Research Article

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A class of subsums of Euler's sum

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Abstract: A class of sums of the type

$$\sum_{\substack{n=1\\ n\equiv a_1,\dots,a_r \ (\mathrm{mod}\ m)}}^{\infty} \frac{1}{n^{2k}}$$

is evaluated, where k, m and r are positive integers with $m \geq 2$ and $a_1, \dots, a_r$ are integers satisfying $1 \leq a_1 < a_2 < \dots < a_r \leq m-1$.

Keywords: Euler's sum, Euler's formula, Bernoulli numbers and polynomials, Kronecker symbol, subsums of Euler's sum

MSC 2010: 11A07, 11A25, 11B25, 11B68, 11F66, 11Y60

1 Introduction

Let $\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$. Let $\mathbb{Q}$, $\bar{\mathbb{Q}}$ and $\mathbb{R}$ denote the fields of rational numbers, algebraic numbers and real numbers, respectively.

In the eighteenth century Euler proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-1} B_{2k} \pi^{2k}}{(2k)!}, \quad k \in \mathbb{N}, \tag{1.1}$$

where B_ℓ ($\ell \in \mathbb{N}_0$) denotes the ℓth Bernoulli number. Euler's formula (1.1) is well known and many proofs of it occur in the literature, see for example [2, 3, 8].

Some subsums of Euler's sum $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ of the type

$$\sum_{\substack{n=1\\ n\equiv a_1,\dots,a_r \ (\mathrm{mod}\ m)}}^{\infty} \frac{1}{n^{2k}}, \quad k, m, r \in \mathbb{N},\ m \geq 2, \tag{1.2}$$

where $a_1, \dots, a_r \in \mathbb{Z}$ satisfy $0 \leq a_1 < a_2 < \dots < a_r \leq m-1$, have been evaluated. One very simple example is

$$\sum_{\substack{n=1\\ n\equiv 0 \ (\mathrm{mod}\ m)}}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-1} B_{2k} \pi^{2k}}{m^{2k}(2k)!}, \quad k, m \in \mathbb{N}, \tag{1.3}$$

which follows immediately from (1.1). Another simple example is

$$\sum_{\substack{n=1\\ n\equiv 1 \ (\mathrm{mod}\ 2)}}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} (2^{2k}-1) B_{2k} \pi^{2k}}{2(2k)!}, \quad k \in \mathbb{N},$$

which follows by subtracting (1.3) with $m = 2$ from (1.1).

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Recently Navas, Ruiz and Varona [6, p. 34, Proposition 3.5] evaluated the subsum

$$\sum_{\substack{n=1 \\ n\equiv \pm s \pmod m}}^{\infty} \frac{1}{n^{2k}}, \quad k, m, s \in \mathbb{N},$$

for $m \equiv 1 \pmod 2$, $m \geq 3$ and $s \in \{1, 2, \ldots, (m-1)/2\}$ in terms of values of trigonometric functions and values of Bernoulli polynomials. They stated that there is a similar evaluation for $m \equiv 0 \pmod 2$ but did not give it. We now state their theorem in a form valid for all $m \in \mathbb{N}$ with $m \geq 3$ and all $s \in \mathbb{Z}$ with $2s \not\equiv 0 \pmod m$, and give a very simple proof of it in Section 2. We recall that the Bernoulli polynomial $B_n(x)$ ($n \in \mathbb{N}_0$, $x \in \mathbb{R}$) is defined by

$$B_n(x) := \sum_{r=0}^{n} \binom{n}{r} B_r x^{n-r},$$

and we note the properties

$$B_n(0) = B_n \quad B_{2k}(x) = B_{2k}(1-x), \quad n, k \in \mathbb{N}_0. \tag{1.4}$$

Theorem 1.1 (Navas, Ruiz and Varona). *Let $k, m \in \mathbb{N}$ with $m \geq 3$. Let $s \in \mathbb{Z}$ be such that $2s \not\equiv 0 \pmod m$. Then*

$$\sum_{\substack{n=1 \\ n\equiv \pm s \pmod m}}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k} \pi^{2k}}{m(2k)!} \sum_{j=0}^{m-1} B_{2k}(j/m) \cos(2\pi s j/m).$$

As $B_{2k}(j/m) \in \mathbb{Q}$ and $\cos(2\pi s j/m) \in \bar{\mathbb{Q}} \cap \mathbb{R}$, we see that

$$\frac{1}{\pi^{2k}} \sum_{\substack{n=1 \\ n\equiv \pm s \pmod m}}^{\infty} \frac{1}{n^{2k}} \in \bar{\mathbb{Q}} \cap \mathbb{R}.$$

Hence a sum of the form (1.2) with

$$\begin{cases} r \equiv 0 \pmod 2, & 1 \leq a_1 < a_2 < \cdots < a_r \leq m-1, \\ (a_j, m) = 1, & a_{r+1-j} = m - a_j, \; j = 1, \ldots, r, \end{cases}$$

is a sum of sums of the type given in Theorem 1.1, namely

$$\sum_{j=1}^{r/2} \sum_{\substack{n=1 \\ n\equiv \pm a_j \pmod m}}^{\infty} \frac{1}{n^{2k}},$$

where each $2a_j \not\equiv 0 \pmod m$, and thus is of the form $\alpha\pi^{2k}$, where $\alpha \in \bar{\mathbb{Q}} \cap \mathbb{R}$. We give a class of subsums of this type for which α can be given explicitly as a rational linear combination of squareroots of positive integers, see Theorem 4.7. The idea of such a result is implicit in the work of Shanks and Wrench [7] and our purpose is to make it completely explicit. Two examples are

$$\sum_{\substack{n=1 \\ n\equiv 11,13 \pmod{24}}}^{\infty} \frac{1}{n^2} = \frac{(8 - 5\sqrt{2} + 4\sqrt{3} - 3\sqrt{6})\pi^2}{288},$$

see Corollary 5.10, and

$$\sum_{\substack{n=1 \\ n\equiv 5,11,13,15,17,23 \pmod{28}}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{49}(3 - \sqrt{7}),$$

see Corollary 5.12.

In Section 2 we prove Theorem 1.1. In Section 3 we define the class of subsums of Euler's sum that we shall evaluate in Section 4. In Section 4 we make use of Theorem 1.1 to prove our main result (Theorem 4.7). In Section 5 we give some examples illustrating Theorem 4.7.

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2 Proof of Theorem 1.1

We make use of the Fourier expansion of the Bernoulli polynomial $B_{2k}(x)$ ($k \in \mathbb{N}$), namely,

$$B_{2k}(x) = \frac{(-1)^{k-1}(2k)!}{2^{2k-1}\pi^{2k}} \sum_{n=1}^{\infty} \frac{\cos 2n\pi x}{n^{2k}}, \quad x \in [0,1], \tag{2.1}$$

see for example [1, p. 805]. Appealing to (2.1), we obtain

$$\begin{aligned}\sum_{j=0}^{m-1} \cos(2\pi sj/m) B_{2k}(j/m) &= \sum_{j=0}^{m-1} \cos(2\pi sj/m) \frac{(-1)^{k-1}(2k)!}{2^{2k-1}\pi^{2k}} \sum_{n=1}^{\infty} \frac{\cos(2\pi jn/m)}{n^{2k}} \\ &= \frac{(-1)^{k-1}(2k)!}{2^{2k-1}\pi^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \sum_{j=0}^{m-1} \cos(2\pi sj/m)\cos(2\pi nj/m) \\ &= \frac{(-1)^{k-1}(2k)!}{2^{2k}\pi^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \sum_{j=0}^{m-1} (\cos(2\pi(n-s)j/m) + \cos(2\pi(n+s)j/m)).\end{aligned}$$

Now

$$\sum_{j=0}^{m-1} \cos(2\pi(n \mp s)j/m) = \begin{cases} m & \text{if } n \equiv \pm s \pmod m, \\ 0 & \text{if } n \not\equiv \pm s \pmod m, \end{cases}$$

and $s \not\equiv -s \pmod m$ (as $2s \not\equiv 0 \pmod m$) so

$$\sum_{j=0}^{m-1} \cos(2\pi sj/m) B_{2k}(j/m) = \frac{(-1)^{k-1}(2k)! m}{2^{2k}\pi^{2k}} \sum_{\substack{n=1 \\ n \equiv \pm s \pmod m}}^{\infty} \frac{1}{n^{2k}}$$

from which the asserted formula follows.

3 A class of subsums of Euler's sum

We begin with some definitions.

Definition 3.1. We call a positive integer d a discriminant if d is not a perfect square and $d \equiv 0$ or $1 \pmod 4$. A discriminant d is called a fundamental discriminant if there is no integer $g > 1$ such that $g^2 | d$ and $d/g^2 \equiv 0$ or $1 \pmod 4$. The conductor $f = f(d)$ of a discriminant d is the largest positive integer such that $f^2 | d$ and $d/f^2 \equiv 0$ or $1 \pmod 4$. The fundamental discriminant $\Delta = \Delta(d)$ associated with the discriminant d is $\Delta = d/f^2$, where f is the conductor of d.

We emphasize that in this paper we are restricting discriminants to be positive integers. The Kronecker symbol for a discriminant d and a positive integer n is written as $(\frac{d}{n})$. Properties of the Kronecker symbol are given in [4, pp. 304–306]. The Kronecker symbol $(\frac{d}{n})$ is a completely multiplicative function of n. Moreover,

$$\left(\frac{d}{n}\right) = \begin{cases} 0 & \text{if } (n,d) > 1, \\ \pm 1 & \text{if } (n,d) = 1. \end{cases} \tag{3.1}$$

Also, if f is the conductor of the discriminant d and $\Delta = d/f^2$ is the fundamental discriminant associated with d then

$$\left(\frac{d}{n}\right) = \begin{cases} 0 & \text{if } (n,f) > 1, \\ (\frac{\Delta}{n}) & \text{if } (n,f) = 1. \end{cases} \tag{3.2}$$

Two further properties of the Kronecker symbol are

$$\left(\frac{d}{n}\right) = \left(\frac{d}{d-n}\right), \quad 1 \le n \le d-1, \tag{3.3}$$

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see [4, p. 305, Theorem 3.3] for a proof, and

$$\sum_{\substack{r=1\\(r,d)=1}}^{d-1}\left(\frac{d}{r}\right)=0. \tag{3.4}$$

By (3.3) we have

$$\sum_{\substack{1\le r<d/2\\(r,d)=1}}\left(\frac{d}{r}\right)=\sum_{\substack{d/2<r\le d-1\\(r,d)=1}}\left(\frac{d}{r}\right)=\frac{1}{2}\sum_{\substack{r=1\\(r,d)=1}}^{d-1}\left(\frac{d}{r}\right)$$

as $d/2$ is not an integer if d is odd and $(d/2, d) = d/2 > 1$ if d is even since $d \ge 8$ in this case. Hence, by (3.4), we deduce

$$\sum_{\substack{1\le r<d/2\\(r,d)=1}}\left(\frac{d}{r}\right)=0. \tag{3.5}$$

The final property of the Kronecker symbol that we need is the identity

$$\sum_{\substack{t=1\\(t,\Delta)=1}}^{\Delta-1}\left(\frac{\Delta}{t}\right)e^{2\pi int/\Delta}=\left(\frac{\Delta}{n}\right)\sqrt{\Delta}, \tag{3.6}$$

which is valid for any positive integer n and any fundamental discriminant Δ, see [5, p. 221, Theorem 215]. As $(\frac{\Delta}{n})\sqrt{\Delta}\in\mathbb{R}$ we have from (3.6)

$$\sum_{\substack{t=1\\(t,\Delta)=1}}^{\Delta-1}\left(\frac{\Delta}{t}\right)\cos(2\pi nt/\Delta)=\left(\frac{\Delta}{n}\right)\sqrt{\Delta}. \tag{3.7}$$

Our next three definitions are of quantities that we need in order to be able to state our main result (Theorem 4.7).

Definition 3.2. For $k, m \in \mathbb{N}$ and a discriminant d, we define

$$P_k(m) := \prod_{p\mid m}\left(1-\frac{1}{p^{2k}}\right)$$

and

$$P_k(m,d) := \prod_{p\mid m}\left(1-\left(\frac{d}{p}\right)\frac{1}{p^{2k}}\right)=\prod_{\substack{p\mid m\\p\nmid d}}\left(1-\left(\frac{d}{p}\right)\frac{1}{p^{2k}}\right),$$

where p runs through the primes satisfying the given conditions.

In particular, we have $P_k(1) = 1$ and $P_k(m, d) = 1$ if $m\mid d$.

Definition 3.3. For $m \in \mathbb{N}_0$ and a discriminant d, we define

$$S_m(d) := \sum_{\substack{1\le t<d/2\\(t,d)=1}}\left(\frac{d}{t}\right)t^m.$$

We note that $S_0(d) = 0$ by (3.5).

Definition 3.4. Let $k \in \mathbb{N}$ and d a discriminant. Let f be the conductor of d and $\Delta = d/f^2$ the fundamental discriminant associated with d. We define

$$H_k(d) := \frac{2}{\Delta^{2k}}P_k(f,\Delta)\sum_{r=0}^{2k-1}\binom{2k}{r}\Delta^r B_r S_{2k-r}(\Delta).$$

We note that in the sum in $H_k(d)$ the terms with r (odd) ≥ 3 vanish as $B_{2n+1} = 0$ for $n \in \mathbb{N}$.

Our final definition defines the class of congruences in (1.2) that we consider.

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Definition 3.5. The set of congruences

$$n \equiv a_1, \ldots, a_r \pmod{m},$$

where m and r are positive integers with $m \geq 2$ and $a_1, \ldots, a_r$ are integers satisfying $1 \leq a_1 < a_2 < \cdots < a_r \leq m-1$, is said to be discriminantly determined if there exist $\epsilon_1 = \pm 1, \ldots, \epsilon_s = \pm 1$ and discriminants $d_1, \ldots, d_s$ with no nonempty product equal to a perfect square such that

$$n \equiv a_1, \ldots, a_r \pmod{m} \quad \text{if and only if} \quad \left(\frac{d_1}{n}\right) = \epsilon_1, \ldots, \left(\frac{d_s}{n}\right) = \epsilon_s.$$

The congruences $n \equiv 7, 17 \pmod{24}$ are discriminantly determined as

$$n \equiv 7, 17 \pmod{24} \quad \text{if and only if} \quad \left(\frac{8}{n}\right) = 1, \ \left(\frac{12}{n}\right) = -1.$$

However the congruence $n \equiv 1 \pmod 4$ is not discriminantly determined. Our main result evaluates the sum (1.2) for the class of congruences which are discriminantly determined.

4 Proof of main result

In this section we evaluate some infinite series and then state and prove our main result Theorem 4.7.

Proposition 4.1. *Let $e, k \in \mathbb{N}$. Let d be a discriminant. Then*

$$\sum_{\substack{n=1 \\ e \mid n}}^{\infty} \left(\frac{d}{n}\right) \frac{1}{n^{2k}} = \left(\frac{d}{e}\right) \frac{1}{e^{2k}} \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) \frac{1}{n^{2k}}.$$

Proof. We have

$$\sum_{\substack{n=1 \\ e \mid n}}^{\infty} \left(\frac{d}{n}\right) \frac{1}{n^{2k}} = \sum_{n=1}^{\infty} \left(\frac{d}{en}\right) \frac{1}{(en)^{2k}} = \sum_{n=1}^{\infty} \left(\frac{d}{e}\right)\left(\frac{d}{n}\right) \frac{1}{e^{2k}} \frac{1}{n^{2k}}$$

and the asserted result now follows. □

Proposition 4.2. *Let $k, m \in \mathbb{N}$. Let d be a discriminant. Then*

$$\sum_{e \mid m} \frac{\mu(e)}{e^{2k}} = P_k(m) \quad \text{and} \quad \sum_{e \mid m} \mu(e) \left(\frac{d}{e}\right) \frac{1}{e^{2k}} = P_k(m, d),$$

where μ denotes the Möbius function.

Proof. We just prove the first formula as the second formula can be proved in a similar manner. As $\mu(e)/e^{2k}$ is a multiplicative function of the positive integer e, and $\mu(p) = -1$ and $\mu(p^2) = \mu(p^3) = \cdots = 0$ for any prime p, we have

$$\sum_{e \mid m} \frac{\mu(e)}{e^{2k}} = \prod_{p^{\nu_p(m)} \parallel m} \left(1 + \frac{\mu(p)}{p^{2k}} + \frac{\mu(p^2)}{p^{4k}} + \cdots + \frac{\mu(p^{\nu_p(m)})}{p^{2\nu_p(m)}}\right) = \prod_{p \mid m} \left(1 - \frac{1}{p^{2k}}\right).$$

The asserted formula now follows by Definition 3.2. □

Proposition 4.3. *Let $k, m \in \mathbb{N}$. Then*

$$\sum_{\substack{n=1 \\ (n,m)=1}}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-1} B_{2k} \pi^{2k}}{(2k)!} P_k(m).$$

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Proof. Appealing to (1.3) and Proposition 4.2, we obtain

$$\begin{aligned}\sum_{\substack{n=1\\(n,m)=1}}^{\infty} \frac{1}{n^{2k}} &= \sum_{n=1}^{\infty}\Big(\sum_{e|(n,m)} \mu(e)\Big)\frac{1}{n^{2k}} \\ &= \sum_{e|m} \mu(e) \sum_{\substack{n=1\\ e|n}}^{\infty} \frac{1}{n^{2k}} \\ &= \sum_{e|m} \mu(e) \frac{(-1)^{k-1}2^{2k-1}B_{2k}\pi^{2k}}{e^{2k}(2k)!} \\ &= \frac{(-1)^{k-1}2^{2k-1}B_{2k}\pi^{2k}}{(2k)!} \sum_{e|m} \frac{\mu(e)}{e^{2k}} \\ &= \frac{(-1)^{k-1}2^{2k-1}B_{2k}\pi^{2k}}{(2k)!} P_k(m)\end{aligned}$$

as asserted. □

Proposition 4.4. *Let $k \in \mathbb{N}$. Let Δ be a fundamental discriminant. Then*

$$\sum_{n=1}^{\infty}\Big(\frac{\Delta}{n}\Big)\frac{1}{n^{2k}} = \frac{(-1)^{k-1}2^{2k}\pi^{2k}}{(2k)!\,\Delta^{2k}\sqrt{\Delta}} \sum_{r=0}^{2k-1}\binom{2k}{r}\Delta^r B_r S_{2k-r}(\Delta).$$

Proof. Let $r_1, \ldots, r_{\phi(\Delta)/2}$ be the integers such that

$$1 \le r_1 < \cdots < r_{\phi(\Delta)/2} \le \Delta - 1, \quad \Big(\frac{\Delta}{r_1}\Big) = \cdots = \Big(\frac{\Delta}{r_{\phi(\Delta)/2}}\Big) = 1,$$

and $s_1, \ldots, s_{\phi(\Delta)/2}$ the integers such that

$$1 < s_1 < \cdots < s_{\phi(\Delta)/2} < \Delta - 1, \quad \Big(\frac{\Delta}{s_1}\Big) = \cdots = \Big(\frac{\Delta}{s_{\phi(\Delta)/2}}\Big) = -1.$$

We note that $\phi(\Delta) \equiv 0 \pmod 4$ and $(r_m, \Delta) = (s_m, \Delta) = 1$, $r_{\phi(\Delta)/2+1-m} = \Delta - r_m$, $s_{\phi(\Delta)/2+1-m} = \Delta - s_m$ for $m = 1, 2, \ldots, \phi(\Delta)/2$. Appealing to (3.1), the theorem of Navas, Ruiz and Varona (Theorem 1.1) and (3.7), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\Big(\frac{\Delta}{n}\Big)\frac{1}{n^{2k}} &= \sum_{\substack{n=1\\(n,\Delta)=1}}^{\infty}\Big(\frac{\Delta}{n}\Big)\frac{1}{n^{2k}} \\ &= \sum_{m=1}^{\phi(\Delta)/2} \sum_{\substack{n=1\\ n\equiv r_m \,(\mathrm{mod}\,\Delta)}}^{\infty} \frac{1}{n^{2k}} - \sum_{m=1}^{\phi(\Delta)/2} \sum_{\substack{n=1\\ n\equiv s_m \,(\mathrm{mod}\,\Delta)}}^{\infty} \frac{1}{n^{2k}} \\ &= \sum_{m=1}^{\phi(\Delta)/4} \sum_{\substack{n=1\\ n\equiv \pm r_m \,(\mathrm{mod}\,\Delta)}}^{\infty} \frac{1}{n^{2k}} - \sum_{m=1}^{\phi(\Delta)/4} \sum_{\substack{n=1\\ n\equiv \pm s_m \,(\mathrm{mod}\,\Delta)}}^{\infty} \frac{1}{n^{2k}} \\ &= \frac{(-1)^{k-1}2^{2k}\pi^{2k}}{(2k)!\,\Delta} \sum_{m=1}^{\phi(\Delta)/4} \sum_{t=0}^{\Delta-1} B_{2k}(t/\Delta)\cos(2\pi r_m t/\Delta) \\ &\quad - \frac{(-1)^{k-1}2^{2k}\pi^{2k}}{(2k)!\,\Delta} \sum_{m=1}^{\phi(\Delta)/4} \sum_{t=0}^{\Delta-1} B_{2k}(t/\Delta)\cos(2\pi s_m t/\Delta) \\ &= \frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{(2k)!\,\Delta} \sum_{t=0}^{\Delta-1} B_{2k}(t/\Delta) \sum_{\substack{u=1\\(u,\Delta)=1}}^{\Delta-1}\Big(\frac{\Delta}{u}\Big)\cos(2\pi u t/\Delta)\end{aligned}$$

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$$= \frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{(2k)!\,\sqrt{\Delta}} \sum_{t=0}^{\Delta-1} \left(\frac{\Delta}{t}\right) B_{2k}(t/\Delta)$$

$$= \frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{(2k)!\,\sqrt{\Delta}} \sum_{\substack{t=1\\(t,\Delta)=1}}^{\Delta-1} \left(\frac{\Delta}{t}\right) B_{2k}(t/\Delta).$$

By (3.3) and (1.4) we have, for $1 \le t \le \Delta - 1$ with $(t, \Delta) = 1$,

$$\left(\frac{\Delta}{t}\right) = \left(\frac{\Delta}{\Delta - t}\right), \quad B_{2k}(t/\Delta) = B_{2k}((\Delta - t)/\Delta).$$

We remark that when Δ is even we have $(\Delta/2, \Delta) = \Delta/2 \ne 1$ as $\Delta \ge 8$, so $t \ne \Delta/2$. Hence, pairing t and $\Delta - t$, we obtain appealing to Definition 3.3

$$\sum_{\substack{t=1\\(t,\Delta)=1}}^{\Delta-1} \left(\frac{\Delta}{t}\right) B_{2k}(t/\Delta) = 2 \sum_{\substack{1\le t<\Delta/2\\(t,\Delta)=1}} \left(\frac{\Delta}{t}\right) B_{2k}(t/\Delta)$$

$$= 2 \sum_{\substack{1\le t<\Delta/2\\(t,\Delta)=1}} \left(\frac{\Delta}{t}\right) \sum_{r=0}^{2k} \binom{2k}{r} B_r \frac{t^{2k-r}}{\Delta^{2k-r}}$$

$$= \frac{2}{\Delta^{2k}} \sum_{r=0}^{2k} \binom{2k}{r} \Delta^r B_r \sum_{\substack{1\le t<\Delta/2\\(t,\Delta)=1}} \left(\frac{\Delta}{t}\right) t^{2k-r}$$

$$= \frac{2}{\Delta^{2k}} \sum_{r=0}^{2k} \binom{2k}{r} \Delta^r B_r S_{2k-r}(\Delta)$$

$$= \frac{2}{\Delta^{2k}} \sum_{r=0}^{2k-1} \binom{2k}{r} \Delta^r B_r S_{2k-r}(\Delta),$$

as $S_0(\Delta) = 0$. The asserted formula now follows. □

Proposition 4.5. *Let $k \in \mathbb{N}$. Let d be a discriminant. Let f be the conductor of d and $\Delta = d/f^2$ the fundamental discriminant associated with d. Then*

$$\sum_{n=1}^{\infty} \left(\frac{d}{n}\right) \frac{1}{n^{2k}} = \frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{(2k)!\sqrt{\Delta}} H_k(d).$$

Proof. By (3.1) we have

$$\sum_{n=1}^{\infty} \left(\frac{d}{n}\right) \frac{1}{n^{2k}} = \sum_{\substack{n=1\\(n,d)=1}}^{\infty} \left(\frac{d}{n}\right) \frac{1}{n^{2k}}. \tag{4.1}$$

Replacing d by Δf^2 in the right-hand sum in (4.1), and noting that $(n, \Delta f^2) = 1$ is equivalent to $(n, \Delta) = (n, f) = 1$, we deduce that

$$\sum_{n=1}^{\infty} \left(\frac{d}{n}\right) \frac{1}{n^{2k}} = \sum_{\substack{n=1\\(n,\Delta)=1\\(n,f)=1}}^{\infty} \left(\frac{\Delta f^2}{n}\right) \frac{1}{n^{2k}}.$$

By (3.2) we have $(\frac{\Delta f^2}{n}) = (\frac{\Delta}{n})$ for $(n, f) = 1$ so

$$\sum_{n=1}^{\infty} \left(\frac{d}{n}\right) \frac{1}{n^{2k}} = \sum_{\substack{n=1\\(n,\Delta)=1\\(n,f)=1}}^{\infty} \left(\frac{\Delta}{n}\right) \frac{1}{n^{2k}}.$$

By (3.1) we have $(\frac{\Delta}{n}) = 0$ for $(n, \Delta) > 1$, so

$$\sum_{n=1}^{\infty} \left(\frac{d}{n}\right) \frac{1}{n^{2k}} = \sum_{\substack{n=1\\(n,f)=1}}^{\infty} \left(\frac{\Delta}{n}\right) \frac{1}{n^{2k}}.$$

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Hence

$$\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n^{2k}}=\sum_{n=1}^{\infty}\Bigg(\sum_{e\mid(n,f)}\mu(e)\Bigg)\left(\frac{\Delta}{n}\right)\frac{1}{n^{2k}}=\sum_{e\mid f}\mu(e)\sum_{\substack{n=1\\ e\mid n}}^{\infty}\left(\frac{\Delta}{n}\right)\frac{1}{n^{2k}}.$$

Appealing to Propositions 4.1, 4.2 and 4.4, as well as Definition 3.4, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n^{2k}}&=\sum_{e\mid f}\mu(e)\left(\frac{\Delta}{e}\right)\frac{1}{e^{2k}}\sum_{n=1}^{\infty}\left(\frac{\Delta}{n}\right)\frac{1}{n^{2k}}\\&=P_k(f,\Delta)\sum_{n=1}^{\infty}\left(\frac{\Delta}{n}\right)\frac{1}{n^{2k}}\\&=P_k(f,\Delta)\frac{(-1)^{k-1}2^{2k}\pi^{2k}}{(2k)!\Delta^{2k}\sqrt{\Delta}}\sum_{r=0}^{2k-1}\binom{2k}{r}\Delta^{r}B_rS_{2k-r}(\Delta)\\&=\frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{(2k)!\sqrt{\Delta}}H_k(d),\end{aligned}$$

which is the asserted result. □

Proposition 4.6. *Let $k,m\in\mathbb{N}$. Let d be a discriminant. Let f be the conductor of d. Let $\Delta=d/f^2$ be the fundamental discriminant associated with d. Then*

$$\sum_{\substack{n=1\\(n,m)=1}}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n^{2k}}=\frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{(2k)!\sqrt{\Delta}}P_k(m,d)H_k(d).$$

Proof. Appealing to Propositions 4.1, 4.2 and 4.5, we deduce

$$\begin{aligned}\sum_{\substack{n=1\\(n,m)=1}}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n^{2k}}&=\sum_{n=1}^{\infty}\Bigg(\sum_{e\mid(n,m)}\mu(e)\Bigg)\left(\frac{d}{n}\right)\frac{1}{n^{2k}}\\&=\sum_{e\mid m}\mu(e)\sum_{\substack{n=1\\e\mid n}}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n^{2k}}\\&=\sum_{e\mid m}\mu(e)\left(\frac{d}{e}\right)\frac{1}{e^{2k}}\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n^{2k}}\\&=P_k(m,d)\frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{(2k)!\ \sqrt{\Delta}}H_k(d),\end{aligned}$$

which is the asserted result. □

We are now ready to state and prove our main result.

Theorem 4.7. *Let m and h be positive integers with $m\geq 2$ and $a_1,\dots,a_h$ integers satisfying $1\leq a_1<a_2<\cdots<a_h\leq m-1$. Suppose that the set of congruences*

$$n\equiv a_1,\dots,a_h \pmod{m}$$

is discriminantly determined, say by discriminants $d_1,\dots,d_r$ (with no nonempty product $d_{j_1}\cdots d_{j_s}$ ($1\leq j_1<\cdots<j_s\leq r$) equal to a perfect square) and $\epsilon_1=\pm1,\dots,\epsilon_r=\pm1$. Then

$$\begin{aligned}\sum_{\substack{n=1\\ n\equiv a_1,\dots,a_h \,(\mathrm{mod}\, m)}}^{\infty}\frac{1}{n^{2k}}&=\frac{(-1)^{k-1}2^{2k-1-r}B_{2k}\pi^{2k}}{(2k)!}P_k(d_1\cdots d_r)\\&\quad+\frac{(-1)^{k-1}2^{2k-1-r}\pi^{2k}}{(2k)!}\sum_{s=1}^{r}\sum_{1\leq j_1<\cdots<j_s\leq r}\epsilon_{j_1}\cdots\epsilon_{j_s}\frac{H_k(d_{j_1}\cdots d_{j_s})}{\sqrt{\Delta(d_{j_1}\cdots d_{j_s})}}P_k(d_1\cdots d_r,d_{j_1}\cdots d_{j_s}).\end{aligned}$$

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Proof. As $d_1, \ldots, d_r$ are discriminants such that no nonempty product $d_{j_1} \cdots d_{j_s}$ $(1 \le j_1 < \cdots < j_s \le r)$ is a perfect square, we deduce that $d_{j_1} \cdots d_{j_s}$ $(1 \le j_1 < \cdots < j_s \le r)$ is a discriminant. We have

$$\begin{aligned}
\sum_{\substack{n=1 \\ n \equiv a_1, \ldots, a_h \pmod{m}}}^{\infty} \frac{1}{n^{2k}} &= \sum_{\substack{n=1 \\ (\frac{d_1}{n})=\epsilon_1, \ldots, (\frac{d_r}{n})=\epsilon_r}}^{\infty} \frac{1}{n^{2k}} \\
&= \sum_{\substack{n=1 \\ (\frac{d_1}{n})=\epsilon_1, \ldots, (\frac{d_r}{n})=\epsilon_r \\ (n, d_1 \cdots d_r)=1}}^{\infty} \frac{1}{n^{2k}} \\
&= \frac{1}{2^r} \sum_{\substack{n=1 \\ (n, d_1 \cdots d_r)=1}}^{\infty} \prod_{j=1}^{r} \Big(1 + \epsilon_j \Big(\frac{d_j}{n}\Big)\Big) \frac{1}{n^{2k}} \\
&= \frac{1}{2^r} \sum_{\substack{n=1 \\ (n, d_1 \cdots d_r)=1}}^{\infty} \Big(1 + \sum_{s=1}^{r} \sum_{1 \le j_1 < \cdots < j_s \le r} \epsilon_{j_1} \cdots \epsilon_{j_s} \Big(\frac{d_{j_1} \cdots d_{j_s}}{n}\Big)\Big) \frac{1}{n^{2k}} \\
&= \frac{1}{2^r} \sum_{\substack{n=1 \\ (n, d_1 \cdots d_r)=1}}^{\infty} \frac{1}{n^{2k}} + \frac{1}{2^r} \sum_{s=1}^{r} \sum_{1 \le j_1 < \cdots < j_s \le r} \epsilon_{j_1} \cdots \epsilon_{j_s} \sum_{\substack{n=1 \\ (n, d_1 \cdots d_r)=1}}^{\infty} \Big(\frac{d_{j_1} \cdots d_{j_s}}{n}\Big) \frac{1}{n^{2k}}.
\end{aligned}$$

The theorem now follows on appealing to Propositions 4.3 and 4.6. □

5 Examples

In this section we give some special cases of Theorem 4.7.

Theorem 5.1. *Let $k \in \mathbb{N}$. Then*

$$\sum_{\substack{n=1 \\ n \equiv 1,4 \pmod{5}}}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-2} \pi^{2k}}{5^{2k+1} (2k)!} A^-, \qquad \sum_{\substack{n=1 \\ n \equiv 2,3 \pmod{5}}}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-2} \pi^{2k}}{5^{2k+1} (2k)!} A^+,$$

where

$$A^{\pm} := 5(5^{2k} - 1) B_{2k} \pm 2 \sum_{r=0}^{2k-1} \binom{2k}{r} 5^r (2^{2k-r} - 1) B_r \sqrt{5}.$$

Proof. The congruences $n \equiv 1, 4 \pmod{5}$ are discriminantly determined as $n \equiv 1, 4 \pmod{5} \Leftrightarrow (\frac{5}{n}) = +1$. By Theorem 4.7 we obtain

$$\sum_{\substack{n=1 \\ n \equiv 1,4 \pmod{5}}}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-2} B_{2k} \pi^{2k}}{(2k)!} P_k(5) + \frac{(-1)^{k-1} 2^{2k-2} \pi^{2k}}{(2k)!} \frac{H_k(5)}{\sqrt{5}} P_k(5, 5).$$

By Definition 3.2 we have $P_k(5) = \frac{5^{2k} - 1}{5^{2k}}$ and $P_k(1, 5) = P_k(5, 5) = 1$. By Definition 3.4 we have

$$H_k(5) = \frac{2}{5^{2k}} \sum_{r=0}^{2k-1} \binom{2k}{r} 5^r (1 - 2^{2k-r}) B_r.$$

The first asserted formula now follows. The second formula follows in a similar manner. □

Taking $k = 1$ and $k = 2$ in Theorem 5.1, we obtain the following corollary.

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Corollary 5.2. *The following four evaluations hold:*

$$\sum_{\substack{n=1\\ n\equiv 1,4 \ (\mathrm{mod}\ 5)}}^{\infty} \frac{1}{n^2} = \frac{2(5+\sqrt{5})\pi^2}{125}, \qquad \sum_{\substack{n=1\\ n\equiv 2,3 \ (\mathrm{mod}\ 5)}}^{\infty} \frac{1}{n^2} = \frac{2(5-\sqrt{5})\pi^2}{125},$$

and

$$\sum_{\substack{n=1\\ n\equiv 1,4 \ (\mathrm{mod}\ 5)}}^{\infty} \frac{1}{n^4} = \frac{4(13+5\sqrt{5})\pi^4}{9375}, \qquad \sum_{\substack{n=1\\ n\equiv 2,3 \ (\mathrm{mod}\ 5)}}^{\infty} \frac{1}{n^4} = \frac{4(13-5\sqrt{5})\pi^4}{9375}.$$

Theorem 5.3. *Let $k \in \mathbb{N}$. Then*

$$\sum_{\substack{n=1\\ n\equiv 1,7 \ (\mathrm{mod}\ 8)}}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}\pi^{2k}}{2^{4k+3}(2k)!}B^{-}, \qquad \sum_{\substack{n=1\\ n\equiv 3,5 \ (\mathrm{mod}\ 8)}}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}\pi^{2k}}{2^{4k+3}(2k)!}B^{+},$$

where

$$B^{\pm} := 2^{4k+1}(2^{2k}-1)B_{2k} \pm \sum_{r=0}^{2k-1}\binom{2k}{r}2^{3r}(3^{2k-r}-1)B_r\sqrt{2}.$$

Proof. The congruences $n \equiv 1,7 \pmod 8$ are discriminantly determined as $n \equiv 1,7 \pmod 8 \Leftrightarrow (\frac{8}{n}) = +1$. By Theorem 4.7 with $r = 1$, $d_1 = 8$ and $\epsilon_1 = +1$, we obtain the first formula. For the second formula we choose $r = 1$, $d_1 = 8$ and $\epsilon_1 = -1$. □

Taking $k = 1$ and $k = 2$ in Theorem 5.3, we obtain the following corollary.

Corollary 5.4. *We have*

$$\sum_{\substack{n=1\\ n\equiv 1,7 \ (\mathrm{mod}\ 8)}}^{\infty} \frac{1}{n^2} = \frac{(2+\sqrt{2})\pi^2}{32}, \qquad \sum_{\substack{n=1\\ n\equiv 3,5 \ (\mathrm{mod}\ 8)}}^{\infty} \frac{1}{n^2} = \frac{(2-\sqrt{2})\pi^2}{32},$$

and

$$\sum_{\substack{n=1\\ n\equiv 1,7 \ (\mathrm{mod}\ 8)}}^{\infty} \frac{1}{n^4} = \frac{(16+11\sqrt{2})\pi^4}{3072}, \qquad \sum_{\substack{n=1\\ n\equiv 3,5 \ (\mathrm{mod}\ 8)}}^{\infty} \frac{1}{n^4} = \frac{(16-11\sqrt{2})\pi^4}{3072}.$$

Theorem 5.5. *Let $k \in \mathbb{N}$. Then*

$$\sum_{\substack{n=1\\ n\equiv 1,11 \ (\mathrm{mod}\ 12)}}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}\pi^{2k}}{2^{2k+2}3^{2k+1}(2k)!}C^{-}, \qquad \sum_{\substack{n=1\\ n\equiv 5,7 \ (\mathrm{mod}\ 12)}}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}\pi^{2k}}{2^{2k+2}3^{2k+1}(2k)!}C^{+},$$

where

$$C^{\pm} := 2^{2k}3(2^{2k}-1)(3^{2k}-1)B_{2k} \pm \sum_{r=0}^{2k-1}\binom{2k}{r}2^{2r}3^{r}(5^{2k-r}-1)B_r\sqrt{3}.$$

Proof. The congruences $n \equiv 1,11 \pmod{12}$ are discriminantly determined as $n \equiv 1,11 \pmod{12} \Leftrightarrow (\frac{12}{n}) = +1$. By Theorem 4.7 with $r = 1$, $d_1 = 12$ and $\epsilon_1 = +1$, we obtain the first formula. For the second formula we choose $r = 1$, $d_1 = 12$ and $\epsilon_1 = -1$. □

Taking $k = 1$ and $k = 2$ in Theorem 5.5, we obtain the following corollary.

Corollary 5.6. *The following four evaluations hold:*

$$\sum_{\substack{n=1\\ n\equiv 1,11 \ (\mathrm{mod}\ 12)}}^{\infty} \frac{1}{n^2} = \frac{(2+\sqrt{3})\pi^2}{36}, \qquad \sum_{\substack{n=1\\ n\equiv 5,7 \ (\mathrm{mod}\ 12)}}^{\infty} \frac{1}{n^2} = \frac{(2-\sqrt{3})\pi^2}{36},$$

and

$$\sum_{\substack{n=1\\ n\equiv 1,11 \ (\mathrm{mod}\ 12)}}^{\infty} \frac{1}{n^4} = \frac{(40+23\sqrt{3})\pi^4}{7776}, \qquad \sum_{\substack{n=1\\ n\equiv 5,7 \ (\mathrm{mod}\ 12)}}^{\infty} \frac{1}{n^4} = \frac{(40-23\sqrt{3})\pi^4}{7776}.$$

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Theorem 5.7. *Let $k \in \mathbb{N}$. Then*

$$\sum_{\substack{n=1 \\ n\equiv 1,9 \pmod{10}}}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}\pi^{2k}}{4\cdot 5^{2k+1}(2k)!}D^-, \qquad \sum_{\substack{n=1 \\ n\equiv 3,7 \pmod{10}}}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}\pi^{2k}}{4\cdot 5^{2k+1}(2k)!}D^+,$$

where

$$D^{\pm} := 5(2^{2k}-1)(5^{2k}-1)B_{2k} \pm 2(2^{2k}+1)\sum_{r=0}^{2k-1}\binom{2k}{r}(2^{2k-r}-1)5^r B_r\sqrt{5}.$$

Proof. As $n \equiv 1, 9 \pmod{10} \Leftrightarrow (\frac{20}{n}) = +1$ we choose $r = 1$, $d_1 = 20$ and $\epsilon_1 = +1$ in Theorem 4.7 to obtain the first formula. For the second formula we choose $r = 1$, $d_1 = 20$ and $\epsilon_1 = -1$. □

Taking $k = 1$ and $k = 2$ in Theorem 5.7, we obtain the following corollary.

Corollary 5.8. *The following four evaluations hold:*

$$\sum_{\substack{n=1 \\ n\equiv 1,9 \pmod{10}}}^{\infty} \frac{1}{n^2} = \frac{(3+\sqrt{5})\pi^2}{50}, \qquad \sum_{\substack{n=1 \\ n\equiv 3,7 \pmod{10}}}^{\infty} \frac{1}{n^2} = \frac{(3-\sqrt{5})\pi^2}{50},$$

and

$$\sum_{\substack{n=1 \\ n\equiv 1,9 \pmod{10}}}^{\infty} \frac{1}{n^4} = \frac{(39+17\sqrt{5})\pi^4}{7500}, \qquad \sum_{\substack{n=1 \\ n\equiv 3,7 \pmod{10}}}^{\infty} \frac{1}{n^4} = \frac{(39-17\sqrt{5})\pi^4}{7500}.$$

Theorem 5.9. *Let $k \in \mathbb{N}$. Then*

$$\sum_{\substack{n=1 \\ n\equiv 1,23 \pmod{24}}}^{\infty} \frac{1}{n^{2k}} = E + F\sqrt{2} + G\sqrt{3} + H\sqrt{6},$$

$$\sum_{\substack{n=1 \\ n\equiv 5,19 \pmod{24}}}^{\infty} \frac{1}{n^{2k}} = E - F\sqrt{2} - G\sqrt{3} + H\sqrt{6},$$

$$\sum_{\substack{n=1 \\ n\equiv 7,17 \pmod{24}}}^{\infty} \frac{1}{n^{2k}} = E + F\sqrt{2} - G\sqrt{3} - H\sqrt{6},$$

$$\sum_{\substack{n=1 \\ n\equiv 11,13 \pmod{24}}}^{\infty} \frac{1}{n^{2k}} = E - F\sqrt{2} + G\sqrt{3} - H\sqrt{6},$$

where

$$E := \frac{(-1)^{k-1}(2^{2k}-1)(3^{2k}-1)B_{2k}\pi^{2k}}{8\cdot 3^{2k}(2k)!},$$

$$F := \frac{(-1)^{k-1}(3^{2k}+1)\pi^{2k}}{2^{4k+4}3^{2k}(2k)!}\sum_{r=0}^{2k-1}\binom{2k}{r}2^{3r}(1-3^{2k-r})B_r,$$

$$G := \frac{(-1)^{k-1}\pi^{2k}}{2^{2k+3}3^{2k+1}(2k)!}\sum_{r=0}^{2k-1}\binom{2k}{r}2^{2r}3^r(1-5^{2k-r})B_r,$$

$$H := \frac{(-1)^{k-1}\pi^{2k}}{2^{4k+4}3^{2k+1}(2k)!}\sum_{r=0}^{2k-1}\binom{2k}{r}2^{3r}3^r(1+5^{2k-r}-7^{2k-r}-11^{2k-r})B_r.$$

Proof. We have

$$n \equiv 1, 23 \pmod{24} \iff \left(\frac{8}{n}\right) = \left(\frac{12}{n}\right) = 1,$$

$$n \equiv 5, 19 \pmod{24} \iff \left(\frac{8}{n}\right) = \left(\frac{12}{n}\right) = -1,$$

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$$n \equiv 7, 17 \pmod{24} \iff \left(\frac{8}{n}\right) = 1,\ \left(\frac{12}{n}\right) = -1,$$
$$n \equiv 11, 13 \pmod{24} \iff \left(\frac{8}{n}\right) = -1,\ \left(\frac{12}{n}\right) = 1.$$

The asserted formulae follow by taking

$$\begin{aligned} &r = 2, \quad d_1 = 8, \quad d_2 = 12, \quad \epsilon_1 = 1, \quad \epsilon_2 = 1,\\ &r = 2, \quad d_1 = 8, \quad d_2 = 12, \quad \epsilon_1 = -1, \quad \epsilon_2 = -1,\\ &r = 2, \quad d_1 = 8, \quad d_2 = 12, \quad \epsilon_1 = 1, \quad \epsilon_2 = -1,\\ &r = 2, \quad d_1 = 8, \quad d_2 = 12, \quad \epsilon_1 = -1, \quad \epsilon_2 = 1,\end{aligned}$$

respectively, in Theorem 4.7. □

Taking $k = 1$ in Theorem 5.9, we obtain the following corollary.

Corollary 5.10. *The following four evaluations hold:*

$$\sum_{\substack{n=1\\ n\equiv 1,23 \pmod{24}}}^{\infty} \frac{1}{n^2} = \frac{(8 + 5\sqrt{2} + 4\sqrt{3} + 3\sqrt{6})\pi^2}{288},$$
$$\sum_{\substack{n=1\\ n\equiv 5,19 \pmod{24}}}^{\infty} \frac{1}{n^2} = \frac{(8 - 5\sqrt{2} - 4\sqrt{3} + 3\sqrt{6})\pi^2}{288},$$
$$\sum_{\substack{n=1\\ n\equiv 7,17 \pmod{24}}}^{\infty} \frac{1}{n^2} = \frac{(8 + 5\sqrt{2} - 4\sqrt{3} - 3\sqrt{6})\pi^2}{288},$$
$$\sum_{\substack{n=1\\ n\equiv 11,13 \pmod{24}}}^{\infty} \frac{1}{n^2} = \frac{(8 - 5\sqrt{2} + 4\sqrt{3} - 3\sqrt{6})\pi^2}{288}.$$

Theorem 5.11. *Let $k \in \mathbb{N}$. Then*

$$\sum_{\substack{n=1\\ n\equiv 1,3,9,19,25,27 \pmod{28}}}^{\infty} \frac{1}{n^{2k}} = J + K\sqrt{7}, \qquad \sum_{\substack{n=1\\ n\equiv 5,11,13,15,17,23 \pmod{28}}}^{\infty} \frac{1}{n^{2k}} = J - K\sqrt{7},$$

where

$$J := \frac{(-1)^{k-1}(2^{2k}-1)(7^{2k}-1)B_{2k}\pi^{2k}}{2^2 7^{2k}(2k)!},$$
$$K := \frac{(-1)^{k-1}\pi^{2k}}{2^{2k+2}7^{2k+1}(2k)!}\sum_{r=0}^{2k-1}\binom{2k}{r}2^{2r}7^r(1^{2k-r} + 3^{2k-r} - 5^{2k-r} + 9^{2k-r} - 11^{2k-r} - 13^{2k-r})B_r.$$

Proof. We have

$$n \equiv 1, 3, 9, 19, 25, 27 \pmod{28} \iff \left(\frac{28}{n}\right) = 1$$

and

$$n \equiv 5, 11, 13, 15, 17, 23 \pmod{28} \iff \left(\frac{28}{n}\right) = -1$$

so the congruences are discriminantly determined and we can apply Theorem 4.7 with $d = \Delta = 28$ and $f = 1$. □

Taking $k = 1$ in Theorem 5.11 we obtain the following result.

Corollary 5.12. *The following two evaluations hold:*

$$\sum_{\substack{n=1\\ n\equiv 1,3,9,19,25,27 \pmod{28}}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{49}(3 + \sqrt{7}), \qquad \sum_{\substack{n=1\\ n\equiv 5,11,13,15,17,23 \pmod{28}}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{49}(3 - \sqrt{7}).$$

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