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# Ternary Quadratic Forms and Eta Quotients

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*Abstract.* Let $\eta(z)$ $(z \in \mathbb{C}, \operatorname{Im}(z) > 0)$ denote the Dedekind eta function. We use a recent product-to-sum formula in conjunction with conditions for the non-representability of integers by certain ternary quadratic forms to give explicitly ten eta quotients

$$f(z) := \eta^{a(m_1)}(m_1 z) \cdots \eta^{a(m_r)}(m_r z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z}, \quad z \in \mathbb{C},\ \operatorname{Im}(z) > 0,$$

such that the Fourier coefficients $c(n)$ vanish for all positive integers $n$ in each of infinitely many non-overlapping arithmetic progressions. For example, we show that if $f(z) = \eta^4(z)\eta^9(4z)\eta^{-2}(8z)$ we have $c(n) = 0$ for all $n$ in each of the arithmetic progressions $\{16k + 14\}_{k \geq 0}$, $\{64k + 56\}_{k \geq 0}$, $\{256k + 224\}_{k \geq 0}$, $\{1024k + 896\}_{k \geq 0}, \ldots$.

## 1 Introduction

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{H}$ denote the Poincaré upper half-plane $\{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$. The Dedekind eta function is defined by

$$\eta(z) := e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}), \quad z \in \mathbb{H}.$$

An eta quotient $f(z)$ is a holomorphic function of the form

$$f(z) := \eta^{a(m_1)}(m_1 z) \cdots \eta^{a(m_r)}(m_r z), \quad z \in \mathbb{H},$$

where $r \in \mathbb{N}$, $m_1, \ldots, m_r \in \mathbb{N}$ satisfy $m_1 < \cdots < m_r$, and $a(m_1), \ldots, a(m_r)$ are nonzero integers. We suppose that

$$m_1 a(m_1) + \cdots + m_r a(m_r) = 24$$

so that the eta quotient $f(z)$ has a Fourier expansion of the form

$$f(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z}, \quad c(1) = 1,$$

where the Fourier coefficients $c(n)$ are integers. We adopt the notation

$$[f(z)]_n := c(n), \quad n \in \mathbb{N}.$$

Many questions concerning the vanishing or non-vanishing of the Fourier coefficients of eta quotients have been addressed; see, for example, [2], [3, p. 133], [4], [5], and [6]. In this note we are interested in determining explicit eta quotients $f(z)$ such that $[f(z)]_n = 0$ for all $n$ in infinitely many non-overlapping arithmetic progressions. We

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do this by using the author's recent product-to-sum formula [7, Theorem 1.1, p. 80] to express the Fourier coefficients $[f(z)]_n$ of certain eta quotients $f(z)$ in the form

$$[f(z)]_n = \sum_{\substack{(x_1,x_2,x_3)\in\mathbb{Z}^3 \\ Q(x_1,x_2,x_3)=n}} P(x_1,x_2,x_3), \quad n \in \mathbb{N}, \tag{1.1}$$

where $P \in \mathbb{Q}[x_1,x_2,x_3]$ and $Q = x_1^2 + ax_2^2 + bx_3^2$ for some $a, b \in \mathbb{N}$ with $1 \le a \le b \le 4$. Classical results on the representability of $n \in \mathbb{N}$ by the ternary quadratic form $Q$ [1, pp. 111–113] give infinitely many arithmetic progressions such that if $n$ belongs to any one of them, then $n$ is not represented by $Q$, and so by (1.1), $[f(z)]_n = 0$ for these $n$. The 10 eta quotients constructed in this manner are given in Theorem 1.1(i)–(x). Theorem 1.1 is proved in Section 2.

***Theorem 1.1*** *For all $e \in \mathbb{N}_0$ we have*

(i) $[\eta^2(z)\eta(2z)\eta^{-2}(3z)\eta(4z)\eta^5(6z)\eta^2(8z)\eta^{-2}(12z)]_n = 0$ *for* $n \equiv 5\cdot 2^{2e+1} \pmod{2^{2e+4}}$,

(ii) $[\eta^{-2}(z)\eta^8(2z)\eta^{-2}(4z)\eta^3(6z)]_n = 0$ *for* $n \equiv 2\cdot 3^{2e+1} \pmod{3^{2e+2}}$,

(iii) $[\eta^3(2z)\eta^{-2}(3z)\eta^8(6z)\eta^{-2}(12z)]_n = 0$ *for* $n \equiv 2\cdot 3^{2e} \pmod{3^{2e+1}}$,

(iv) $[\eta^3(2z)\eta^{-2}(4z)\eta^3(6z)\eta^5(8z)\eta^{-2}(16z)]_n = 0$ *for* $n \equiv 5\cdot 3^{2e+1} \pmod{2\cdot 3^{2e+2}}$,

(v) $[\eta^{-2}(3z)\eta^6(4z)\eta^5(6z)\eta^{-2}(12z)]_n = 0$ *for* $n \equiv 8\cdot 3^{2e+1} \pmod{4\cdot 3^{2e+2}}$,

(vi) $[\eta^{-2}(3z)\eta^6(4z)\eta^5(6z)\eta^{-2}(12z)]_n = 0$ *for* $n \equiv 11\cdot 3^{2e+1} \pmod{4\cdot 3^{2e+2}}$,

(vii) $[\eta^2(z)\eta^7(2z)\eta^2(4z)]_n = 0$ *for* $n \equiv 7\cdot 2^{2e} \pmod{2^{2e+3}}$,

(viii) $[\eta^4(z)\eta^9(4z)\eta^{-2}(8z)]_n = 0$ *for* $n \equiv 7\cdot 2^{2e+1} \pmod{2^{2e+4}}$,

(ix) $[\eta^4(z)\eta^2(2z)\eta^{-2}(3z)\eta^4(4z)\eta^5(6z)\eta^{-2}(12z)]_n = 0$ *for* $n \equiv 2\cdot 3^{2e+1} \pmod{3^{2e+2}}$,

(x) $[\eta^4(z)\eta^2(2z)\eta^2(4z)\eta^5(8z)\eta^{-2}(16z)]_n = 0$ *for* $n \equiv 7\cdot 2^{2e} \pmod{2^{2e+3}}$.

## 2 Proof of Theorem 1.1

For $k \in \mathbb{N}$ and $q \in \mathbb{C}$ with $|q| < 1$, we define

$$E_k = E_k(q) := \prod_{n=1}^{\infty}(1 - q^{kn}),$$

so that $E_k = E_1(q^k)$.

We begin with a lemma that enables us to eliminate some uninteresting cases that arise in the proof of Theorem 1.1 as well as assisting in the formulation of some conjectures in Section 3.

***Lemma 2.1*** *Let $n \in \mathbb{N}$.*

(i) *If* $n \equiv 0, 3 \pmod 4$, *then* $[qE_1^2E_2^{-1}E_4^6]_n = 0$.

(ii) *If* $n \equiv 0, 6 \pmod 8$, *then* $[qE_1^2E_2E_4^{-1}E_8^7E_{16}^{-2}]_n = 0$.

(iii) *If* $n \equiv 0, 3 \pmod 4$, *then* $[qE_1^{-2}E_2^5E_4^4]_n = 0$.

(iv) *If* $n \equiv 2, 3 \pmod 4$, *then* $[qE_3^{-2}E_4^6E_6^5E_{12}^{-2}]_n = 0$.

(v) *If* $n \equiv 3 \pmod 4$, *then* $[qE_1^4E_2^2E_4^2E_8^5E_{16}^{-2}]_n = 0$.

**Proof** (i) By a classical theorem of Jacobi, we have

$$E_1^2E_2^{-1} = 1 + 2\sum_{m=1}^{\infty}(-1)^m q^{m^2}.$$

Thus for $n \in \mathbb{N}$, we have

$$[qE_1^2E_2^{-1}]_n = \begin{cases} 1 & \text{if } n = 1, \\ 2(-1)^m & \text{if } n = m^2+1, m \in \mathbb{N}, \\ 0 & \text{if } n \neq m^2+1, m \in \mathbb{N}_0. \end{cases}$$

If $n \equiv 0, 3 \pmod 4$, then we have $n \neq m^2+1$ ($m \in \mathbb{N}_0$) so that $[qE_1^2E_2^{-1}]_n = 0$. Hence,

$$[qE_1^2E_2^{-1}E_4^6]_n = 0 \text{ if } n \equiv 0, 3 \pmod 4.$$

(ii) Replacing $q$ by $q^2$ in Jacobi's identity, we obtain

$$E_2^2E_4^{-1} = 1 + 2\sum_{m=1}^{\infty}(-1)^m q^{2m^2}.$$

Multiplying the series for $E_1^2E_2^{-1}$ and $E_2^2E_4^{-1}$ together, we deduce

$$qE_1^2E_2E_4^{-1} = q + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2+1} + 2\sum_{n=1}^{\infty}(-1)^n q^{2n^2+1} + 4\sum_{n=1}^{\infty}\left(\sum_{\substack{\ell, m \geq 1 \\ \ell^2+2m^2+1=n}}(-1)^{\ell+m}\right)q^n.$$

As $n^2+1 \equiv 1, 2, 5 \pmod 8$, $2n^2+1 \equiv 1, 3 \pmod 8$ and $\ell^2+2m^2+1 \equiv 1, 2, 3, 4, 5, 7 \pmod 8$, we have $[qE_1^2E_2E_4^{-1}]_n = 0$ for $n \equiv 0, 6 \pmod 8$. Thus,

$$[qE_1^2E_2E_4^{-1}E_8^7E_{16}^{-2}]_n = 0 \text{ if } n \equiv 0, 6 \pmod 8.$$

(iii) By another classical identity of Jacobi, we have

$$E_1^{-2}E_2^5E_4^{-2} = 1 + 2\sum_{m=1}^{\infty} q^{m^2}.$$

Thus for $n \in \mathbb{N}$ we have

$$[qE_1^{-2}E_2^5E_4^{-2}]_n = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n = m^2+1, m \in \mathbb{N}, \\ 0 & \text{if } n \neq m^2+1, m \in \mathbb{N}_0. \end{cases}$$

If $n \equiv 0, 3 \pmod 4$ then $n \neq m^2+1$ ($m \in \mathbb{N}_0$) so $[qE_1^{-2}E_2^5E_4^{-2}]_n = 0$. Thus,

$$[qE_1^{-2}E_2^5E_4^4]_n = [qE_1^{-2}E_2^5E_4^{-2} \cdot E_4^6]_n = 0 \text{ if } n \equiv 0, 3 \pmod 4.$$

(iv) Mapping $q$ to $q^3$ in the identity of Jacobi given in the proof of (iii), we have

$$E_3^{-2}E_6^5E_{12}^{-2} = 1 + 2\sum_{m=1}^{\infty} q^{3m^2}.$$

Hence for $n \in \mathbb{N}$, we have

$$[qE_3^{-2}E_6^5E_{12}^{-2}]_n = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n = 3m^2+1, m \in \mathbb{N}, \\ 0 & \text{if } n \neq 3m^2+1, m \in \mathbb{N}_0. \end{cases}$$

If $n \equiv 2, 3 \pmod 4$, then $n \neq 3m^2 + 1$ $(m \in \mathbb{N}_0)$ so $[qE_3^{-2}E_6^5E_{12}^{-2}]_n = 0$. Thus

$$[qE_3^{-2}E_4^6E_6^5E_{12}^{-2}]_n = 0 \text{ if } n \equiv 2, 3 \pmod 4.$$

(v) Ramanujan defined the theta function $\phi$ by

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$$

and proved that

$$\phi^2(q) + \phi^2(-q) = 2\phi^2(q^2), \quad \phi(q)\phi(-q) = \phi^2(-q^2).$$

Hence,

$$\begin{aligned}
&\sum_{j=0}^{3} (-1)^j \phi(i^j q)\phi^3(-i^j q) \\
&\quad = \phi(q)\phi(-q)\big(\phi^2(q) + \phi^2(-q)\big) - \phi(iq)\phi(-iq)\big(\phi^2(iq) + \phi^2(-iq)\big) \\
&\quad = 2\phi^2(-q^2)\phi^2(q^2) - 2\phi^2(q^2)\phi^2(-q^2) = 0.
\end{aligned}$$

Thus,

$$[\phi(q)\phi^3(-q)]_n = 0 \text{ for } n \equiv 2 \pmod 4.$$

Now $\phi(q) = E_1^{-2}E_2^5E_4^{-2}$ and $\phi(-q) = E_1^2E_2^{-1}$ so $\phi(q)\phi^3(-q) = E_1^4E_2^2E_4^{-2}$. Hence,

$$[E_1^4E_2^2E_4^{-2}]_n = 0 \text{ for } n \equiv 2 \pmod 4.$$

Then

$$[E_1^4E_2^2E_4^2E_8^5E_{16}^{-2}]_n = [E_1^4E_2^2E_4^{-2} \cdot E_4^4E_8^5E_{16}^{-2}]_n = [E_1^4E_2^2E_4^{-2}]_n = 0 \text{ for } n \equiv 2 \pmod 4,$$

and finally

$$[qE_1^4E_2^2E_4^2E_8^5E_{16}^{-2}]_n = 0 \text{ for } n \equiv 3 \pmod 4.$$

This completes the proof of the lemma. ■

Taking $q = e^{2\pi i z}$ $(z \in \mathbb{H})$ so that $|q| < 1$, we have

$$\eta(kz) = e^{\pi i k z/12} E_k = q^{k/24} E_k.$$

We now state the product-to-sum formula proved by the author in [7, Theorem 1.1, p. 80], which we will use in the proof of Theorem 1.1.

***Product-to-Sum Formula*** *Let $k \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$. Let $r, s, t, u \in \mathbb{N}_0$ be such that $r + s + t + u = k$. Let $v, w, x, y \in \mathbb{N}_0$ be such that $v + w + x + y = \ell$. Set $m = k + 2\ell$ so that $m \in \mathbb{N}$ and $m \geq 2$. Let*

$$\begin{aligned}
P(x_1, \ldots, x_m) = \frac{1}{2^\ell} &\prod_{g=r+1}^{r+v} (x_g^2 - 2x_{g+s+\ell+y}^2) \prod_{g=r+v+1}^{r+v+w} (x_g^2 - 3x_{g+s+t+\ell+y}^2) \\
&\times \prod_{g=r+v+w+1}^{r+v+w+x} (x_g^2 - 4x_{g+s+t+\ell+y+u}^2) \prod_{g=r+v+w+x+1}^{r+\ell} (x_g^4 - 3x_g^2 x_{g+y}^2)
\end{aligned}$$

*and*

$$\begin{aligned}
Q(x_1, \ldots, x_m) = x_1^2 + \cdots + x_{r+\ell+y}^2 + 2x_{r+\ell+y+1}^2 + \cdots + 2x_{r+s+\ell+v+y}^2 \\
+ 3x_{r+s+\ell+v+y+1}^2 + \cdots + 3x_{r+s+t+\ell+v+w+y}^2 + 4x_{r+s+t+\ell+v+w+y+1}^2 + \cdots + 4x_m^2.
\end{aligned}$$

*Let*

$$a_1 = -2r + 2v + 4y, \qquad a_6 = 5t + 3w,$$
$$a_2 = 5r - 2s + v + 3w + 2y, \qquad a_8 = -2s + 5u + 2v,$$
$$a_3 = -2t, \qquad a_{12} = -2t,$$
$$a_4 = -2r + 5s - 2u + v + 6x + 4y, \qquad a_{16} = -2u.$$

*Then for* $n \in \mathbb{N}$ *with* $n \geq \ell$, *we have*

$$\left[ q^{\ell} E_1^{a_1} E_2^{a_2} E_3^{a_3} E_4^{a_4} E_6^{a_6} E_8^{a_8} E_{12}^{a_{12}} E_{16}^{a_{16}} \right]_n = \sum_{\substack{(x_1,\ldots,x_m)\in\mathbb{Z}^m \\ Q(x_1,\ldots,x_m)=n}} P(x_1,\ldots,x_m)$$

*and*

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 6a_6 + 8a_8 + 12a_{12} + 16a_{16} = 24\ell.$$

**Proof of Theorem 1.1** We use the special case of the product-to-sum formula when $k = \ell = 1$, so that $m = 3$ and $Q$ is a positive diagonal ternary quadratic form all of whose coefficients are 1, 2, 3, or 4 with at least one of the coefficients equal to 1. Let $r, s, t, u, v, w, x, y \in \mathbb{N}_0$ satisfy

(2.1) $$r + s + t + u = 1, \quad v + w + x + y = 1.$$

Define $A \in \mathbb{N}$ and $B, C, D \in \mathbb{N}_0$ by

(2.2) $$A := r + y + 1, \quad B := s + v, \quad C := t + w, \quad D := u + x,$$

so that

(2.3) $$A + B + C + D = 3.$$

Then

$$P(x_1, x_2, x_3) = \frac{1}{2} \prod_{g=r+1}^{r+v} (x_g^2 - 2x_{g+s+y+1}^2) \prod_{g=r+v+1}^{r+v+w} (x_g^2 - 3x_{g+s+t+y+1}^2) \times \prod_{g=r+v+w+1}^{r+v+w+x} (x_g^2 - 4x_{g+s+t+y+u+1}^2) \prod_{g=r+v+w+x+1}^{r+1} (x_g^4 - 3x_g^2 x_{g+y}^2)$$

and

(2.4) $$Q(x_1, x_2, x_3) = \sum_{i=1}^{A} x_i^2 + 2\sum_{i=A+1}^{A+B} x_i^2 + 3\sum_{i=A+B+1}^{A+B+C} x_i^2 + 4\sum_{i=A+B+C+1}^{A+B+C+D} x_i^2.$$

Define the integers $a_1, a_2, a_3, a_4, a_6, a_8, a_{12}$ and $a_{16}$ as in the product-to-sum formula. Then, as $\ell = 1$, by the product-to-sum formula we have

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 6a_6 + 8a_8 + 12a_{12} + 16a_{16} = 24$$

and

(2.5) $$\left[ \eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_3}(3z)\eta^{a_4}(4z)\eta^{a_6}(6z)\eta^{a_8}(8z)\eta^{a_{12}}(12z)\eta^{a_{16}}(16z) \right]_n = \left[ qE_1^{a_1} E_2^{a_2} E_3^{a_3} E_4^{a_4} E_6^{a_6} E_8^{a_8} E_{12}^{a_{12}} E_{16}^{a_{16}} \right]_n = \sum_{\substack{(x_1,x_2,x_3)\in\mathbb{Z}^3 \\ Q(x_1,x_2,x_3)=n}} P(x_1, x_2, x_3).$$

We next examine the 16 possible values of the vector $(r, s, t, u, v, w, x, y) \in \mathbb{N}_0^8$ subject to the restrictions in (2.1). The ternary form $Q$ corresponding to each such vector is determined from (2.2), (2.3), and (2.4) and each eta quotient

$$\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_3}(3z)\eta^{a_4}(4z)\eta^{a_6}(6z)\eta^{a_8}(8z)\eta^{a_{12}}(12z)\eta^{a_{16}}(16z)$$

from the formulae for $a_1, a_2, a_3, a_4, a_6, a_8, a_{12}, a_{16}$ in terms of $r, s, t, u, v, w, x, y$ given in the product-to-sum formula. The values are given in Table 1.

It is known from the work of Dickson and Jones (see, for example, [1, pp. 111–112]) that the positive integers $n$ for which $Q(x_1, x_2, x_3) = n$ is not solvable in integers $x_1, x_2$, and $x_3$ are as given in Table 2, where $k, \ell \in \mathbb{N}_0$. Appealing to (2.5), Table 1 and Table 2, we obtain the following results.

Case 1 gives

$$[\eta^6(2z)\eta^{-1}(4z)\eta^2(8z)]_n = 0 \tag{2.6}$$

for $n \equiv 7 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ for all $e \in \mathbb{N}_0$, but this is not interesting, as clearly (2.6) holds for all $n \in \mathbb{N}$ with $n \equiv 0 \pmod 2$.

Case 2 gives

$$[\eta^2(z)\eta^{-1}(2z)\eta^6(4z)]_n = 0 \tag{2.7}$$

for $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ for all $e \in \mathbb{N}_0$, which is again not interesting, as by Lemma 2.1(i) (2.7) holds for all $n \equiv 0, 3 \pmod 4$.

Case 3 gives

$$[\eta^2(z)\eta(2z)\eta^{-2}(3z)\eta(4z)\eta^5(6z)\eta^2(8z)\eta^{-2}(12z)]_n = 0$$

for $n \equiv 5 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ for all $e \in \mathbb{N}_0$, which is Theorem 1.1(i).

Case 4 gives

$$[\eta^2(z)\eta(2z)\eta^{-1}(4z)\eta^7(8z)\eta^{-2}(16z)]_n = 0 \tag{2.8}$$

for $n \equiv 7 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ for all $e \in \mathbb{N}_0$, but this is not interesting, as (2.8) holds for all $n \equiv 0, 6 \pmod 8$ by Lemma 2.1(ii).

Case 5 gives

$$[\eta^{-2}(z)\eta^8(2z)\eta^{-2}(4z)\eta^3(6z)]_n = 0$$

for $n \equiv 2 \cdot 3^{2e+1} \pmod{3^{2e+2}}$ for all $e \in \mathbb{N}_0$, which is Theorem 1.1(ii).

Case 6 gives

$$[\eta(2z)\eta^5(4z)\eta^3(6z)\eta^{-2}(8z)]_n = 0 \tag{2.9}$$

for $n \equiv 5 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ for all $e \in \mathbb{N}_0$, which is not interesting, as (2.9) holds for all $n \equiv 0 \pmod 2$.

Case 7 gives

$$[\eta^3(2z)\eta^{-2}(3z)\eta^8(6z)\eta^{-2}(12z)]_n = 0$$

for $n \equiv 2 \cdot 3^{2e} \pmod{3^{2e+1}}$ for all $e \in \mathbb{N}_0$, which is Theorem 1.1(iii).

| case | $(r,s,t,u,v,w,x,y)$ | $(a_1,a_2,a_3,a_4,a_6,a_8,a_{12},a_{16})$ | $Q(x_1,x_2,x_3)$ |
|---|---|---|---|
| 1 | $(1,0,0,0,1,0,0,0)$ | $(0,6,0,-1,0,2,0,0)$ | $x_1^2+x_2^2+2x_3^2$ |
| 2 | $(0,1,0,0,1,0,0,0)$ | $(2,-1,0,6,0,0,0,0)$ | $x_1^2+2x_2^2+2x_3^2$ |
| 3 | $(0,0,1,0,1,0,0,0)$ | $(2,1,-2,1,5,2,-2,0)$ | $x_1^2+2x_2^2+3x_3^2$ |
| 4 | $(0,0,0,1,1,0,0,0)$ | $(2,1,0,-1,0,7,0,-2)$ | $x_1^2+2x_2^2+4x_3^2$ |
| 5 | $(1,0,0,0,0,1,0,0)$ | $(-2,8,0,-2,3,0,0,0)$ | $x_1^2+x_2^2+3x_3^2$ |
| 6 | $(0,1,0,0,0,1,0,0)$ | $(0,1,0,5,3,-2,0,0)$ | $x_1^2+2x_2^2+3x_3^2$ |
| 7 | $(0,0,1,0,0,1,0,0)$ | $(0,3,-2,0,8,0,-2,0)$ | $x_1^2+3x_2^2+3x_3^2$ |
| 8 | $(0,0,0,1,0,1,0,0)$ | $(0,3,0,-2,3,5,0,-2)$ | $x_1^2+3x_2^2+4x_3^2$ |
| 9 | $(1,0,0,0,0,0,1,0)$ | $(-2,5,0,4,0,0,0,0)$ | $x_1^2+x_2^2+4x_3^2$ |
| 10 | $(0,1,0,0,0,0,1,0)$ | $(0,-2,0,11,0,-2,0,0)$ | $x_1^2+2x_2^2+4x_3^2$ |
| 11 | $(0,0,1,0,0,0,1,0)$ | $(0,0,-2,6,5,0,-2,0)$ | $x_1^2+3x_2^2+4x_3^2$ |
| 12 | $(0,0,0,1,0,0,1,0)$ | $(0,0,0,4,0,5,0,-2)$ | $x_1^2+4x_2^2+4x_3^2$ |
| 13 | $(1,0,0,0,0,0,0,1)$ | $(2,7,0,2,0,0,0,0)$ | $x_1^2+x_2^2+x_3^2$ |
| 14 | $(0,1,0,0,0,0,0,1)$ | $(4,0,0,9,0,-2,0,0)$ | $x_1^2+x_2^2+2x_3^2$ |
| 15 | $(0,0,1,0,0,0,0,1)$ | $(4,2,-2,4,5,0,-2,0)$ | $x_1^2+x_2^2+3x_3^2$ |
| 16 | $(0,0,0,1,0,0,0,1)$ | $(4,2,0,2,0,5,0,-2)$ | $x_1^2+x_2^2+4x_3^2$ |

Table 1: Eta quotients and ternary forms corresponding to $(r,s,t,u,v,w,x,y)$

| $Q(x_1,x_2,x_3)$ | integers not represented by $Q$ | $Q(x_1,x_2,x_3)$ | integers not represented by $Q$ |
|---|---|---|---|
| $x_1^2+x_2^2+x_3^2$ | $4^k(8\ell+7)$ | $x_1^2+2x_2^2+3x_3^2$ | $4^k(16\ell+10)$ |
| $x_1^2+x_2^2+2x_3^2$ | $4^k(16\ell+14)$ | $x_1^2+2x_2^2+4x_3^2$ | $4^k(16\ell+14)$ |
| $x_1^2+x_2^2+3x_3^2$ | $9^k(9\ell+6)$ | $x_1^2+3x_2^2+3x_3^2$ | $9^k(3\ell+2)$ |
| $x_1^2+x_2^2+4x_3^2$ | $8\ell+3,\ 4^k(8\ell+7)$ | $x_1^2+3x_2^2+4x_3^2$ | $4\ell+2,\ 9^k(9\ell+6)$ |
| $x_1^2+2x_2^2+2x_3^2$ | $4^k(8\ell+7)$ | $x_1^2+4x_2^2+4x_3^2$ | $4\ell+2,\ 4\ell+3,\ 4^k(8\ell+7)$ |

Table 2: Integers not represented by ternary quadratic forms

Case 8 gives

$$[\eta^3(2z)\eta^{-2}(4z)\eta^3(6z)\eta^5(8z)\eta^{-2}(16z)]_n = 0 \tag{2.10}$$

for $n \equiv 2 \pmod 4$ and $n \equiv 2\cdot 3^{2e+1} \pmod{3^{2e+2}}$ for all $e \in \mathbb{N}_0$. Clearly (2.10) holds for all $n \equiv 0 \pmod 2$ so the former is not interesting while the latter is interesting only when $n \equiv 1 \pmod 2$, that is, when $n \equiv 5 \cdot 3^{2e+1} \pmod{2 \cdot 3^{2e+2}}$, which is Theorem 1.1(iv).

Case 9 gives

$$[\eta^{-2}(z)\eta^{5}(2z)\eta^{4}(4z)]_n = 0 \tag{2.11}$$

for $n \equiv 3 \pmod 8$ and $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ for all $e \in \mathbb{N}_0$. But this is not interesting, as (2.11) holds for all $n \equiv 0, 3 \pmod 4$ by Lemma 2.1(iii).

Case 10 gives

$$[\eta^{-2}(2z)\eta^{11}(4z)\eta^{-2}(8z)]_n = 0 \tag{2.12}$$

for $n \equiv 7 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ for all $e \in \mathbb{N}_0$. But this is not interesting as (2.12) holds for all $n \equiv 0 \pmod 2$.

Case 11 gives

$$[\eta^{-2}(3z)\eta^{6}(4z)\eta^{5}(6z)\eta^{-2}(12z)]_n = 0 \tag{2.13}$$

for $n \equiv 2 \pmod 4$ and $n \equiv 2 \cdot 3^{2e+1} \pmod{3^{2e+2}}$ for all $e \in \mathbb{N}_0$. By Lemma 2.1(iv) (2.13) holds for all $n \equiv 2, 3 \pmod 4$. Thus (2.13) is only interesting for those $n$ satisfying $n \equiv 2 \cdot 3^{2e+1} \pmod{3^{2e+2}}$ and $n \equiv 0, 1 \pmod 4$, that is, for $n \equiv 8 \cdot 3^{2e+1} \pmod{4 \cdot 3^{2e+2}}$ and $n \equiv 11 \cdot 3^{2e+1} \pmod{4 \cdot 3^{2e+2}}$. These are parts (v) and (vi) of Theorem 1.1, respectively.

Case 12 gives

$$[\eta^{4}(4z)\eta^{5}(8z)\eta^{-2}(16z)]_n = 0 \tag{2.14}$$

for $n \equiv 2 \pmod 4$, $n \equiv 3 \pmod 4$ and $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ for all $e \in \mathbb{N}_0$, but again this is not interesting as (2.14) clearly holds for all $n \not\equiv 1 \pmod 4$.

Case 13 gives

$$[\eta^{2}(z)\eta^{7}(2z)\eta^{2}(4z)]_n = 0$$

for $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ for all $e \in \mathbb{N}_0$, which is Theorem 1.1(vii).

Case 14 gives

$$[\eta^{4}(z)\eta^{9}(4z)\eta^{-2}(8z)]_n = 0$$

for $n \equiv 7 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ for all $e \in \mathbb{N}_0$, which is Theorem 1.1(viii).

Case 15 gives

$$[\eta^{4}(z)\eta^{2}(2z)\eta^{-2}(3z)\eta^{4}(4z)\eta^{5}(6z)\eta^{-2}(12z)]_n = 0$$

for $n \equiv 2 \cdot 3^{2e+1} \pmod{3^{2e+2}}$ for all $e \in \mathbb{N}_0$, which is Theorem 1.1(ix).

Finally, Case 16 gives

$$[\eta^{4}(z)\eta^{2}(2z)\eta^{2}(4z)\eta^{5}(8z)\eta^{-2}(16z)]_n = 0$$

for $n \equiv 3 \pmod 8$ and $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ for all $e \in \mathbb{N}_0$. By Lemma 2.1(v) the equality holds for all $n \equiv 3 \pmod 4$. The latter congruence is Theorem 1.1(x). ∎

## 3 Final Remarks

We briefly discuss whether or not the criteria in each of the ten parts of Theorem 1.1 form a complete description of the set of vanishing coefficients for the corresponding eta quotient.

(i) As

$$\left[\eta^2(z)\eta(2z)\eta^{-2}(3z)\eta(4z)\eta^5(6z)\eta^2(8z)\eta^{-2}(12z)\right]_n = 0$$

for $n = 6, 24, 29, 39, 54, 60, 78, \ldots$, the congruences $n \equiv 5 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ do not form a complete description of when the coefficients of the eta quotient vanish and it does not seem easy to formulate such a description. We note that the data up to $n = 2000$ suggest the following conjecture.

***Conjecture 3.1***

$$\left[\eta^2(z)\eta(2z)\eta^{-2}(3z)\eta(4z)\eta^5(6z)\eta^2(8z)\eta^{-2}(12z)\right]_n = 0$$

$$\textit{for } n \equiv 2 \cdot 3^{2e+1} \pmod{2 \cdot 3^{2e+2}}.$$

(ii) The data up to $n = 1000$ suggest the following conjecture.

***Conjecture 3.2*** *If*

$$\left[\eta^{-2}(z)\eta^8(2z)\eta^{-2}(4z)\eta^3(6z)\right]_n = 0$$

*then* $n \equiv 2 \cdot 3^{2e+1} \pmod{3^{2e+2}}$ *for some* $e \in \mathbb{N}_0$.

If Conjecture 3.2 is true, then by Theorem 1.1(ii) we have

$$\left[\eta^{-2}(z)\eta^8(2z)\eta^{-2}(4z)\eta^3(6z)\right]_n = 0$$

if and only if $n \equiv 2 \cdot 3^{2e+1} \pmod{3^{2e+2}}$ for some $e \in \mathbb{N}_0$.

(iii) The data up to $n = 1000$ suggest the following conjecture.

***Conjecture 3.3*** *If*

$$\left[\eta^3(2z)\eta^{-2}(3z)\eta^8(6z)\eta^{-2}(12z)\right]_n = 0,$$

*then* $n \equiv 2 \cdot 3^{2e} \pmod{3^{2e+1}}$ *for some* $e \in \mathbb{N}_0$.

If Conjecture 3.3 is true then by Theorem 1.1(iii) we have

$$\left[\eta^3(2z)\eta^{-2}(3z)\eta^8(6z)\eta^{-2}(12z)\right]_n = 0$$

if and only if

$$n \equiv 2 \cdot 3^{2e} \pmod{3^{2e+1}} \quad \text{for some } e \in \mathbb{N}_0.$$

(iv) The data up to $n = 1000$ suggest the following conjecture.

***Conjecture 3.4*** *If*

$$\left[\eta^3(2z)\eta^{-2}(4z)\eta^3(6z)\eta^5(8z)\eta^{-2}(16z)\right]_n = 0$$

*and n is odd, then* $n \equiv 5 \cdot 3^{2e+1} \pmod{2 \cdot 3^{2e+2}}$ *for some* $e \in \mathbb{N}_0$.

If Conjecture 3.4 is true, then by Theorem 1.1(iv) we have

$$\left[\eta^3(2z)\eta^{-2}(4z)\eta^3(6z)\eta^5(8z)\eta^{-2}(16z)\right]_n = 0$$

if and only if $n \equiv 0 \pmod 2$ or $n \equiv 5 \cdot 3^{2e+1} \pmod{2 \cdot 3^{2e+2}}$ for some $e \in \mathbb{N}_0$.

(v)(vi) The data up to $n = 1000$ suggest the following conjecture.

***Conjecture 3.5*** *If $n \equiv 0, 1 \pmod 4$ and*

$$\left[\eta^{-2}(3z)\eta^6(4z)\eta^5(6z)\eta^{-2}(12z)\right]_n = 0,$$

*then $n \equiv 8 \cdot 3^{2e+1} \pmod{4 \cdot 3^{2e+2}}$ or $n \equiv 11 \cdot 3^{2e+1} \pmod{4 \cdot 3^{2e+2}}$ for some $e \in \mathbb{N}_0$.*

If Conjecture 3.5 is true, then by Theorem 1.1(v)(vi) and Lemma 2.1(iv) we have

$$\left[\eta^{-2}(3z)\eta^6(4z)\eta^5(6z)\eta^{-2}(12z)\right]_n = 0$$

if and only if $n \equiv 2 \pmod 4$, $n \equiv 3 \pmod 4$, $n \equiv 8 \cdot 3^{2e+1} \pmod{4 \cdot 3^{2e+2}}$ or $n \equiv 11 \cdot 3^{2e+1} \pmod{4 \cdot 3^{2e+2}}$ for some $e \in \mathbb{N}_0$.

(vii) The data up to $n = 1000$ suggest the following conjecture.

***Conjecture 3.6*** *If*

$$\left[\eta^2(z)\eta^7(2z)\eta^2(4z)\right]_n = 0,$$

*then $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ for some $e \in \mathbb{N}_0$.*

If Conjecture 3.6 is true, then by Theorem 1.1(vii) we have

$$\left[\eta^2(z)\eta^7(2z)\eta^2(4z)\right]_n = 0$$

if and only if $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ for some $e \in \mathbb{N}_0$.

(viii) The data up to $n = 1000$ suggest the following conjecture.

***Conjecture 3.7*** *If*

$$\left[\eta^4(z)\eta^9(4z)\eta^{-2}(8z)\right]_n = 0,$$

*then $n \equiv 7 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ for some $e \in \mathbb{N}_0$.*

If Conjecture 3.7 is true, then by Theorem 1.1(viii) we have

$$\left[\eta^4(z)\eta^9(4z)\eta^{-2}(8z)\right]_n = 0$$

if and only if $n \equiv 7 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ for some $e \in \mathbb{N}_0$.

(ix) Theorem 1.1(ix) and the data up to $n = 1000$ suggest the following conjecture.

***Conjecture 3.8***

$$\left[\eta^4(z)\eta^2(2z)\eta^{-2}(3z)\eta^4(z)\eta^5(6z)\eta^{-2}(12z)\right]_n = 0$$

*if and only if $n \equiv 2 \cdot 3^{2e+1} \pmod{3^{2e+2}}$ or $n = 3^{2e+1}$ for some $e \in \mathbb{N}_0$.*

(x) The data up to $n = 1000$ suggest the following conjecture.

***Conjecture 3.9*** *If* $n \not\equiv 3 \pmod 4$ *and*

$$\left[\eta^4(z)\eta^2(2z)\eta^2(4z)\eta^5(8z)\eta^{-2}(16z)\right]_n = 0$$

*then* $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ *for some* $e \in \mathbb{N}$.

If Conjecture 3.9 is true, then Lemma 2.1(v) and Theorem 1.1(x) give

$$\left[\eta^4(z)\eta^2(2z)\eta^2(4z)\eta^5(8z)\eta^{-2}(16z)\right]_n = 0$$

if and only if $n \equiv 3 \pmod 4$ or $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ for some $e \in \mathbb{N}$.

Over the past twenty years or so, many new results concerning the representability and non-representability of positive integers by ternary quadratic forms have been proved by a number of authors, for example, Berkovich, Bhargava, Duke, Jagy, Kaplansky, and Oh, as well as many others. However, as the product-to-sum formula used in this paper applies only to the ten ternaries $x_1^2 + ax_2^2 + bx_3^2$ with $1 \le a \le b \le 4$, we cannot use these new results in conjunction with the product-to-sum formula to obtain further results like those in Theorem 1.1.

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