The Parents of Jacobi's Four Squares Theorem Are Unique

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Abstract. Jacobi's four squares theorem asserts that the number of representations of a positive integer n as a sum of four squares is 8 times the sum of the positive divisors of n, which are not multiples of 4. A formula expressing an infinite product as an infinite sum is called a product-to-sum identity. The product-to-sum identities in a single complex variable q from which Jacobi's four squares formula can be deduced by equating coefficients of q^n (the "parents") are explored using some amazing identities of Ramanujan, and are shown to be unique in a certain sense, thereby justifying the title of this article. The same is done for Legendre's four triangular numbers theorem. Finally, a general uniqueness result is proved.

1. INTRODUCTION. A formula expressing an infinite product as an infinite series is called a product-to-sum identity. Many such identities are known. Perhaps the most famous product-to-sum identity was given by Jacobi in 1829, in his monumental work on elliptic functions, *Fundamenta Nova Theoriae Functionum Ellipticarum*, which is reproduced in his *Gesammelte Werke* [**6**, Vol. I, pp. 49–239]; see page 234. Jacobi's formula, now known as Jacobi's triple product identity, asserts that

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+aq^{2n-1})(1+a^{-1}q^{2n-1}) = \sum_{n=-\infty}^{\infty} a^n q^{n^2} \tag{1}$$

for all complex numbers a and q satisfying $a \neq 0$ and $|q| < 1$. A wide variety of interesting proofs of this identity appear in the literature. We refer the reader to the easy-to-read proof given in [**4**, Theorem 352, pp. 282–283].

In this article we are particularly interested in the two product-to-sum identities (valid for $|q| < 1$)

$$\prod_{n=1}^{\infty}(1-q^n)^{-8}(1-q^{2n})^{20}(1-q^{4n})^{-8} = 1 + \sum_{n=1}^{\infty}(8\sigma(n) - 32\sigma(n/4))q^n \tag{2}$$

and

$$\prod_{n=1}^{\infty}(1-q^n)^{8}(1-q^{2n})^{-4} = 1 + \sum_{n=1}^{\infty}(-8\sigma(n) + 48\sigma(n/2) - 64\sigma(n/4))q^n, \tag{3}$$

where $\sigma(m)$ denotes the sum of the positive divisors of m if m is a positive integer, and $\sigma(m) = 0$ if m is not a positive integer but is a rational number. In Section 2 we sketch proofs of these two identities, by introducing the reader to some amazing identities due to Ramanujan. In Section 3, we show that these two identities are the "parents" of Jacobi's arithmetic formula, for the number $r_4(n)$ of representations of a positive integer n as a sum of four squares of integers (in the sense that Jacobi's formula can

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be deduced from them). Jacobi's formula is implicit in his work [**6**, Vol. I, p. 239] and states that $r_4(n)$ is 8 times the sum of the positive divisors of n that are not multiples of 4; that is,

$$\begin{aligned} r_4(n) &:= \operatorname{card}\left\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = x_1^2 + x_2^2 + x_3^2 + x_4^2\right\} \\ &= 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d = 8\sigma(n) - 32\sigma(n/4), \end{aligned} \tag{4}$$

where, as usual, $\mathbb{Z}$ denotes the set of all integers. Since the only representation of 0 as the sum of four squares of integers is $0 = 0^2 + 0^2 + 0^2 + 0^2$, we set $r_4(0) = 1$. An elementary arithmetic proof of (4), in the spirit of Liouville, is given in [**9**] and [**11**, p. 116]. Proofs in the spirit of Ramanujan are given in [**2**, pp. 59, 62]. A modern proof using modular forms is given in [**11**, pp. 266–268]. In Section 4 we give an elementary argument to show that, in a certain sense, there are no other formulas like (2) and (3), so that the "parents" of Jacobi's four squares theorem are unique; see Theorem 1. In Section 5, from Jacobi's triple product identity, we deduce a "sibling" formula to (2) and (3), namely

$$q \prod_{n=1}^{\infty} (1 - q^{2n})^{-4} (1 - q^{4n})^8 = \sum_{n=1}^{\infty} (\sigma(n) - 3\sigma(n/2) + 2\sigma(n/4)) q^n. \tag{5}$$

In Section 6 we show that (5) is the "single parent" of Legendre's arithmetic formula, for the number $t_4(n)$ of representations of a nonnegative integer n as a sum of four triangular numbers. Recall that a triangular number is a (necessarily nonnegative) integer of the form $\frac{1}{2}x(x+1)$ for some nonnegative integer x. Legendre's formula, namely

$$\begin{aligned} t_4(n) &:= \operatorname{card}\Big\{(x_1, x_2, x_3, x_4) \in \mathbb{N}_0^4 \mid n = \frac{1}{2}x_1(x_1+1) + \frac{1}{2}x_2(x_2+1) \\ &\qquad + \frac{1}{2}x_3(x_3+1) + \frac{1}{2}x_4(x_4+1)\Big\} \\ &= \sum_{d \mid 2n+1} d = \sigma(2n+1), \end{aligned} \tag{6}$$

where $\mathbb{N}_0$ denotes the set of nonnegative integers, appears in his book on elliptic functions *Traité des fonctions elliptiques* [**7**, Vol. III, pp. 133–134]. A proof using modular forms has been given by Ono, Robins, and Wahl [**8**, pp. 79–80]. Elementary arithmetic proofs are given in [**5**, pp. 259–262] and [**11**, p. 209]. In Section 7 we prove, in an elementary fashion, that there are no other formulas like (5), so that the "parent" of Legendre's four triangular numbers theorem is unique; see Theorem 2. In Section 8 we show that there are no identities of a certain type similar to (2), (3) and (5), that is, there are no further "siblings"; see Theorem 3. In Section 9 we put together the previous results to obtain a general uniqueness theorem (Theorem 4). Section 10 contains some concluding remarks and problems, as well as some ideas for further investigation.

2. PROOFS OF PRODUCT-TO-SUM FORMULAS (2) AND (3). In [**6**, Vol. I, p. 235] Jacobi introduced his famous theta functions. One of these, in the notation of Whittaker and Watson's classic text, *A Course of Modern Analysis* [**10**, p. 464], is defined for complex numbers z and q with $|q| < 1$ by

$$\theta_3(z, q) := 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nz.$$

On the other hand, Ramanujan studied the theta function defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2},$$

where a and b are complex numbers satisfying $|ab| < 1$; see, for example, [**2**, p. 6]. Our interest is in the theta function given by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{7}$$

which is a special case of Jacobi's theta function $\theta_3(z, q)$ as $\varphi(q) = \theta_3(0, q)$ and a special case of Ramanujan's theta function $f(a, b)$ as $\varphi(q) = f(q, q)$. Both Jacobi and Ramanujan determined the fundamental relations satisfied by $\varphi(q)$, namely

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \tag{8}$$

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \tag{9}$$

and

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2). \tag{10}$$

Taking $a = 1$ in Jacobi's triple product identity (1), and noting that

$$\begin{aligned}
\prod_{n=1}^{\infty}(1 + q^{2n-1}) &= \prod_{n=1}^{\infty} \frac{(1 + q^{2n-1})(1 + q^{2n})}{(1 + q^{2n})} \\
&= \prod_{n=1}^{\infty} \frac{(1 + q^n)}{(1 + q^{2n})} \\
&= \prod_{n=1}^{\infty} \frac{(1 - q^{2n})/(1 - q^n)}{(1 - q^{4n})/(1 - q^{2n})} \\
&= \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^n)(1 - q^{4n})},
\end{aligned}$$

we deduce from (1) and (7) that

$$\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^5}{(1 - q^n)^2(1 - q^{4n})^2} = \sum_{n=-\infty}^{\infty} q^{n^2} = \varphi(q). \tag{11}$$

Formula (11) expresses $\varphi(q)$ as an infinite product and ensures that $\varphi(q) \neq 0$ for $|q| < 1$. Following Berndt [**2**, p. 120], it is convenient to introduce the notation

$$x = x(q) := 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}, \quad z = z(q) := \varphi^2(q). \tag{12}$$

Making use of the basic properties of $\varphi(q)$ given in (8)–(10), we can determine how $x(q)$ and $z(q)$ transform under the mapping $q \to q^2$. We find

$$x(q^2) = \left(\frac{1-(1-x)^{1/2}}{1+(1-x)^{1/2}}\right)^2, \quad \text{and} \quad z(q^2) = \frac{1}{2}(1+(1-x)^{1/2})z. \tag{13}$$

The second formula in (13) follows easily from (9) and (12) as

$$\begin{aligned} z(q^2) &= \varphi^2(q^2) \\ &= \frac{1}{2}(\varphi^2(q) + \varphi^2(-q)) \\ &= \frac{1}{2}\left(1 + \frac{\varphi^2(-q)}{\varphi^2(q)}\right)\varphi^2(q) \\ &= \frac{1}{2}(1+(1-x)^{1/2})z. \end{aligned}$$

The first formula in (13) can be proved similarly. The transformation (13) is due to Jacobi and is known as Jacobi's duplication principle. Ramanujan was the first mathematician to investigate the relationship between the theta function $\varphi(q)$ and Eisenstein series. We now describe this connection. The classical Eisenstein series $E_{2k}(q)$ is defined for a positive integer k by the Lambert series

$$E_{2k}(q) := 1 - \frac{4k}{B_{2k}} \sum_{m=1}^{\infty} \frac{m^{2k-1} q^m}{1-q^m},$$

where B_{2k} is the $2k$-th Bernoulli number; see, for example, [**2**, p. 87]. When $k = 1$, we have

$$E_2(q) = 1 - 24 \sum_{m=1}^{\infty} \frac{mq^m}{1-q^m} = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n, \tag{14}$$

as $B_2 = \frac{1}{6}$. Ramanujan's deep insight allowed him to express $E_2(q)$ in terms of the theta function $\varphi(q)$, namely

$$E_2(q) = (1-5x)z^2 + 12x(1-x)z\frac{dz}{dx}; \tag{15}$$

see [**2**, p. 125]. Applying the duplication principle to (15), we obtain (after some calculation)

$$E_2(q^2) = (1-2x)z^2 + 6x(1-x)z\frac{dz}{dx}; \tag{16}$$

see [**2**, p. 125]. Applying the duplication principle again, this time to (16), we deduce

$$E_2(q^4) = \left(1 - \frac{5}{4}x\right)z^2 + 3x(1-x)z\frac{dz}{dx}; \tag{17}$$

see [**2**, p. 128]. The reader interested in the details of these calculations should consult [**3**, p. 60]. We are now ready to prove (2). We have

$$\prod_{n=1}^{\infty}(1-q^n)^{-8}(1-q^{2n})^{20}(1-q^{4n})^{-8}$$

$$\begin{aligned}
&= \varphi^4(q) \qquad \text{(by (11))} \\
&= z^2 \qquad \text{(by (12))} \\
&= \frac{4}{3}\left((1-\frac{5}{4}x)z^2 + 3x(1-x)z\frac{dz}{dx}\right) - \frac{1}{3}\left((1-5x)z^2 + 12x(1-x)z\frac{dz}{dx}\right) \\
&= \frac{4}{3}E_2(q^4) - \frac{1}{3}E_2(q) \qquad \text{(by (15) and (17))} \\
&= 1 + 8\sum_{n=1}^{\infty}\sigma(n)q^n - 32\sum_{n=1}^{\infty}\sigma(n)q^{4n} \qquad \text{(by (14))} \\
&= 1 + \sum_{n=1}^{\infty}(8\sigma(n) - 32\sigma(n/4))q^n,
\end{aligned}$$

which is (2).

The product-to-sum formula (3) can be proved in exactly the same way as (2). The only difference is that, instead of working with $\varphi(q)$, we use

$$\varphi(-q) = \sum_{n=-\infty}^{\infty}(-q)^{n^2} = \prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^{2n})}, \tag{18}$$

where the infinite product in (18) comes from taking $a = -1$ in Jacobi's triple product identity (1). However, it is easier just to map $q \to -q$ in (2). The relation (3) then follows from (2) in view of the easily-proved relations

$$\prod_{n=1}^{\infty}(1-(-q)^n) = \prod_{n=1}^{\infty}(1-q^n)^{-1}(1-q^{2n})^3(1-q^{4n})^{-1} \tag{19}$$

and for a positive integer n

$$(-1)^n(8\sigma(n) - 32\sigma(n/4)) = -8\sigma(n) + 48\sigma(n/2) - 64\sigma(n/4). \tag{20}$$

The "parents" of (4) are thus in a relationship.

3. DEDUCTION OF JACOBI'S FOUR SQUARES THEOREM FROM THE PRODUCT-TO-SUM FORMULAS (2) AND (3). Jacobi's four squares theorem (equation (4)) follows from the product-to-sum identity (2) by equating coefficients of q^n, where n is a positive integer, as

$$\begin{aligned}
1 + \sum_{n=1}^{\infty} r_4(n)q^n = \sum_{n=0}^{\infty} r_4(n)q^n &= \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^4 \\
&= \prod_{n=1}^{\infty}(1-q^n)^{-8}(1-q^{2n})^{20}(1-q^{4n})^{-8} \qquad \text{(by (11))} \\
&= 1 + \sum_{n=1}^{\infty}(8\sigma(n) - 32\sigma(n/4))q^n \qquad \text{(by (2))}
\end{aligned}$$

so that

$$r_4(n) = 8\sigma(n) - 32\sigma(n/4) = 8\sum_{\substack{d|n \\ 4\nmid d}} d. \qquad (21)$$

Equation (21) is the required formula (4). Similarly, Jacobi's four squares theorem follows from the product-to-sum identity (3) by equating coefficients of $(-q)^n$, where n is a positive integer, as

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} r_4(n)(-q)^n &= \sum_{n=0}^{\infty} r_4(n)(-q)^n = \left(\sum_{n=-\infty}^{\infty} (-q)^{n^2}\right)^4 \\ &= \prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^{-4} \qquad \text{(by (18))} \\ &= 1 + \sum_{n=1}^{\infty}(-8\sigma(n) + 48\sigma(n/2) - 64\sigma(n/4))q^n \qquad \text{(by (3))} \\ &= 1 + \sum_{n=1}^{\infty}(8\sigma(n) - 32\sigma(n/4))(-q)^n \qquad \text{(by (20))}. \end{aligned}$$

4. PARENTS OF JACOBI'S FOUR SQUARES THEOREM ARE UNIQUE. The natural question arises: "Are there any more product-to-sum identities like (2) and (3)?" We show that, in a certain sense, there are no others. We prove that the "parents" of Jacobi's four squares theorem (formulas (2) and (3)) are unique in the sense that there are no other product-to-sum formulas of the type

$$\prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c = 1 + \sum_{n=1}^{\infty}(x\sigma(n) + y\sigma(n/2) + z\sigma(n/4))q^n,$$

for integers a, b, c, x, y, z with $(a, b, c) \neq (0, 0, 0)$. We exclude $(a, b, c) = (0, 0, 0)$ as this possibility gives only the trivial solution $(a, b, c, x, y, z) = (0, 0, 0, 0, 0, 0)$. We prove the following.

Theorem 1. *If a, b, c, x, y, z are integers with $(a, b, c) \neq (0, 0, 0)$ such that*

$$\prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c = 1 + \sum_{n=1}^{\infty}(x\sigma(n) + y\sigma(n/2) + z\sigma(n/4))q^n \qquad (22)$$

holds, then

$$(a, b, c, x, y, z) = (-8, 20, -8, 8, 0, -32) \quad \textit{or} \quad (8, -4, 0, -8, 48, -64).$$

Proof. Using MAPLE, we find that the left-hand side of (22) is

$$\begin{aligned} 1 - aq &+ \left(\frac{1}{2}a^2 - \frac{3}{2}a - b\right)q^2 + \left(-\frac{1}{6}a^3 + \frac{3}{2}a^2 - \frac{4}{3}a + ab\right)q^3 \\ &+ \left(\frac{1}{24}a^4 - \frac{3}{4}a^3 + \frac{59}{24}a^2 - \frac{7}{4}a - \frac{1}{2}a^2b + \frac{3}{2}ab + \frac{1}{2}b^2 - \frac{3}{2}b - c\right)q^4 \end{aligned}$$

$$+\left(-\frac{1}{120}a^5+\frac{1}{4}a^4-\frac{43}{24}a^3+\frac{15}{4}a^2-\frac{6}{5}a+\frac{1}{6}a^3b-\frac{3}{2}a^2b\right.$$
$$\left.+\frac{17}{6}ab-\frac{1}{2}ab^2+ac\right)q^5$$
$$+\left(\frac{1}{720}a^6-\frac{1}{16}a^5+\frac{113}{144}a^4-\frac{55}{16}a^3+\frac{1697}{360}a^2-2a-\frac{1}{24}a^4b\right.$$
$$+\frac{3}{4}a^3b-\frac{77}{24}a^2b+4ab+\frac{1}{4}a^2b^2-\frac{3}{4}ab^2-\frac{1}{6}b^3+\frac{3}{2}b^2$$
$$\left.-\frac{4}{3}b-\frac{1}{2}a^2c+\frac{3}{2}ac+bc\right)q^6$$
$$+\left(-\frac{1}{5040}a^7+\frac{1}{80}a^6-\frac{35}{144}a^5+\frac{89}{48}a^4-\frac{2021}{360}a^3+\frac{92}{15}a^2-\frac{8}{7}a+\frac{1}{120}a^5b\right.$$
$$-\frac{1}{4}a^4b+\frac{49}{24}a^3b-6a^2b+\frac{68}{15}ab-\frac{1}{12}a^3b^2+\frac{3}{4}a^2b^2-\frac{13}{6}ab^2+\frac{1}{6}ab^3$$
$$\left.+\frac{1}{6}a^3c-\frac{3}{2}a^2c+\frac{4}{3}ac-abc\right)q^7+\cdots.$$

The right-hand side of (22) is

$$1+xq+(3x+y)q^2+4xq^3+(7x+3y+z)q^4+6xq^5$$
$$+(12x+4y)q^6+8xq^7+\cdots.$$

Equating coefficients of q, q^3, q^5, and q^7, we obtain

$$-a=x, \tag{23}$$

$$-\frac{1}{6}a^3+\frac{3}{2}a^2-\frac{4}{3}a+ab=4x, \tag{24}$$

$$-\frac{1}{120}a^5+\frac{1}{4}a^4-\frac{43}{24}a^3+\frac{15}{4}a^2-\frac{6}{5}a+\frac{1}{6}a^3b-\frac{3}{2}a^2b+\frac{17}{6}ab-\frac{1}{2}ab^2+ac=6x, \tag{25}$$

and

$$-\frac{1}{5040}a^7+\frac{1}{80}a^6-\frac{35}{144}a^5+\frac{89}{48}a^4-\frac{2021}{360}a^3+\frac{92}{15}a^2-\frac{8}{7}a+\frac{1}{120}a^5b$$
$$-\frac{1}{4}a^4b+\frac{49}{24}a^3b-6a^2b+\frac{68}{15}ab-\frac{1}{12}a^3b^2+\frac{3}{4}a^2b^2-\frac{13}{6}ab^2+\frac{1}{6}ab^3$$
$$+\frac{1}{6}a^3c-\frac{3}{2}a^2c+\frac{4}{3}ac-abc=8x. \tag{26}$$

Assume now that $a \neq 0$. The possibility that $a = 0$ will be treated at the end of the proof. From (23) and (24), we deduce that

$$b=\frac{1}{6}a^2-\frac{3}{2}a-\frac{8}{3}. \tag{27}$$

From (23), (25), and (27), we obtain

$$c=-\frac{1}{180}a^4+\frac{7}{36}a^2+\frac{1}{2}a+\frac{284}{45}. \tag{28}$$

Using (23), (27), and (28) in (26), we deduce that

$$-\frac{18584}{2835}a - \frac{64}{135}a^3 - \frac{1}{2835}a^7 + \frac{4}{135}a^5 = -8a$$

so that

$$a^7 - 84a^5 + 1344a^3 - 4096a = 0.$$

That is,

$$a(a-2)(a+2)(a-4)(a+4)(a-8)(a+8) = 0.$$

Hence, as we are assuming here that $a \neq 0$, we have

$$a = 2, -2, 4, -4, 8, \text{ or } -8. \tag{29}$$

Equating coefficients of q^2 in (22), we obtain

$$\frac{1}{2}a^2 - \frac{3}{2}a - b = 3x + y.$$

Appealing to (23) and (27), we deduce that

$$y = \frac{1}{3}a^2 + 3a + \frac{8}{3}. \tag{30}$$

Equating coefficients of q^4 in (22), we obtain

$$\frac{1}{24}a^4 - \frac{3}{4}a^3 + \frac{59}{24}a^2 - \frac{7}{4}a - \frac{1}{2}a^2b + \frac{3}{2}ab + \frac{1}{2}b^2 - \frac{3}{2}b - c = 7x + 3y + z.$$

Appealing to (23), (27), (28), and (30), we have

$$z = -\frac{1}{45}a^4 + \frac{7}{9}a^2 - 2a - \frac{304}{45}. \tag{31}$$

From (29), (27), (28), (23), (30), and (31), we deduce that the only possible values of (a, b, c, x, y, z) are

$$\begin{aligned}(a, b, c, x, y, z) = {} & (2, -5, 8, -2, 10, -8),\ (-2, 1, 6, 2, -2, 0),\\ & (4, -6, 10, -4, 20, -8),\ (-4, 6, 6, 4, -4, 8),\\ & (8, -4, 0, -8, 48, -64),\ (-8, 20, -8, 8, 0, -32).\end{aligned}$$

For each of these six possibilities we determine the coefficients of q^8 in

$$X := \prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c$$

and

$$Y := 1 + \sum_{n=1}^{\infty} (x\sigma(n) + y\sigma(n/2) + z\sigma(n/4))q^n.$$

We obtain the following table.

(a, b, c, x, y, z)	Coefficient of q^8 in X	Coefficient of q^8 in Y
$(2, -5, 8, -2, 10, -8)$	25	16
$(-2, 1, 6, 2, -2, 0)$	25	16
$(4, -6, 10, -4, 20, -8)$	20	56
$(-4, 6, 6, 4, -4, 8)$	20	56
$(8, -4, 0, -8, 48, -64)$	24	24
$(-8, 20, -8, 8, 0, -32)$	24	24

Thus $(a, b, c, x, y, z) = (8, -4, 0, -8, 48, -64)$ and $(-8, 20, -8, 8, 0, -32)$ are the only two viable possibilities, because they are the product-to-sum formulae (3) and (2) respectively proved in Section 2.

We now turn to the case $a = 0$. Assume that there is a formula of the type (22) with $a = 0$, that is,

$$\prod_{n=1}^{\infty} (1 - q^{2n})^b (1 - q^{4n})^c = 1 + \sum_{n=1}^{\infty} (x\sigma(n) + y\sigma(n/2) + z\sigma(n/4))q^n.$$

Equating the coefficients of q on both sides, we deduce that $x = 0$, so that

$$\begin{aligned}\prod_{n=1}^{\infty} (1 - q^{2n})^b (1 - q^{4n})^c &= 1 + \sum_{n=1}^{\infty} (y\sigma(n/2) + z\sigma(n/4))q^n \\ &= 1 + \sum_{n=1}^{\infty} (y\sigma(n) + z\sigma(n/2))q^{2n}.\end{aligned}$$

Replacing q^2 by q, we deduce that

$$\prod_{n=1}^{\infty} (1 - q^n)^b (1 - q^{2n})^c = 1 + \sum_{n=1}^{\infty} (y\sigma(n) + z\sigma(n/2))q^n.$$

If $b \neq 0$, then this falls under the case treated at the outset and we see that there are no solutions. If $b = 0$, then (as $(a, b, c) \neq (0, 0, 0)$) we have $c \neq 0$ and

$$\prod_{n=1}^{\infty} (1 - q^{2n})^c = 1 + \sum_{n=1}^{\infty} (y\sigma(n) + z\sigma(n/2))q^n.$$

Equating coefficients of q, we deduce that $y = 0$, so

$$\prod_{n=1}^{\infty} (1 - q^{2n})^c = 1 + \sum_{n=1}^{\infty} z\sigma(n/2)q^n = 1 + z\sum_{n=1}^{\infty} \sigma(n)q^{2n}.$$

Replacing q^2 by q, we obtain

$$\prod_{n=1}^{\infty}(1-q^n)^c = 1 + z\sum_{n=1}^{\infty}\sigma(n)q^n.$$

As $c \neq 0$, this again is covered by the first treated case and there are no solutions.

This completes the proof of Theorem 1. ■

5. A SIBLING TO (2) AND (3). Replacing both q and a by $q^{1/2}$ in Jacobi's triple product identity (1), we obtain

$$\prod_{n=1}^{\infty}(1-q^n)(1+q^n)(1+q^{n-1}) = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2}. \tag{32}$$

Since

$$\prod_{n=1}^{\infty}(1+q^{n-1}) = 2\prod_{n=1}^{\infty}(1+q^n), \quad \prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^n)},$$

and

$$\sum_{n=-\infty}^{\infty} q^{n(n+1)/2} = 2\sum_{n=0}^{\infty} q^{n(n+1)/2},$$

we deduce from (32) that

$$\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2}{(1-q^n)} = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{33}$$

Replacing q by q^8 in (33), and then completing the square in the exponent of q, we obtain

$$q\prod_{n=1}^{\infty}\frac{(1-q^{16n})^2}{(1-q^{8n})} = \sum_{n=0}^{\infty} q^{(2n+1)^2} = \frac{1}{2}\sum_{n=-\infty}^{\infty} q^{(2n+1)^2}. \tag{34}$$

Now, by (7) and (8), we have

$$\sum_{n=-\infty}^{\infty} q^{(2n+1)^2} = \sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{(2n)^2} = \varphi(q) - \varphi(q^4) = \frac{1}{2}(\varphi(q) - \varphi(-q))$$

so that

$$\varphi(q) - \varphi(-q) = 4q\prod_{n=1}^{\infty}\frac{(1-q^{16n})^2}{(1-q^{8n})}. \tag{35}$$

Next, by (8) and (11), we have

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4) = 2\prod_{n=1}^{\infty}\frac{(1-q^{8n})^5}{(1-q^{4n})^2(1-q^{16n})^2} \tag{36}$$

and, by (9) and (11),

$$\varphi^2(q)+\varphi^2(-q)=2\varphi^2(q^2)=2\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{10}}{(1-q^{2n})^4(1-q^{8n})^4}. \tag{37}$$

Multiplying (35), (36), and (37) together, we obtain

$$\varphi^4(q)-\varphi^4(-q)=16q\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{8}}{(1-q^{2n})^4}. \tag{38}$$

From (12) and (38) we have

$$xz^2=\left(1-\frac{\varphi^4(-q)}{\varphi^4(q)}\right)\varphi^4(q)=\varphi^4(q)-\varphi^4(-q)=16q\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{8}}{(1-q^{2n})^4}.$$

Now, by (15), (16), (17), and (14), we have

$$\begin{aligned}xz^2&=-\frac{2}{3}\left((1-5x)z^2+12x(1-x)z\frac{dz}{dx}\right)\\&\quad+2\left((1-2x)z^2+6x(1-x)z\frac{dz}{dx}\right)\\&\quad-\frac{4}{3}\left((1-\frac{5}{4}x)z^2+3x(1-x)z\frac{dz}{dx}\right)\\&=-\frac{2}{3}E_2(q)+2E_2(q^2)-\frac{4}{3}E_2(q^4)\\&=16\sum_{n=1}^{\infty}(\sigma(n)-3\sigma(n/2)+2\sigma(n/4))q^n\end{aligned}$$

so that

$$q\prod_{n=1}^{\infty}(1-q^{2n})^{-4}(1-q^{4n})^8=\sum_{n=1}^{\infty}(\sigma(n)-3\sigma(n/2)+2\sigma(n/4))q^n,$$

which is formula (5). Thus formula (5) is the "uncle" of Jacobi's four squares theorem. It is very similar to both the "parents" of Jacobi's four squares theorem (identities (2) and (3)) except that it has a "q" on the left-hand side and no constant term on the right-hand side. Perhaps it too is a "parent" of an arithmetic formula. Indeed, we show in Section 6 that it is the "parent" of Legendre's four triangular numbers theorem (identity (6)). Formula (5) is a "single parent" of Legendre's four triangular numbers theorem, as mapping q to $-q$ in (5) does not yield a new formula as

$$\sigma(n)-3\sigma(n/2)+2\sigma(n/4)=0 \text{ for } n \text{ even.}$$

6. DEDUCTION OF LEGENDRE'S FOUR TRIANGULAR NUMBERS THEOREM FROM FORMULA (5). We prove Legendre's four triangular numbers theorem (identity (6)) by deducing it from the product-to-series formula (5), and so (5) is the "parent" of (6). We have by (33) and (5) that

$$\begin{aligned}\sum_{n=0}^{\infty} t_4(n) q^{2n+1} &= q \sum_{n=0}^{\infty} t_4(n) q^{2n} \\ &= q \left(\sum_{x=0}^{\infty} (q^2)^{x(x+1)/2} \right)^4 \\ &= q \prod_{n=1}^{\infty} \frac{(1-q^{4n})^8}{(1-q^{2n})^4} \\ &= \sum_{n=1}^{\infty} (\sigma(n) - 3\sigma(n/2) + 2\sigma(n/4)) q^n \\ &= \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sigma(n) q^n \\ &= \sum_{n=0}^{\infty} \sigma(2n+1) q^{2n+1}.\end{aligned}$$

Equating coefficients of q^{2n+1}, we obtain

$$t_4(n) = \sigma(2n+1), \quad n = 0, 1, 2, \ldots,$$

which is (6).

7. THE PARENT OF LEGENDRE'S FOUR TRIANGULAR NUMBERS THEOREM IS UNIQUE. We show that the "parent" of Legendre's four triangular numbers theorem (identity (5)) is unique in the sense that there are no other product-to-sum formulas of the type

$$q \prod_{n=1}^{\infty} (1-q^n)^a (1-q^{2n})^b (1-q^{4n})^c = \sum_{n=1}^{\infty} (x\sigma(n) + y\sigma(n/2) + z\sigma(n/4)) q^n$$

for integers a, b, c, x, y, z. The proof is similar to that of Theorem 1, so we abbreviate the details.

Theorem 2. *If a, b, c, x, y, z are integers such that*

$$q \prod_{n=1}^{\infty} (1-q^n)^a (1-q^{2n})^b (1-q^{4n})^c = \sum_{n=1}^{\infty} (x\sigma(n) + y\sigma(n/2) + z\sigma(n/4)) q^n \quad (39)$$

holds, then

$$(a, b, c, x, y, z) = (0, -4, 8, 1, -3, 2).$$

Proof. Using MAPLE, we find that the left hand side of (39) is

$$q - aq^2 + \left(\frac{1}{2}a^2 - \frac{3}{2}a - b \right) q^3 + \left(-\frac{1}{6}a^3 + \frac{3}{2}a^2 - \frac{4}{3}a + ab \right) q^4$$

$$+\left(\frac{1}{24}a^4-\frac{3}{4}a^3+\frac{59}{24}a^2-\frac{7}{4}a-\frac{1}{2}a^2b+\frac{3}{2}ab+\frac{1}{2}b^2-\frac{3}{2}b-c\right)q^5$$
$$+\left(-\frac{1}{120}a^5+\frac{1}{4}a^4-\frac{43}{24}a^3+\frac{15}{4}a^2-\frac{6}{5}a+\frac{1}{6}a^3b-\frac{3}{2}a^2b\right.$$
$$\left.+\frac{17}{6}ab-\frac{1}{2}ab^2+ac\right)q^6$$
$$+\left(\frac{1}{720}a^6-\frac{1}{16}a^5+\frac{113}{144}a^4-\frac{55}{16}a^3+\frac{1697}{360}a^2-2a-\frac{1}{24}a^4b+\frac{3}{4}a^3b-\frac{77}{24}a^2b\right.$$
$$\left.+4ab+\frac{1}{4}a^2b^2-\frac{3}{4}ab^2-\frac{1}{6}b^3+\frac{3}{2}b^2-\frac{4}{3}b-\frac{1}{2}a^2c+\frac{3}{2}ac+bc\right)q^7+\cdots.$$

The right-hand side of (39) is

$$xq+(3x+y)q^2+4xq^3+(7x+3y+z)q^4+6xq^5+(12x+4y)q^6+8xq^7+\cdots.$$

Equating coefficients of q, q^3, and q^5, we obtain

$$x=1, \tag{40}$$
$$b=\frac{1}{2}a^2-\frac{3}{2}a-4, \tag{41}$$

and

$$c=-\frac{1}{12}a^4+\frac{7}{12}a^2+\frac{1}{2}a+8. \tag{42}$$

Equating coefficients of q^7, we obtain the equation $a^6-20a^4+64a^2=0$ so that

$$a=0, 2, -2, 4, \text{ or } -4. \tag{43}$$

From the coefficients of q^2 we deduce

$$y=-a-3. \tag{44}$$

From the coefficients of q^4 we obtain

$$z=\frac{1}{3}a^3-\frac{7}{3}a+2. \tag{45}$$

Hence, we have the following possibilities:

$$\begin{aligned}(a,b,c,x,y,z)=&(0,-4,8,1,-3,2),\ (2,-5,10,1,-5,0),\\&(-2,1,8,1,-1,4),\ (4,-2,-2,1,-7,-14),\\&(-4,10,-6,1,1,-10).\end{aligned} \tag{46}$$

For each of these five possibilities we determine the coefficients of q^8 in

$$U:=q\prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c$$

and

$$V := \sum_{n=1}^{\infty}(x\sigma(n) + y\sigma(n/2) + z\sigma(n/4))q^n.$$

We obtain the folowing table.

(a, b, c, x, y, z)	Coefficient of q^8 in U	Coefficient of q^8 in V
$(0, -4, 8, 1, -3, 2)$	0	0
$(2, -5, 10, 1, -5, 0)$	0	-20
$(-2, 1, 8, 1, -1, 4)$	0	20
$(4, -2, -2, 1, -7, 14)$	0	8
$(-4, 10, -6, 1, 1, -10)$	0	-8

Thus $(a, b, c, x, y, z) = (0, -4, 8, 1, -3, 2)$ is the only viable possibility. It is valid, as it is the identity (5), which was proved in Section 5.

This completes the proof of Theorem 2. ■

8. A NON-EXISTENT IDENTITY. We now seek identities of the form (39), with the q on the left-hand side replaced by q^r with $r \geq 2$.

Theorem 3. *There are no integers r, a, b, c, x, y, z with $r \geq 2$ such that*

$$q^r \prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c = \sum_{n=1}^{\infty}(x\sigma(n) + y\sigma(n/2) + z\sigma(n/4))q^n. \quad (47)$$

Proof. Suppose there exist integers r, a, b, c, x, y, and z with $r \geq 2$ such that an identity of the type (47) exists. Then we have

$$\begin{aligned} xq &+ (3x+y)q^2 + 4xq^3 + (7x+3y+z)q^4 + \cdots \\ &= \sum_{n=1}^{\infty}(x\sigma(n) + y\sigma(n/2) + z\sigma(n/4))q^n \\ &= q^r \prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c \\ &= q^r - aq^{r+1} + \left(\frac{1}{2}a^2 - \frac{3}{2}a - b\right)q^{r+2} + \cdots . \end{aligned}$$

Thus, if $r \geq 5$, we must have

$$x = 3x + y = 4x = 7x + 3y + z = 0$$

so that

$$x = y = z = 0.$$

Hence, the right-hand side of (47) is 0, a contradiction. Thus we have only to examine $r = 2$, 3, and 4.

If $r = 2$, then we have

$$q^2 - aq^3 + \cdots = xq + (3x + y)q^2 + 4xq^3 + \cdots ,$$

so that

$$x = 0, \quad y = 1, \quad a = 0.$$

Thus (47) takes the form

$$q^2 \prod_{n=1}^{\infty}(1 - q^{2n})^b(1 - q^{4n})^c = \sum_{n=1}^{\infty}(\sigma(n/2) + z\sigma(n/4))q^n = \sum_{n=1}^{\infty}(\sigma(n) + z\sigma(n/2))q^{2n}.$$

Replacing q^2 by q, we obtain

$$q \prod_{n=1}^{\infty}(1 - q^n)^b(1 - q^{2n})^c = \sum_{n=1}^{\infty}(\sigma(n) + z\sigma(n/2))q^n.$$

By Theorem 2 we see that this cannot occur.

If $r = 3$, then we have

$$q^3 - \cdots = xq + (3x + y)q^2 + 4xq^3 + \cdots$$

so that

$$x = 0, \quad y = 0, \quad 4x = 1,$$

which is clearly impossible.

If $r = 4$, then we have

$$q^4 - aq^5 + \left(\frac{1}{2}a^2 - \frac{3}{2}a - b\right) q^6 + \cdots = xq + (3x + y)q^2 + 4xq^3 + (7x + 3y + z)q^4 + 6xq^5 + (12x + 4y)q^6 + \cdots$$

so that

$$x = 0, \quad y = 0, \quad z = 1, \quad a = 0, \quad b = 0.$$

Thus (47) takes the form

$$q^4 \prod_{n=1}^{\infty}(1 - q^{4n})^c = \sum_{n=1}^{\infty}\sigma(n/4)q^n = \sum_{n=1}^{\infty}\sigma(n)q^{4n}.$$

Replacing q^4 by q, we deduce

$$q \prod_{n=1}^{\infty}(1 - q^n)^c = \sum_{n=1}^{\infty}\sigma(n)q^n.$$

By Theorem 2 this cannot occur.

This completes the proof of Theorem 3. ■

9. A UNIQUENESS THEOREM. Putting together Theorems 1, 2, and 3, we obtain the following uniqueness theorem.

Theorem 4. *Let r, u, a, b, c, x, y, and z be integers with $r \geq 0$ such that*

$$q^r \prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b(1-q^{4n})^c = u + \sum_{n=1}^{\infty}(x\sigma(n) + y\sigma(n/2) + z\sigma(n/4))q^n.$$

Then

$$\begin{aligned}(r,u,a,b,c,x,y,z) = {} & (0,1,0,0,0,0,0,0),\ (0,1,-8,20,-8,8,0,-32),\\ & (0,1,8,-4,0,-8,48,-64),\\ \text{or}\quad & (1,0,0,-4,8,1,-3,2).\end{aligned}$$

Proof. This theorem follows immediately from Theorems 1, 2, and 3, on noting that we have included the trivial case $(a, b, c) = (0, 0, 0)$, which was excluded in Theorem 1, and that putting $q = 0$ in the identity gives $u = 1$ if $r = 0$ and $u = 0$ if $r \geq 1$.

This completes the proof of Theorem 4. ■

10. CONCLUDING REMARKS. The integers $1, 2, 4$ occurring in Theorem 4 are precisely the (positive) divisors of 4. If they are replaced by the divisors of another positive integer h, then there may well be many product-to-sum formulas of the type

$$q^r \prod_{n=1}^{\infty}\prod_{d|h}(1-q^{dn})^{a_d} = u + \sum_{n=1}^{\infty}\sum_{d|h} b_d\sigma(n/d)q^n \tag{48}$$

with integers $r(\geq 0), u, a_d$, and b_d. The author has found 126 such formulas when $h = 12$ (and there could be more!); see [**12**]. Two examples are

$$\prod_{n=1}^{\infty}(1-q^n)^{-2}(1-q^{2n})^2(1-q^{3n})^{-2}(1-q^{4n})^4(1-q^{6n})^6(1-q^{12n})^{-4}$$
$$= 1 + \sum_{n=1}^{\infty}(2\sigma(n) - 3\sigma(n/2) + 4\sigma(n/4) + 9\sigma(n/6) - 36\sigma(n/12))q^n$$

and

$$q^3\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})^{-1}(1-q^{3n})^{-3}(1-q^{4n})^{-2}(1-q^{6n})^3(1-q^{12n})^6$$
$$= \sum_{n=1}^{\infty}(\sigma(n/3) - \sigma(n/4) - \sigma(n/6) + \sigma(n/12))q^n.$$

It appears to be quite complicated to use the elementary method of this article to determine the number of such product-to-sum formulas of the type (48), even with $h = 12$. Since the product $\prod_{n=1}^{\infty}(1-q^n)$ is intimately related to the Dedekind eta function, which is a modular form, and certain linear combinations of Eisenstein series are also modular forms, it seems likely that the theory of modular forms could be used to determine the number of such identities of the type (48).

We leave it to the reader to investigate the parents of Jacobi's two, six, and eight squares theorems [**2**, pp. 56, 63, 67].

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