1 Enumeration with restricted components.

Let $F(x, y)$ be the bivariate OGF for unlabelled objects whose components are all distinct. Then we have

$$\begin{aligned} F(x) &= \prod_{j\geq 1}(1+yx^j)^{c_j} \\ &= \exp\left(-\sum_{j\geq 1} c_j \ln(1+yx^j)^{-1}\right) \\ &= \exp\left(-\sum_{j\geq 1} c_j \sum_{k\geq 1}(-1)^k y^k x^{jk}/k\right) \\ &= \exp\left(\sum_{k\geq 1}(-1)^{k-1} y^k C(x^k)/k\right). \end{aligned}$$

Example 1 *Let a_n be the number of polynomials which do not contain repeated factors. Then its OGF is given by*

$$A(x) = \prod_{j\geq 1}(1+x^j)^{I_j} = \exp\left(\sum_{k\geq 1}(-1)^{k-1} I(x^k)/k\right).$$

Example 2 *Let $p_d(n)$ be the number of partitions of n into distinct parts, and $p_o(n)$ be the number of partitions of n into odd parts. Then we have the following expressions for the OGF's*

$$\begin{aligned} \sum_{n\geq 0} p_d(n)x^n &= \prod_{j\geq 1}\left(1+x^j\right) \\ \sum_{n\geq 0} p_o(n)x^n &= \prod_{j\geq 1}\frac{1}{1-x^{2j-1}} \end{aligned}$$

We note

$$\begin{aligned} \prod_{j\geq 1}\frac{1}{1-x^{2j-1}} &= \prod_{j\geq 1}\frac{1}{1-x^{2j-1}}\frac{1-x^{2j}}{1-x^{2j}} \\ &= \prod_{j\geq 1}\frac{1-x^{2j}}{1-x^j} \\ &= \prod_{j\geq 1}\left(1+x^j\right) \end{aligned}$$

Hence we have $p_d(n) = p_o(n)$.

2 Dirichlet generating function and Möbius inversion.

The Dirichlet generating function of a sequence $\{a(n)\}_{n\geq 1}$ is

$$A(s) = \sum_{n\geq 1} \frac{a(n)}{n^s}$$

The Dirichlet generating function of the sequence of 1's is

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \cdots + \frac{1}{n^s} + \cdots$$

Let

$$\left(\sum_{n\geq 1} \frac{a(n)}{n^s}\right)\left(\sum_{n\geq 1} \frac{b(n)}{n^s}\right) = \sum_{n\geq 1} \frac{c(n)}{n^s},$$

then

$$c(n) = \sum_{rs=n} a(r)b(s) = \sum_{r|n} a(r)b(n/r).$$

We note

$$\zeta^2(s) = \sum_{n\geq 1} n^{-s} \sum_{r|n} 1 \times 1 = \sum_{n\geq 1} d(n)n^{-s},$$

where $d(n)$ denotes the number of divisors of n.

We note that $d(n)$ has the following important property

$$d(mn) = d(m)d(n) \text{ for any positive integers } m \text{ and } n \text{ with } gcd(m,n) = 1.$$

Such a function is called *multiplicative*. Since a positive integer has a unique prime factorization

$$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k},$$

if $f(n)$ is a multiplicative function, then we have

$$f(n) = f(p_1^{n_1}) f(p_2^{n_2}) \cdots f(p_k^{n_k}),$$

and hence

$$\begin{aligned}\sum_{n\geq 1} f(n)n^{-s} &= \sum_{n_1,n_2,\ldots} \sum_{p_1,p_2,\ldots} f(p_1^{n_1}) f(p_2^{n_2}) \cdots p_1^{-sn_1} p_2^{-sn_2} \cdots \\ &= \prod_{\text{prime } p} (1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots).\end{aligned}$$

For example, we have

$$\begin{aligned}\zeta(s) = \sum_{n\geq 1} n^{-s} &= \prod_p \frac{1}{1-p^{-s}},\\ \zeta^2(s) = \sum_{n\geq 1} d(n)n^{-s} &= \prod_p \frac{1}{(1-p^{-s})^2}.\end{aligned}$$

We note the Möbius function $\mu(n)$ is multiplicative. Hence

$$\sum_{n\geq 1} \mu(n)n^{-s} = \prod_p (1-p^{-s}) = \frac{1}{\zeta(s)}.$$

Suppose two sequences $\{f(n)\}$ and $\{g(n)\}$ are related by the relation

$$f(n) = \sum_{r|n} g(r), \quad n \geq 1,$$

how do we express $g(n)$ in terms of $f(n)$?

Well, using Dirichlet generating function, the above relation is equivalent to

$$F(s) = G(s)\zeta(s).$$

Hence

$$G(s) = F(s)/\zeta(s) = F(s)\sum_{n\geq 1} \mu(n)n^{-s},$$

or

$$g(n) = \sum_{r|n} \mu(r) f(n/r).$$