G. Huard, Z. M. Ou, B. K. Spearman and K. S. Williams, Elementary evaluation of certain convolution sums involving divisor functions, in *Number Theory for the Millennium II*, eds. M. A. Bennett, B. C. Berndt, N. Boston, H. G. Diamond, A. J. Hildebrand and W. Philipp (A. K. Peters, Natick, Massachusetts, 2002), pp. 229–274.

[9] C. G. J. Jacobi, Fundamenta nova theoriae functionum ellipticarum, 1829, in *Gesammelte Werke, Erster Band* (Chelsea Publishing Co., New York, 1969), pp. 49–239.

[10] J. Lützen, *Joseph Liouville 1809–1882: Master of Pure and Applied Mathematics* (Springer-Verlag, Berlin-Heidelberg, 1990).
[11] K. S. Williams, An arithmetic proof of Jacobi's eight squares theorem, *Far East J. Math. Sci.* **3** (2001) 1001–1005.
[12] K. S. Williams, The convolution sum $\sum_{m<n/8} \sigma(m)\sigma(n-8m)$, *Pacific J. Math.* **228** (2006) 387–396.