Solutions Manual for

ROBERT B. COOPER'S
Introduction to
QUEUEING THEORY
Second Edition

by **Børge Tilt**

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Foreword

As stated by Børge Tilt in his Preface, he has written this solutions manual with the objective of maximal utility for the instructor, which requires that the solutions be presented in a detailed and orderly fashion. As the reader will see, this objective has been easily met.

I should add that Dr. Tilt's contribution goes far beyond merely solving a large number of (often difficult) exercises. He made many helpful suggestions and corrections, not only to the exercises, but to the text as well. There is no question that his efforts have considerably enhanced the value of my book, and for this I am deeply grateful.

Robert B. Cooper

PREFACE

It is my hope that this solutions manual will prove a valuable supplement to Robert B. Cooper's Introduction to Queueing Theory, second edition, and that both teachers and students will benefit from its availibility. As far as the teacher is concerned, it offers substantial savings of time, if nothing else. Solutions to the exercises have been written from the point of view that to be of maximal usefulness to the teacher they should contain a detailed and orderly exposition of steps in the solution.

The given solutions are based almost exclusively on information in the book, and such knowledge of mathematics that may reasonably be assumed on the part of the student.

On some of the exercises I have profited from discussions with the author. However, any errors or inaccuracies that may exist are solely my responsibility.

Børge Tilt, Copenhagen, Denmark

November 1980

Chapter 1, Exercise 1

'In what ways...'

No comment.

Chapter 1, Exercise 2

'List some applications of the Erlang loss model.'

No comment.

Chapter 1, Exercise 3

'Discuss ways...'

No comment.

Chapter 1, Exercise 4

'Extend the heuristic conservation-of-flow argument...'

In the present case only one-step state transitions are effected by arrivals or service completions. The conservation-of-flow principle then leads to the conclusion that in the long run there will be the same number of transitions $E_j \to E_{j+1}$ and $E_{j+1} \to E_j$, per unit time. Also, under certain conditions, the mean rate of transitions $E_j \to E_{j+1}$ is λP_j (true for Poisson arrivals), and the mean rate of transitions $E_{j+1} \to E_j$ is $(j+1)\tau^{-1} P_{j+1}$ for $j = 0, 1, \ldots, s-1$ and $s\tau^{-1} P_{j+1}$ for $j = s, s+1, \ldots$ (true for exponential service times). If these conditions hold, the conservation-of-flow equations extending equations (1.1) become

$$\lambda P_j = \begin{cases} (j+1)\tau^{-1} P_{j+1} & (j = 0, 1, \ldots, s-1), \\ s\tau^{-1} P_{j+1} & (j = s, s+1, \ldots). \end{cases} \qquad (*)$$

In the following eq. (*) are supposed to hold.

(Chap. 1, Ex. 4a)

$\boxed{a}$ Recurrent solution of (*) results in

$$P_j = \begin{cases} \frac{a^j}{j!} P_0 & (j = 1, 2, \ldots, s-1), \\ \frac{a^j}{s!\, s^{j-s}} P_0 & (j = s, s+1, \ldots), \end{cases} \tag{1}$$

where $a = \lambda\tau$. Using (1) and $\sum_{j=0}^{\infty} P_j = 1$, we find

$$P_0 = \left(\sum_{k=0}^{s-1} \frac{a^k}{k!} + \frac{a^s}{s!(1 - a/s)} \right)^{-1}. \tag{2}$$

$\boxed{b}$ In calculating (2) we set $\sum_{j=0}^{\infty} \left(\frac{a}{s}\right)^j = 1/(1 - a/s)$. However, this presupposes $a < s$. If $a \geq s$, then eq. (2) is incorrect and should be replaced by $P_0 = 0$.

The offered load $a = \lambda\tau$ equals the number of servers that on the average (in the long run) must be in service in order to dispose of the work load in such a way that customer orders do not pile up infinitely. Thus $a < s$ is necessary and sufficient for disposal of the work load without infinite delays.

$\boxed{c}$ $$C(s,a) = \sum_{j=s}^{\infty} P_j = P_0 \sum_{j=s}^{\infty} \frac{a^j}{s!\, s^{j-s}} = P_0 \frac{a^s}{s!(1 - a/s)}.$$

By (2),

$$C(s,a) = \frac{\frac{a^s}{s!(1 - a/s)}}{\sum_{k=0}^{s-1} \frac{a^k}{k!} + \frac{a^s}{s!(1 - a/s)}}. \tag{3}$$

$\boxed{d}$ In general, when $s > 1$, $C(s,a)$ will depend on the order in which waiting customers are selected from the queue. For example, if the customer with the shortest service time is always selected for service, then $C(s,a)$ will be different than when the converse policy is adopted. It is therefore necessary to specify service order. The usual assumption, making the model amenable to analysis, is that customers are selected without regard to service time required, as in

(Chap. 1, Ex. 4d)

order-of-arrival service. Without this assumption, the conservation-of-flow equations will not hold, even with Poisson arrivals and exponential service times.

$\boxed{e}$ Clearly, $p_j = P_{s+j}/\sum_{k=0}^{\infty} P_{s+k} = P_{s+j}/C(s,a)$ for $j = 0,1,\ldots$. Hence, when $a < s$, by (1), (2) and (3),

$$p_j = (1-\tfrac{a}{s})(\tfrac{a}{s})^j = (1-\rho)\rho^j \quad (j = 0,1,\ldots) \tag{4}$$

where $\rho = a/s$.

$\boxed{f}$

$$\hat{p}_j = \frac{P_{s+k+j}}{\sum_{i=0}^{\infty} P_{s+k+i}} = \frac{\frac{a^{s+k+j}}{s!\,s^{k+j}} P_0}{\sum_{i=0}^{\infty} \frac{a^{s+k+i}}{s!\,s^{k+i}} P_0}$$

$$= \frac{\frac{a^j}{s^j}}{\sum_{i=0}^{\infty} \frac{a^i}{s^i}} = (1-\tfrac{a}{s})(\tfrac{a}{s})^j.$$

Thus,

$$\hat{p}_j = (1-\rho)\rho^j = p_j, \qquad (j = 0,1,\ldots)$$

independently of k.

$\boxed{g}$ Assume $s = 1$. By (1)

$$P_j = a^j P_0 \qquad (j = 0,1,\ldots),$$

where $P_0 = (1 + a + a^2 + \ldots)^{-1} = 1 - a$, for $a < s = 1$. Since $\rho = a$,

$$P_j = p_j = (1-\rho)\rho^j \qquad (j = 0,1,\ldots).$$

Finally, by (3),

$$C(1,a) = \frac{a/(1-a)}{1 + a/(1-a)} = a.$$

$\square$

Chapter 1, Exercise 5

'Consider the so-called loss-delay system ...'

As in Exercise 4, we assume that the mean number of transitions $E_j \to E_{j+1}$ and $E_{j+1} \to E_j$ will be estimated correctly. That is the case with Poisson arrivals and exponential service times, respectively.

$\boxed{a}$ The conservation-of-flow argument leads to

$$\lambda P_j = \begin{cases} (j+1)\tau^{-1} P_{j+1} & (j = 0, 1, \ldots, s-1), \\ s\tau^{-1} P_{j+1} & (j = s, \ldots, s+n). \end{cases}$$

Hence,

$$P_j = \begin{cases} \frac{a^j}{j!} P_0 & (j = 1, 2, \ldots, s-1), \\ \frac{a^j}{s!\, s^{j-s}} P_0 & (j = s, \ldots, s+n), \end{cases}$$

where $a = \lambda\tau$, and, for all a, $P_0 = \left(\sum_{k=0}^{s-1} \frac{a^k}{k!} + \frac{a^s}{s!} \sum_{i=0}^{n} \left(\frac{a}{s}\right)^i\right)^{-1}$.

$\boxed{b}$ $\underline{n = 0:}$

$$P_j = \frac{a^j}{j!} P_0 \quad (j = 0, 1, \ldots, s),$$

$$P_0 = \left(\sum_{k=0}^{s} \frac{a^k}{k!}\right)^{-1}.$$

$$B(s,a) = P_s = \frac{a^s/s!}{\sum_{k=0}^{s} a^k/k!}. \qquad (1.6)$$

$\underline{n = \infty:}$

$$P_j = \begin{cases} \frac{a^j}{j!} P_0 & (j = 1, 2, \ldots, s-1), \\ \frac{a^j}{s!\, s^{j-s}} P_0 & (j = s, s+1, \ldots), \end{cases}$$

$$P_0 = \left(\sum_{k=0}^{s-1} \frac{a^k}{k!} + \frac{a^s}{s!\,(1 - a/s)}\right)^{-1}.$$

$$C(s,a) = \sum_{i=0}^{\infty} P_{s+i} = \frac{a^s/[s!\,(1 - a/s)]}{\sum_{k=0}^{s-1} a^k/k! + a^s/[s!\,(1-a/s)]}.$$

$\boxed{c}$ When $n < \infty$, there is no restriction on a (the sums involved in the formula for P_0 are finite). When $n = \infty$, the restriction is $a < s$ as discussed in Exercise 4. $\square$

Chapter 1, Exercise 6

'Consider a queueing model with two servers and one waiting position...'

a

$$\lambda P_0 = 1\tau^{-1} P_1,$$
$$\lambda P_1 = 2\tau^{-1} P_2,$$
$$p\lambda P_2 = 2\tau^{-1} P_3.$$

b

$$P_1 = a P_0, \qquad (a = \lambda\tau)$$
$$P_2 = \frac{a^2}{2} P_0,$$
$$P_3 = p\frac{a^3}{4} P_0,$$
$$P_0 = \left(1 + a + \frac{a^2}{2} + p\frac{a^3}{4}\right)^{-1}.$$

c

$$B = (1-p)P_2 + P_3$$

d Let $\lambda = 2$ and $\tau = 1$, so that $a = 2$, and let $p = 1/2$. Then, by part (b),

$$(P_0, P_1, P_2, P_3) = \left(\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{6}\right),$$

and by part (c),

$$B = \tfrac{1}{3}.$$

e

$$R = 2.00\,\lambda(P_0 + P_1) + 1.00\,p\lambda P_2$$
$$= 2\tfrac{1}{3}\ \$/h.$$

f

$$C = 0.50 \cdot 2 + 0.25[1 \cdot P_1 + 2(P_2 + P_3)]$$
$$= 1\tfrac{1}{3}\ \$/h.$$

Hence,

$$\text{Profit rate} = R - C = 2\tfrac{1}{3} - 1\tfrac{1}{3} = 1\ \$/h.$$

□

Chapter 2, Exercise 1

'In the model considered above, suppose that it costs c dollars...'

$$E\left(\frac{c}{Y}\right) = \sum_{j=1}^{\infty} \frac{c}{j} P\{Y=j\}$$

$$= \sum_{j=1}^{\infty} \frac{c}{j} \frac{j\,P\{X=j\}}{E(X)} \qquad [\text{by } (1.5)]$$

$$= \frac{c}{E(X)}.$$

Chapter 2, Exercise 2

'Consider a population modeled as a pure birth process...', cf. ex. 6

If $N(0)=0$, then, trivially, $P_0(t)=1$ for all t. Assume therefore $N(0)>0$. For notational convenience, let $n=N(0)$. The differential-difference equations (2.3) specialize to

$$\frac{d}{dt} P_j(t) = (j-1)\lambda P_{j-1}(t) - j\lambda P_j(t) \qquad (j=n, n+1, \dots;\ P_{n-1}(t)=0)$$

with initial conditions (2.4): $P_n(0)=1$ and $P_j(0)=0$ for $j\neq n$. Differential-difference equations are given only for $j \geq n$ as evidently $P_j(t)=0$ for $t\geq 0$ when $j<n$.

In the case $n=1$, we have

$$\frac{d}{dt} P_j(t) = (j-1)\lambda P_{j-1}(t) - j\lambda P_j(t) \qquad (j=1,2,\dots;\ P_0(t)=0)$$

with $P_1(0)=1$, $P_j(0)=0$ for $j\neq 1$.

For $j=1$, $\frac{d}{dt}P_1(t) = -\lambda P_1(t)$, so that

$$P_1(t) = e^{-\lambda t}.$$

Hence, $\frac{d}{dt}P_2(t) = \lambda e^{-\lambda t} - 2\lambda P_2(t)$. Applying standard methods in the solution of this linear differential equation one derives

$$P_2(t) = e^{-\lambda t}(1-e^{-\lambda t}).$$

The general formula, obtained by induction, is

$$P_j(t) = e^{-\lambda t}(1-e^{-\lambda t})^{j-1} \qquad (j=1,2,\dots). \qquad \square$$

Chapter 2, Exercise 3

'Consider a birth-and-death process with $\mu_k = 0$ and $\mu_j > 0$ when $j > k$...'

First assume an initial state E_j with $j \geq k$. Then states E_j for $j = 0, 1, \ldots, k-1$ are impossible. A relabeling of the states $(j' = j-k)$ and an application of the above theorem, followed by a reverse relabeling, results in the stated equilibrium distribution $\{P_j\}$.

If, on the other hand, the initial state is E_j with $j < k$, then E_k will be reached eventually (with probability 1) and an application of the theorem leads, again, to the indicated limiting distribution. Thus, unconditionally, the equilibrium distribution is as stated for $S < \infty$. It also follows that $P_j = 0$ for $S = \infty$.

Chapter 2, Exercise 4

'Compound distributions.' — cf. Chap. 5, Ex. 5

$\boxed{a}$ By the theorem of total probability,

$$P\{S_N = k\} = \sum_{n=0}^{\infty} P\{N = n\} P\{\sum_{j=1}^{n} X_j = k\}.$$

The probability generating function of S_N is

$$\begin{aligned} h(z) &= \sum_{k=0}^{\infty} P\{S_N = k\} z^k = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} P\{N=n\} P\{\sum_{j=1}^{n} X_j = k\} z^k \\ &= \sum_{n=0}^{\infty} P\{N = n\} \sum_{k=0}^{\infty} P\{\sum_{j=1}^{n} X_j = k\} z^k \\ &= \sum_{n=0}^{\infty} P\{N = n\} [f(z)]^n = g(f(z)). \end{aligned}$$

$\boxed{b}$ Differentiating $h(z)$ twice,

$$h'(z) = g'(f(z)) \cdot f'(z),$$

$$h''(z) = g'(f(z)) f''(z) + g''(f(z))[f'(z)]^2.$$

Hence,

$$h'(1) = g'(f(1)) f'(1) = g'(1) f'(1),$$

$$h''(1) = g'(f(1)) f''(1) + g''(f(1))[f'(1)]^2 = g'(1) f''(1) + g''(1)[f'(1)]^2.$$

(Chap. 2, Ex. 4 b)

By (4.5),

$$E(S_N) = h'(1) = g'(1)f'(1) = E(N)E(X).$$

By (4.5) and (4.8),

$$\begin{aligned} V(S_N) &= h''(1) + h'(1) - [h'(1)]^2 \\ &= g'(1)f''(1) + g''(1)[f'(1)]^2 + g'(1)f'(1) - [g'(1)]^2[f'(1)]^2 \\ &= g'(1)\left(f''(1) + f'(1) - [f'(1)]^2\right) + \left(g''(1) + g'(1) - [g'(1)]^2\right)[f'(1)]^2 \\ &= E(N)V(X) + V(N)E^2(X) \end{aligned}$$

Chapter 2, Exercise 5

'Let N_1 and N_2 be...' — cf. Ex. 23

Under procedure (a) each of the N_1 balls will be left unmarked with probability x, so by its definition $g(x)$ is the probability that none of the N_1 balls is marked. Similarly, $g(y)$ is the probability that none of the N_2 balls is marked. Hence, under procedure (a), $g(x)g(y)$ is the probability that none of the N_1+N_2 balls will be marked, provided that a ball placed in cell 1 (2) is left unmarked with probability x (y).

Under procedure (b) a ball from either batch will be left unmarked with probability $\frac{1}{2}x + \frac{1}{2}y$. It follows that the probability that none of the N_ν ($\nu = 1,2$) balls is marked is $g\left(\frac{x+y}{2}\right)$. Hence, under procedure (b), $g\left(\frac{x+y}{2}\right)g\left(\frac{x+y}{2}\right)$ is the probability that none of the N_1+N_2 balls will be marked.

We take equivalence to mean that the probability distribution of balls in the two cells is the same for both procedures. If the procedures are equivalent in this sense, then, whatever x and y, the probability that no ball is marked must be the same under both procedures. Thus equivalence implies

$$g(x)\,g(y) = g^2\left(\tfrac{x+y}{2}\right).$$

□

$\boxed{\text{Chapter 2, Exercise 6}}$

'For the model of Exercise 2, Section 2.2, define...' — cf. Ex. 8

$\boxed{a}$ For $n = N(0) = 1$, we have found $P_0(t) = 0$ and

$$\frac{d}{dt} P_j(t) = (j-1)\lambda P_{j-1}(t) - j\lambda P_j(t) \quad (j = 1, 2, \ldots).$$

Hence,

$$\sum_{j=1}^{\infty} \frac{d}{dt} P_j(t) z^j = \sum_{j=1}^{\infty} (j-1)\lambda P_{j-1}(t) z^j - \sum_{j=1}^{\infty} j\lambda P_j(t) z^j,$$

$$\begin{aligned}\frac{\partial}{\partial t} P(z,t) &= \lambda z^2 \sum_{j=2}^{\infty} (j-1) P_{j-1}(t) z^{j-2} - \lambda z \sum_{j=1}^{\infty} j P_j(t) z^{j-1} \\ &= (\lambda z^2 - \lambda z) \sum_{j=1}^{\infty} j P_j(t) z^{j-1} \\ &= \lambda z(z-1) \frac{\partial}{\partial z} P(z,t).\end{aligned}$$

$\boxed{b}$ We shall verify that the above partial differential equation as well as the initial condition $n = N(0) =$ are satisfied by

$$P(z,t) = \frac{z e^{-\lambda t}}{1 - z + z e^{-\lambda t}}.$$

Differentiation results in

$$\frac{\partial}{\partial z} P(z,t) = \frac{e^{-\lambda t}}{(1 - z + z e^{-\lambda t})^2},$$

$$\frac{\partial}{\partial t} P(z,t) = \lambda z(z-1) \frac{e^{-\lambda t}}{(1 - z + z e^{-\lambda t})^2}.$$

It is seen that the expression for $P(z,t)$ satisfies the partial differential equation derived in part (a). The initial condition $P_1(0) = 1$ translates into the requirement $P(z,0) = \sum_{j=0}^{\infty} P_j(0) z^j = z$, which sure enough is met by the proposed expression for $P(z,t)$.

In Exercise 2 it was found that if $n = N(0) = 1$, then

$$P_j(t) = \begin{cases} 0 & (j = 0), \\ e^{-\lambda t}(1 - e^{-\lambda t})^{j-1} & (j = 1, 2, \ldots). \end{cases}$$

To this probability distribution corresponds the generating function

(Chap. 2, Ex. 6 b)

$$g(z,t) = \sum_{j=0}^{\infty} P_j(t) z^j = \sum_{j=1}^{\infty} e^{-\lambda t}(1-e^{-\lambda t})^{j-1} z^j$$

$$= z e^{-\lambda t} \sum_{j=0}^{\infty} [z(1-e^{-\lambda t})]^j = \frac{z e^{-\lambda t}}{1 - z + z e^{-\lambda t}}.$$

Since $g(z,t)$ is identical to the given $P(z,t)$ and because of the one-to-one correspondence between distribution and generating function, it is true that $P(z,t)$ generates the distribution found in Exercise 2.

$\boxed{c}$ In the general case $N(0) = n \geq 1$, it was found in Exercise 2 that $P_0(t) = P_1(t) = \dots = P_{n-1}(t) = 0$ for all t, and

$$\frac{d}{dt} P_j(t) = (j-1)\lambda P_{j-1}(t) - j\lambda P_j(t) \qquad (j = n, n+1, \dots).$$

In the same way as in part (a) for $n=1$, we find

$$\frac{\partial}{\partial t} P_n(z,t) = \lambda z(z-1) \frac{\partial}{\partial z} P_n(z,t),$$

where $P_n(z,t)$ is the generating function for $P_j(t)$, given that $P_n(0) = 1$. We shall verify that

$$P_n(z,t) = P^n(z,t).$$

First, we will show that the proposed solution satisfies the above partial differential equation. This follows from

$$\frac{\partial}{\partial z} P_n(z,t) = n P^{n-1}(z,t) \frac{\partial}{\partial z} P(z,t),$$

$$\frac{\partial}{\partial t} P_n(z,t) = n P^{n-1}(z,t) \frac{\partial}{\partial t} P(z,t),$$

and the previously verified result $\frac{\partial}{\partial t} P(z,t) = \lambda z(z-1) \frac{\partial}{\partial z} P(z,t)$.

The proposed solution also meets the initial condition. For $t = 0$, $P_n(z,0) = P^n(z,0) = z^n$. By definition, $P_n(z,0) = P_0(0) + P_1(0) z^1 + \dots + P_n(0) z^n + \dots$. Equating the coefficients of the two polynomials we see that $P_n(0) = 1$ and $P_j(0) = 0$ for $j \neq n$ as required.

We conclude that $P_n(z,t) = P^n(z,t)$ is the unique solution. The result, $P_n(z,t) = P^n(z,t)$, should not come as a surprise.

(Chap. 2, Ex. 6c)

Clearly, the process with $N(0) = n$ may be interpreted as the sum of n independent processes, each with $N(0) = 1$. That is, the state $N(t)$, given $N(0) = n$, equals the sum of the states $N_1(t), N_2(t), \ldots, N_n(t)$ of independent processes with $N_1(0) = \cdots = N_n(0) = 1$. By a fundamental property of generating functions, then $P_n(z,t) = P^n(z,t)$.

Chapter 2, Exercise 7

'Suppose S_{n_1} has the binomial distribution ...'

S_{n_1} is the sum of n_1 independent Bernoulli variables, and S_{n_2} is the sum of n_2 independent Bernoulli variables, all of which are independent and have parameter p. Hence, the sum $S_{n_1} + S_{n_2}$ is the sum of $n_1 + n_2$ independent Bernoulli variables with parameter p. That is, $S_{n_1} + S_{n_2}$ has the binomial distribution (5.1) with $n = n_1 + n_2$.

Alternatively, the generating functions of S_{n_1} and S_{n_2} are $(q+pz)^{n_1}$ and $(q+pz)^{n_2}$, respectively. Hence, $S_{n_1} + S_{n_2}$ has the generating function $(q+pz)^{n_1}(q+pz)^{n_2} = (q+pz)^{n_1+n_2}$, which is recognized as the generating function of a binomial distribution with parameters $n = n_1 + n_2$ and p.

Chapter 2, Exercise 8

'Verify the parenthetical statement of part c of Exercise 6'

$$P^n(z,t) = \left[\frac{z e^{-\lambda t}}{1 - z(1-e^{-\lambda t})}\right]^n = e^{-n\lambda t} z^n (1 - z(1-e^{-\lambda t}))^{-n}$$

$$= e^{-n\lambda t} z^n \Big(1 + n[z(1-e^{-\lambda t})] + \frac{n(n+1)}{2!}[z(1-e^{-\lambda t})]^2 + \cdots$$
$$\cdots + \frac{(n+r-1)!}{(n-1)!\,r!}[z(1-e^{-\lambda t})]^r + \cdots\Big)$$

$$= e^{-n\lambda t} z^n \sum_{r=0}^{\infty} \binom{n+r-1}{r} (1-e^{-\lambda t})^r z^r$$

$$= \sum_{j=n}^{\infty} \binom{j-1}{j-n} e^{-n\lambda t}(1-e^{-\lambda t})^{j-n} z^j .$$

Thus, $P_j(t) = \binom{j-1}{j-n} e^{-n\lambda t}(1-e^{-\lambda t})^{j-n}$ for $j \geq n$, and $P_j(t) = 0$ for $j < n$. $\square$

Chapter 2, Exercise 9

'Repeat Exercise 7, with the phrase...'

S_{n_ν} ($\nu = 1,2$) is the sum of n_ν independent, identically distributed random variables following a geometric distribution. Hence, $S_{n_1} + S_{n_2}$ is the sum of $n_1 + n_2$ independent, identically distributed random variables with a geometric distribution. Thus, $S_{n_1} + S_{n_2}$ has the negative binomial distribution (5.8) with $n = n_1 + n_2$.

Alternatively, the probability generating function (p.g.f.) of S_{n_1} equals $(p/(1-qz))^{n_1}$ and the p.g.f. of S_{n_2} equals $(p/(1-qz))^{n_2}$. Hence, $S_{n_1} + S_{n_2}$ has the p.g.f. $(p/(1-qz))^{n_1}(p/(1-qz))^{n_2} = (p/(1-qz))^{n_1+n_2}$, which is the p.g.f. of a variable with the negative binomial distribution (5.8) with $n = n_1 + n_2$.

Chapter 2, Exercise 10

'Feller [1971]. Find the distribution function of the length of...'

Let T $(0 \leq T < c)$ be the length of the covering arc.

$$F(t) = P\{T \leq t\} = \left(\frac{t}{c}\right)^2 \qquad (0 \leq t \leq c).$$

$$f(t) = \frac{dF(t)}{dt} = \frac{2t}{c^2} \qquad (0 \leq t \leq c).$$

$$E(T) = \int_0^c t f(t)\,dt = \frac{2}{3}c.$$

Chapter 2, Exercise 11

'Let $X_1, \ldots, X_n$ $(n \geq 2)$ be...'

R is the maximum of the $n-1$ residual variables at $t = X_{(1)}$. $R \leq x$ if and only if all of these $n-1$ exponential variables are less than or equal to x. Hence,

$$P\{R \leq x\} = (1 - e^{-\mu x})^{n-1}.$$

□

Chapter 2, Exercise 12

'Let $X_1, \ldots, X_n$ be a sequence of...'

Suppose S_n is the maximum of n independent exponential variables with mean μ^{-1}. Then $P\{S_n \leq t\} = (1-e^{-\mu t})^n$.

Now, S_n may be decomposed into n successive time intervals of lengths $X_n, X_{n-1}, \ldots, X_1$ such that $\{X_n\}$ is a set of independent exponential variables and X_j has mean $j\mu$. Since $\Sigma X_j = S_n$, $P\{\Sigma X_j \leq t\} = P\{S_n \leq t\} = (1-e^{-\mu t})^n$.

Chapter 2, Exercise 13

'In reliability theory the failure rate function $r(t)$...'

Suppose $F(t)$ is continuous and differentiable. The probability of a failure in $(t, t+\Delta t)$, given that no failure has occurred before t, equals

$$R(t, t+\Delta t) = \frac{F(t+\Delta t) - F(t)}{1 - F(t)},$$

whereby

$$r(t) \equiv \lim_{\Delta t \to 0} \frac{R(t, t+\Delta t)}{\Delta t} = \frac{f(t)}{1 - F(t)}.$$

Thus, $r(t)dt$ has the desired interpretation. If $F(t) = 1 - e^{-\lambda t}$ $(t \geq 0)$, then clearly $r(t) = \lambda$.

Chapter 2, Exercise 14

'Let $X_1, X_2, \ldots, X_n$ be independent exponential random variables...'

By an easy generalization of (5.21), $P\{\min(X_1, X_2, \ldots, X_n) > x\} = e^{-(\Sigma \mu_j)x}$. Thus $Y_i = \min(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ is exponentially distributed with parameter $\Sigma_{j \neq i} \mu_j$. Now write (5.23) as $P\{X_i = \min(X_1, X_2)\} = \mu_i/(\mu_1 + \mu_2)$ $(i = 1, 2)$. Using the fact that X_i and Y_i are independent exponential variables, we find that

$$P\{X_i = \min(X_1, X_2, \ldots, X_n)\} = P\{X_i = \min(X_i, Y_i)\} = \frac{\mu_i}{\mu_i + \sum_{j \neq i} \mu_j} = \frac{\mu_i}{\mu_1 + \mu_2 + \cdots + \mu_n}.$$

$\square$

Chapter 2, Exercise 15

'Let X_1 and X_2 be independent exponential variables.'

Direct proof

Clearly,

$$P\{t < X_i < t+dt, X_i = \min(X_1, X_2)\} = e^{-(\mu_1+\mu_2)t}\mu_i dt.$$

Hence,

$$P\{X_i > t, X_i = \min(X_1, X_2)\} = \int_t^\infty e^{-(\mu_1+\mu_2)x}\mu_i dx = \frac{\mu_i}{\mu_1+\mu_2} e^{-(\mu_1+\mu_2)t}.$$

For $t = 0$,

$$P\{X_i = \min(X_1, X_2)\} = \frac{\mu_i}{\mu_1+\mu_2} \qquad (5.23)$$

Thus,

$$\begin{aligned} P\{X_i > t \mid X_i = \min(X_1, X_2)\} &= \frac{P\{X_i > t, X_i = \min(X_1, X_2)\}}{P\{X_i = \min(X_1, X_2)\}} \\ &= e^{-(\mu_1+\mu_2)t} \\ &= P\{\min(X_1, X_2) > t\}. \end{aligned}$$

Proof by use of Markov property

The Markov property of the two exponential distributions implies

$$P\{X_i = \min(X_1, X_2) \mid \min(X_1, X_2) > t\} = P\{X_i = \min(X_1, X_2)\}.$$

By this and the formula $P\{A|B\} = P\{A\}P\{B|A\}/P\{B\}$,

$$\begin{aligned} P\{\min(X_1, X_2) > t \mid X_i = \min(X_1, X_2)\} &= \frac{P\{\min(X_1, X_2) > t\}\, P\{X_i = \min(X_1, X_2) \mid \min(X_1, X_2) > t\}}{P\{X_i = \min(X_1, X_2)\}} \\ &= P\{\min(X_1, X_2) > t\} \end{aligned}$$

which is the same as

$$P\{X_i > t \mid X_i = \min(X_1, X_2)\} = P\{\min(X_1, X_2) > t\}$$

Generalization to $n \geq 2$ independent exponential variables is straightforward. $\square$

Chapter 2, Exercise 16

'At $t=0$ a customer (the test customer) places a request...'

[a] The departure rate from system, and thus from line into service, equals $s\mu$ as long as any customer is in the waiting line. Hence, $X_1, X_2, \ldots, X_{j+1}$ are independent exponential variables with mean $(s\mu)^{-1}$.

[b] $$E(X) = E\Big(\sum_{i=1}^{j+1} X_i\Big) = \sum_{i=1}^{j+1} E(X_i) = (j+1)(s\mu)^{-1}.$$

[c] $$E(T) = E(X) + \frac{1}{s\mu} + \frac{1}{(s-1)\mu} + \cdots + \frac{1}{\mu} = \frac{1}{\mu}\Big(\frac{j+1}{s} + \sum_{i=1}^{s} \frac{1}{i}\Big)$$

[d] $$P\{X=m\} = \begin{cases} 0 & (m = 1, 2, \ldots, j+1), \\ \frac{1}{s} & (m = 1+j+1,\ 2+j+1, \ldots, s+j+1). \end{cases}$$

[e] $$P = \Big(1 - \frac{1}{s}\Big)\frac{1}{2}.$$

Chapter 2, Exercise 17

'Suppose customers arrive at instants $T_1, T_2, \ldots$'

Clearly, $P_0(t) = 1 - G(t)$, and $P_j(t) = \int_0^t P_{j-1}(t-\xi)\,dG(\xi)$ for $j = 1, 2, \ldots$. Assume $G(x) = 1 - e^{-\lambda x}$. Then we have $P_0(t) = e^{-\lambda t}$, and

$$P_1(t) = \int_0^t P_0(t-\xi)\,dG(\xi) = \int_0^t e^{-\lambda(t-\xi)} \lambda e^{-\lambda\xi}\,d\xi = \lambda t e^{-\lambda t}.$$

It is seen that (5.25) holds for $j=0$ and $j=1$. Suppose it holds for $j = k$, so that $P_k(t) = [(\lambda t)^k/k!]\,e^{-\lambda t}$. Then

$$\begin{aligned} P_{k+1}(t) &= \int_0^t P_k(t-\xi)\,dG(\xi) = \int_0^t \frac{[\lambda(t-\xi)]^k}{k!} e^{-\lambda(t-\xi)} \lambda e^{-\lambda\xi}\,d\xi \\ &= \frac{\lambda^{k+1}}{k!} e^{-\lambda t} \int_0^t (t-\xi)^k\,d\xi = \frac{\lambda^{k+1}}{k!} e^{-\lambda t} \int_0^t y^k\,dy \\ &= \frac{(\lambda t)^{k+1}}{(k+1)!} e^{-\lambda t}. \end{aligned}$$

We conclude that if $G(x) = 1 - e^{-\lambda x}$, then $\{P_j(t)\}$ is the Poisson distribution with parameter λt. $\square$

Chapter 2, Exercise 18

'Prove equation (5.37)' – cf. Ex. 3 of Chap. 5.

$t \leqq y$: The event $I_t > y$ can occur in two mutually exclusive ways: (1) No arrivals in $[0, y]$; (2) An arrival at $\tau \in [0,t]$ and no arrivals in $(\tau, \tau+y]$. Thus

$$P\{I_t > y\} = e^{-\lambda y} + \int_0^t \lambda e^{-\lambda y} d\tau = e^{-\lambda y} + \lambda t e^{-\lambda y}.$$

Hence,

$$P\{I_t \leqq y\} = 1 - e^{-\lambda y} - \lambda t e^{-\lambda y}. \qquad (t \leqq y)$$

$t > y$: The event $I_t > y$ can occur in three mutually exclusive ways: (1) No arrivals in $[0, t]$; (2) An arrival at $\tau \in [0, t-y)$ and no arrivals in $(\tau, t]$; (3) An arrival at $\tau \in [t-y, t]$ and no arrivals in $(\tau, \tau+y]$. Thus

$$\begin{aligned} P\{I_t > y\} &= e^{-\lambda t} + \int_0^{t-y} \lambda e^{-\lambda(t-\tau)} d\tau + \int_{t-y}^{t} \lambda e^{-\lambda y} d\tau \\ &= e^{-\lambda t} + [e^{-\lambda y} - e^{-\lambda t}] + \lambda y e^{-\lambda y} \\ &= e^{-\lambda y} + \lambda y e^{-\lambda y}. \end{aligned}$$

Hence,

$$P\{I_t \leqq y\} = 1 - e^{-\lambda y} - \lambda y e^{-\lambda y} \qquad (t > y)$$

The two equations may be combined into

$$P\{I_t \leqq y\} = 1 - e^{-\lambda y} - \lambda \min(y, t) e^{-\lambda y}. \qquad (5.37)$$

Chapter 2, Exercise 19

'Let $F(x,y)$ be the limiting joint distribution function ...'

The formula

$$F(x,y) \equiv \lim_{t \to \infty} P\{R_t \leqq x, I_t \leqq y\} = 1 - e^{-\lambda x} - \lambda x e^{-\lambda y} \qquad (0 \leqq x \leqq y)$$

may be derived from eq. (7.20) of Chapter 5.

(Chap. 2, Ex. 19)

$\boxed{a}$

$$\lim_{t\to\infty} P\{I_t \le y\} = \lim_{t\to\infty} P\{R_t \le y, I_t \le y\}$$
$$= 1 - e^{-\lambda y} - \lambda y e^{-\lambda y}.$$

$\boxed{b}$

$$\lim_{t\to\infty} P\{R_t > x\} = \lim_{t\to\infty}(1 - P\{R_t \le x\})$$
$$= 1 - \lim_{t\to\infty} P\{R_t \le x\}$$
$$= 1 - \lim_{t\to\infty} P\{R_t \le x, I_t \le \infty\}$$
$$= e^{-\lambda x}.$$

$\boxed{c}$ It may be shown that

$$\lim_{t\to\infty} P\{R_t > x, A_t > y\} = \int_x^\infty \left(\int_{\xi+y}^\infty f(\xi,\eta)\,d\eta\right) d\xi$$

where $f(x,y) = dF(x,y)/dxdy$ is the density function. By differentiation we find $f(x,y) = \lambda^2 e^{-\lambda y}$. Hence,

$$\lim_{t\to\infty} P\{R_t > x, A_t > y\} = \int_x^\infty \left(\int_{\xi+y}^\infty \lambda^2 e^{-\lambda\eta}\,d\eta\right) d\xi$$
$$= \int_x^\infty \lambda e^{-\lambda(\xi+y)}\,d\xi$$
$$= e^{-\lambda x} e^{-\lambda y}.$$

$\boxed{d}$ $$\lim_{t\to\infty} P\{A_t > y\} = \lim_{t\to\infty} P\{R_t > 0, A_t > y\} = e^{-\lambda 0} e^{-\lambda y} = e^{-\lambda y} \qquad (5.34)$$

$$\lim_{t\to\infty} P\{R_t > x, A_t > y\} = e^{-\lambda x} e^{-\lambda y} \qquad [\text{by (c)}]$$
$$= \lim_{t\to\infty} P\{R_t > x\} \lim_{t\to\infty} P\{A_t > y\} \qquad [\text{by (b) \& (5.34)}]$$
$$= \lim_{t\to\infty} \left(P\{R_t > x\} P\{A_t > y\}\right).$$

$\boxed{e}$ $f(x,y) = \lambda^2 e^{-\lambda y}$ is constant on the interval $0 \le x \le y$ for any given y. Thus R_t is uniformly distributed throughout the covering interval. $\square$

Chapter 2, Exercise 20

'A bus shuttles back and forth..'

By Eq. (1.5), the probability that an arbitrary passenger is one of a bus load of j people equals $P\{Y=j\} = jP_j/a$, where $a = \Sigma jP_j$. Evidently, $\Pi_j = P\{Y=j+1\}$. It follows that, for any distribution $\{P_j\}$,

$$\Pi_j = \frac{(j+1)P_{j+1}}{a} \qquad (j=0,1,2,\ldots).$$

The condition, $P_j = \Pi_j$ for all j, is therefore equivalent to

$$\begin{aligned} P_0 &= \Pi_0 = \tfrac{1}{a}P_1, \\ P_1 &= \Pi_1 = \tfrac{2}{a}P_2, \\ &\vdots \\ P_{j-1} &= \Pi_{j-1} = \tfrac{j}{a}P_j, \\ &\vdots \end{aligned}$$

or,

$$\begin{aligned} P_1 &= \tfrac{a}{1}P_0 \\ P_2 &= \tfrac{a^2}{2!}P_0 \\ &\vdots \\ P_j &= \tfrac{a^j}{j!}P_0 \\ &\vdots \end{aligned}$$

As $\Sigma_{j=0}^{\infty} P_j = 1$, we must have $P_0 = (\Sigma_{j=0}^{\infty} \frac{a^j}{j!})^{-1} = e^{-a}$. The inference is

$$P_j = \Pi_j \ (j=0,1,2,\ldots) \Leftrightarrow P_j = \frac{a^j}{j!}e^{-a} \ (j=0,1,2,\ldots).$$

Chapter 2, Exercise 21

'Suppose customers arrive according to a Poisson process.'

$\boxed{a}$ By the theorem of total probability, for $j = 0,1,\ldots,$

$$P\{M=j\} = \int_0^\infty P\{M=j \mid X=t\}\,dH(t) = \int_0^\infty \frac{(\lambda t)^j}{j!}e^{-\lambda t}\,dH(t).$$

(Chap. 2, Ex. 21 a)

We need the conditional means

$$E(M|X) = \lambda X,$$
$$E(M^2|X) = V(M|X) + E^2(M|X) = \lambda X + (\lambda X)^2.$$

Unconditioning, we derive

$$E(M) = E(E(M|X)) = E(\lambda X) = \lambda E(X) = \lambda\tau$$
$$E(M^2) = E(E(M^2|X)) = E(\lambda X + (\lambda X)^2) = \lambda E(X) + \lambda^2 E(X^2)$$
$$= \lambda\tau + \lambda^2(\sigma^2 + \tau^2),$$

$$V(M) = E(M^2) - E^2(M) = \lambda\tau + \lambda^2\sigma^2.$$

$\boxed{b}$

$$d\,P\{X \leqq t, M = j\} = \frac{(\lambda t)^j}{j!} e^{-\lambda t} dH(t),$$
$$d\,P\{X \leqq t \,|\, M = j\} = \frac{d\,P\{X \leqq t, M = j\}}{P\{M = j\}} = \frac{1}{P\{M=j\}} \frac{(\lambda t)^j}{j!} e^{-\lambda t} dH(t),$$

$$E(X|M=j) = \int_0^\infty t\, d\,P\{X \leqq t \,|\, M = j\}$$
$$= \frac{1}{P\{M=j\}} \int_0^\infty t \frac{(\lambda t)^j}{j!} e^{-\lambda t} dH(t)$$
$$= \frac{j+1}{\lambda} \frac{1}{P\{M=j\}} \int_0^\infty \frac{(\lambda t)^{j+1}}{(j+1)!} e^{-\lambda t} dH(t).$$

By part (a) then,

$$E(X|M=j) = \frac{j+1}{\lambda} \frac{P\{M=j+1\}}{P\{M=j\}} \qquad (j = 0, 1, \ldots).$$

$\boxed{c}$ "If" — We assume that $P\{M=j\} = \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau}$ $(j = 0, 1, \ldots)$.

By part (b),

$$E(X|M=j) = \frac{j+1}{\lambda} \frac{P\{M=j+1\}}{P\{M=j\}}$$
$$= \frac{j+1}{\lambda}\left[\frac{(\lambda\tau)^{j+1}}{(j+1)!} e^{-\lambda\tau}\right] \Big/ \left[\frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau}\right]$$
$$= \tau \equiv E(X).$$

(Chap. 2, Ex. 21 c)

"Only if" – We assume that $E(X|M=j)=E(X)\equiv\tau \quad (j=0,1,\ldots)$

By part (b),

$$\tau=\frac{1}{\lambda}\frac{P\{M=1\}}{P\{M=0\}}=\frac{2}{\lambda}\frac{P\{M=2\}}{P\{M=1\}}=\cdots=\frac{j}{\lambda}\frac{P\{M=j\}}{P\{M=j-1\}}\cdots.$$

Hence, $P\{M=j\}=\frac{\lambda\tau}{j}P\{M=j-1\}=\frac{(\lambda\tau)^j}{j!}P\{M=0\}$ for $j\geq 1$. Utilizing $\sum_{j=0}^{\infty}P\{M=j\}=1$ we find $P\{M=0\}=e^{-\lambda\tau}$. Thus,

$$P\{M=j\}=\frac{(\lambda\tau)^j}{j!}e^{-\lambda\tau} \quad (j=0,1,\ldots).$$

$\boxed{d}$ Let t meet the condition $0<H(t)<1$. Then

$$P\{X\leq t|M=j\}=\frac{P\{X\leq t,M=j\}}{P\{M=j\}}=\frac{\int_0^t\frac{(\lambda x)^j}{j!}e^{-\lambda x}dH(x)}{\int_0^\infty\frac{(\lambda x)^j}{j!}e^{-\lambda x}dH(x)}=\frac{\int_0^t x^j e^{-\lambda x}dH(x)}{\int_0^\infty x^j e^{-\lambda x}dH(x)}.$$

Now define

$$A_j=\int_0^t x^j e^{-\lambda x}dH(x)\,;\quad B_j=\int_t^\infty x^j e^{-\lambda x}dH(x).$$

Thus,

$$P\{X\leq t|M=j\}=\frac{A_j}{A_j+B_j} \quad (j=0,1,\ldots).$$

Clearly,

$$A_{j+1}=\int_0^t x^{j+1}e^{-\lambda x}dH(x)\leq\int_0^t t\,x^j e^{-\lambda x}dH(x)=tA_j,$$

$$B_{j+1}=\int_t^\infty x^{j+1}e^{-\lambda x}dH(x)>\int_t^\infty t\,x^j e^{-\lambda x}dH(x)=tB_j.$$

Hence, $A_{j+1}/A_j\leq t<B_{j+1}/B_j$, whereby $A_{j+1}/A_j<B_{j+1}/B_j$. Thus, $B_{j+1}>(A_{j+1}/A_j)B_j$, so that

$$P\{X\leq t|M=j+1\}=\frac{A_{j+1}}{A_{j+1}+B_{j+1}}<\frac{A_{j+1}}{A_{j+1}+(A_{j+1}/A_j)B_j}=\frac{A_j}{A_j+B_j}=P\{X\leq t|M=j\}.$$

This proves that for all t such that $0<H(t)<1$,

$$P\{X\leq t|M=j+1\}<P\{X\leq t|M=j\} \quad (j=0,1,\ldots). \qquad (*)$$

The variable X, given $M=j+1$, stochastically dominates X, given $M=j$.

(Chap. 2, Ex. 21 d)

It remains to be shown that the stochastic dominance implies a higher mean. A simple proof is this:

$$\begin{aligned} E(X|M=j+1) &= \int_0^\infty [1-P\{X\le t|M=j+1\}]\,dt \\ &> \int_0^\infty [1-P\{X\le t|M=j\}]\,dt \qquad [\text{by } (*)] \\ &= E(X|M=j) \qquad (j=0,1,\dots). \end{aligned}$$

We have used the fact that the mean of a nonnegative variable X with distribution function $F(t)=P\{X\le t\}$ is given by the formula $E(X)=\int_0^\infty t\,dF(t)=\int_0^\infty [1-F(t)]\,dt$, which may be proved by integration by parts.

$\boxed{e}$ $H(t)=1-e^{-\mu t}$. Given exponential "service time", the process may be viewed as an exponential race, repeated until the μ-variable wins. By (5.23), the winning probabilities are $\lambda/(\lambda+\mu)$ and $\mu/(\lambda+\mu)$, respectively, for the λ- and μ-variable. We deduce that in this case

$$P\{M=j\} = \left(\frac{\lambda}{\lambda+\mu}\right)^j \frac{\mu}{\lambda+\mu} \qquad (j=0,1,\dots).$$

$\boxed{f}$ $H(t)=1-e^{-\mu t}$. Inserting the above expression for $P\{M=j\}$ into the formula in part (b) we easily derive $E(X|M=j)=(j+1)/(\lambda+\mu)$. Alternatively, if $M=j$, then the "service time" X is composed of $j+1$ intervals resulting from exponential races. By Ex. 15 these intervals are independent, exponential variables with mean $1/(\lambda+\mu)$. Proved!

Chapter 2, Exercise 22

'Customers request service from a group of s servers...'

$\boxed{a}$ $$P = \frac{\lambda}{\lambda+s\mu},$$

$\boxed{b}$ $$P\{N=j\} = \left(\frac{\lambda}{\lambda+s\mu}\right)^j \frac{s\mu}{\lambda+s\mu},$$

$\boxed{c}$ $$P = \frac{s\mu}{\lambda+s\mu}\times\frac{(s-1)\mu}{\lambda+(s-1)\mu}.$$ □

Chapter 2, Exercise 23

'Continuation of Exercise 5'.

In Exercise 5 it was shown that

"If procedures (a) and (b) are equivalent, then $g(x)g(y) = g^2(\frac{x+y}{2})$" (*)

Using (*) we shall prove that

"Procedures (a) and (b) are equivalent if and only if N_ν ($\nu = 1,2$) has a Poisson distribution" (**)

"If": We assume that N_ν ($\nu = 1,2$) has a Poisson distribution with mean a. By procedure (a), the contents of the cells will be $J = N_1$ and $K = N_2$, respectively, and so J and K are independent Poisson variables with means a. By procedure (b), $N_1 + N_2$ is a Poisson variable with mean $2a$. The decomposition property, expressed by (5.45), implies that J and K will be independent Poisson variables with means $\frac{1}{2} \times 2a = a$. Hence, procedures (a) and (b) are equivalent.

"Only if": We assume that procedures (a) and (b) are equivalent. By (*),

$$g(x)\,g(y) = g^2(\tfrac{x+y}{2}) \tag{1}$$

Hence,

$$g(x)\,g(y) = g(x+y)\,g(0) \tag{2}$$

This implies $g(0) > 0$. Now put

$$g(z) = u(z)\,g(0) \tag{3}$$

Inserting (3) into (2) yields

$$u(x)\,u(y) = u(x+y) \tag{4}$$

Since $g(z)$ is increasing in z, so is $u(z)$, by (3). Eq. (4) is identical to Eq. (5.20). It follows that the only increasing function u satisfying the functional equation (4) is of the form

$$u(z) = e^{az} \quad (a > 0).$$

(Chap. 2, Ex. 23)

Thus, by (3),

$$g(z) = e^{az} g(0) \tag{5}$$

But $g(1)=1$, so by (5) $1 = e^{a} g(0)$, whence $g(0) = e^{-a}$ Thus

$$g(z) = e^{-a(1-z)} \qquad (a>0). \tag{6}$$

This is recognized as the p.g.f. of a Poisson variable with mean a. Thus N_ν $(\nu = 1,2)$ has a Poisson distribution.

Chapter 2, Exercise 24

'Consider the single-server queue with an unlimited number..'

$\boxed{a}$ First multiply equation j $(j=0,1,\ldots)$ of equation system (1) by z^j:

$$\Pi_0^* z^0 = p_0 z^0 \Pi_0^* + (p_0 \Pi_1^*) z^0$$

$$\Pi_1^* z^1 = p_1 z^1 \Pi_0^* + (p_0 \Pi_2^* + p_1 \Pi_1^*) z^1$$

$$\Pi_2^* z^2 = p_2 z^2 \Pi_0^* + (p_0 \Pi_3^* + p_1 \Pi_2^* + p_2 \Pi_1^*) z^2$$

$$\vdots$$

Adding all these equations results in

$$g(z) = h(z)\Pi_0^* + z^{-1}(p_0 + p_1 z + p_2 z^2 + \cdots) \sum_{i=1}^{\infty} \Pi_i^* z^i$$

$$= h(z)\Pi_0^* + z^{-1} h(z) [g(z) - \Pi_0^*]$$

Hence,

$$g(z) = \frac{(z-1)\,h(z)}{z - h(z)} \Pi_0^*. \tag{3}$$

$\boxed{b}$

$$h(z) = \sum_{j=0}^{\infty} p_j z^j = \sum_{j=0}^{\infty} \left(\int_0^\infty \frac{(\lambda \xi)^j}{j!} e^{-\lambda\xi} dH(\xi) \right) z^j$$

$$= \sum_{j=0}^{\infty} \int_0^\infty \frac{(\lambda z \xi)^j}{j!} e^{-\lambda \xi} dH(\xi) = \int_0^\infty \left(\sum_{j=0}^{\infty} \frac{(\lambda z \xi)^j}{j!} \right) e^{-\lambda\xi} dH(\xi)$$

$$= \int_0^\infty e^{-(\lambda - \lambda z)\xi} dH(\xi) = \eta(\lambda - \lambda z).$$

(Chap. 2, Ex. 24 b)

Substitution of $h(z) = \eta(\lambda - \lambda z)$ into (3) yields

$$g(z) = \frac{(z-1)\eta(\lambda-\lambda z)}{z - \eta(\lambda - \lambda z)} \Pi_0^* \tag{4}$$

$\boxed{c}$ By the application of l'Hospital's rule to Eq. (3),

$$\begin{aligned} g(1) = \lim_{z\to 1} g(z) &= \frac{\frac{d}{dz}\left((z-1)h(z)\right)\big|_{z=1}}{\frac{d}{dz}\left(z - h(z)\right)\big|_{z=1}} \Pi_0^* \\ &= \frac{(z-1)h'(z) + h(z)\big|_{z=1}}{1 - h'(z)\big|_{z=1}} \Pi_0^* \\ &= \frac{h(1)}{1-h'(1)} \Pi_0^* = \frac{\Pi_0^*}{1-h'(1)}. \end{aligned}$$

Clearly,

$$g(1) = 1. \tag{5}$$

Hence,

$$\Pi_0^* = 1 - h'(1).$$

Now, $h'(1) = \Sigma\, j\, p_j$ is the mean number of arrivals during a service time, which in Exercise 21 a was shown to be equal to $\lambda\tau$. That is, $h'(1) = \lambda\tau$. For $\lambda\tau = \rho < 1$ then,

$$\Pi_0^* = 1 - \rho. \tag{6}$$

Thus Eq. (4) becomes

$$g(z) = \frac{(z-1)\eta(\lambda-\lambda z)}{z - \eta(\lambda-\lambda z)}(1-\rho) \qquad (\rho < 1). \tag{7}$$

$\boxed{d}$ By (4.5), $E(N^*) = g'(1)$. Differentiation of (3), with Π_0^* replaced by $1-\rho$, gives

$$g'(z) = \frac{A(z)}{B(z)}(1-\rho),$$

where

$$A(z) = h(z) - h^2(z) - z(1-z)h'(z),$$

and

$$B(z) = (z - h(z))^2.$$

(Chap. 2, Ex. 24 d)

Since $h(1)=1$, $A(1)=0$ and $B(1)=0$. Thus, evaluation of $g'(1)$ requires the application of L'Hospital's rule. Differentiation yields

$$A'(z) = 2z\,h'(z) - 2h(z)h'(z) - z(1-z)h''(z),$$

$$B'(z) = 2(z-h(z))(1-h'(z)).$$

As $A'(1)=0$ and $B'(1)=0$ we differentiate once more:

$$A''(z) = 2h'(z)(1-h'(z)) + (4z-1-2h(z))h''(z) - z(1-z)h'''(z),$$

$$B''(z) = 2(1-h'(z))^2 - 2(z-h(z))h''(z).$$

$A''(z)$ and $B''(z)$ have to be evaluated at $z=1$. We already know that $h(1)=1$ and $h'(1)=\rho$. In order to find $h''(1)$, recall that by part (b) $h(z)=\eta(\lambda-\lambda z)$. Hence $h'(z)=-\lambda\eta'(\lambda-\lambda z)$ and $h''(z)=\lambda^2\eta''(\lambda-\lambda z)$. Thus $h''(1)=\lambda^2\eta''(0)$. By definition, $\eta(s)=\int_0^\infty e^{-s\xi}dH(\xi)$. Hence, $\eta'(s)=\int_0^\infty(-\xi)e^{-s\xi}dH(\xi)$ and $\eta''(s)=\int_0^\infty\xi^2e^{-s\xi}dH(\xi)$. Thus $\eta''(0)=\int_0^\infty\xi^2dH(\xi)=\sigma^2+\tau^2$. It follows that $h''(1)=\lambda^2(\sigma^2+\tau^2)$. By these results,

$$A''(1) = 2\rho(1-\rho) + \lambda^2(\sigma^2+\tau^2),$$

$$B''(1) = 2(1-\rho)^2.$$

By l'Hospital's rule, $g'(1)=\frac{A''(1)}{B''(1)}(1-\rho)$. Hence

$$E(N^*) = \rho + \frac{\rho^2(1+(\sigma^2/\tau^2))}{2(1-\rho)}. \qquad (8)$$

$\boxed{e}$ Let $H(\xi)=1-e^{-\mu\xi}$. Then $\eta(s)=\int_0^\infty e^{-s\xi}dH(\xi)=\int_0^\infty e^{-s\xi}\mu e^{-\mu\xi}d\xi=\mu/(\mu+s)$. Thus $\eta(\lambda-\lambda z)=\mu/(\mu+\lambda-\lambda z)$ and, by (7),

$$g(z) = \frac{(z-1)\mu/(\mu+\lambda-\lambda z)}{z-\mu/(\mu+\lambda-\lambda z)}(1-\rho) = \frac{(z-1)\mu}{(z-1)(\mu-\lambda z)}(1-\rho)$$

$$= (1-\rho)/(1-\rho z) = \sum_{j=0}^{\infty}(1-\rho)\rho^j z^j \qquad [\rho=\lambda\tau=\lambda/\mu]$$

Hence,

$$\pi_j^* = (1-\rho)\rho^j \qquad (j=0,1,\dots). \qquad (9)$$

□

Chapter 2, Exercise 25

'An operations research consultant ...'

No comment.

Chapter 2, Exercise 26

'In the model of Exercise 25, let X be the merging time ...'

In the model of Exercise 25 the (a priori) interarrival times $U_1, U_2, \dots$ are i.i.d. exponential variables with mean α^{-1}, and the required acceleration times $V_1, V_2, \dots$ are i.i.d. exponential variables with mean β^{-1}. As a consequence, the merging time $X = U_1 + U_2 + \cdots U_{n-1} + V_n$ is exponentially distributed with mean β^{-1} and does not depend on α (see preceding pages of book).

In the model of this exercise, it will be assumed that $V_1 = V_2 = \cdots = V$, where V is an exponential variable with mean β^{-1}. In this case the mean and the variance of X are a function of α. Both mean and variance are greater than in the model of Exercise 25 for identical α and β.

[a] Let V equal the constant c. Then

$$X = c + \sum_{i=1}^{n-1} U_i, \tag{1}$$

where each of the observed U_i's $(i = 1, \dots, n-1)$ has the conditional distribution of U, given $U \leq c$. Since these U_i's are i.i.d. variables and independent of $n-1$,

$$E(X|V=c) = c + E(n-1)E(U|U \leq c). \tag{2}$$

Now, $n-1$ has the geometric distribution with parameter $p = \text{Prob}\{U>c\} = e^{-\alpha c}$. Hence $E(n-1) = (1-p)/p = e^{\alpha c} - 1$. Substitution into (2) yields the desired expression

$$E(X|V=c) = c + (e^{\alpha c}-1)E(U|U \leq c). \tag{3}$$

(Chap. 2, Ex. 26 b)

$\boxed{b}$ First we determine $E(U|U\leq c)$. By assumption, U is exponentially distributed with mean α^{-1}, i.e. $\text{Prob}\{U\leq u\} = 1-e^{-\alpha u}$. Hence, $F_c(u) \equiv \text{Prob}\{U\leq u\,|\,U\leq c\} = (1-e^{-\alpha u})/(1-e^{-\alpha c})$, and the density function of U, given $U\leq c$, is $f_c(u) = dF_c(u)/du = \alpha e^{-\alpha u}/(1-e^{-\alpha c})$, for $0\leq u\leq c$. Consequently,

$$E(U|U\leq c) = \int_0^c u f_c(u)du = \frac{1}{1-e^{-\alpha c}}\int_0^c u\alpha e^{-\alpha u}du = \frac{1}{1-e^{-\alpha c}}\Big(-\frac{(1+\alpha u)e^{-\alpha u}}{\alpha}\Big|_0^c\Big).$$

Hence,

$$E(U|U\leq c) = \frac{1}{\alpha} - \frac{c}{e^{\alpha c}-1}. \qquad (4)$$

Substitution of Equation (4) into Equation (3) results in

$$E(X|V=c) = \frac{e^{\alpha c}-1}{\alpha}. \qquad (5)$$

We now assume that V is exponentially distributed with mean β^{-1}. Unconditioning on V we find

$$E(X) = \int_0^\infty E(X|V=c)\beta e^{-\beta c}dc = \frac{\beta}{\alpha}\Big(\int_0^\infty e^{-(\beta-\alpha)c}dc - \int_0^\infty e^{-\beta c}dc\Big),$$

by which

$$E(X) = \begin{cases}(\beta-\alpha)^{-1} & \text{when } \beta>\alpha,\\ \infty & \text{when } \beta\leq\alpha.\end{cases} \qquad (6)$$

$\boxed{c}$ For the purpose of calculating $V(X)$ we shall employ the decomposition formula

$$V(X) = E_c(V(X|V=c)) + V_c(E(X|V=c)), \qquad (7)$$

which gives $V(X)$ as the sum of the mean of the conditional variance, given V, and the variance of the conditional mean, given V. Clearly,

$$V(X|V=c) = V\Big(c+\sum_{i=1}^{n-1}U_i\Big) = V\Big(\sum_{i=1}^{n-1}U_i\Big).$$

The distribution of $\sum_{i=1}^{n-1}U_i$ is a compound distribution. Since the formulas in part (b) of Exercise 4 also hold for nondiscrete variables (see Chap. 5, Ex. 5) we have

(Chap. 2, Ex. 26 c)

$$V(X|V=c) = E(n-1)V(U|U\leq c) + V(n-1)E^2(U|U\leq c) \qquad (8)$$

By part (a), $n-1$ is geometrically distributed with parameter $p=e^{-\alpha c}$. Hence, $E(n-1)=e^{\alpha c}-1$ and $V(n-1)=(1-p)/p^2=e^{\alpha c}(e^{\alpha c}-1)$. $E(U|U\leq c)$ is given by Eq. (4). Only $V(U|U\leq c)$ remains to be calculated. Following the development in part (b) we find

$$\begin{aligned} V(U|U\leq c) &= E(U^2|U\leq c) - E^2(U|U\leq c) \\ &= \int_0^c u^2 f_c(u)\,du - \left(\frac{1}{\alpha} - \frac{c}{e^{\alpha c}-1}\right)^2 \\ &= -\left.\frac{(\alpha^2u^2+2\alpha u+2)e^{-\alpha u}}{\alpha^2(1-e^{-\alpha c})}\right|_0^c - \left(\frac{1}{\alpha} - \frac{c}{e^{\alpha c}-1}\right)^2. \end{aligned}$$

Hence,

$$V(U|U\leq c) = \frac{1}{\alpha^2} - \frac{c^2e^{\alpha c}}{(e^{\alpha c}-1)^2}. \qquad (9)$$

Substitution of the various expressions into (8) gives

$$V(X|V=c) = (e^{\alpha c}-1)\left(\frac{1}{\alpha^2} - \frac{c^2e^{\alpha c}}{(e^{\alpha c}-1)^2}\right) + e^{\alpha c}(e^{\alpha c}-1)\left(\frac{1}{\alpha} - \frac{c}{e^{\alpha c}-1}\right)^2,$$

that reduces to

$$V(X|V=c) = \frac{1}{\alpha^2}\left(e^{2\alpha c} - 2\alpha c e^{\alpha c} - 1\right). \qquad (10)$$

Assuming that V is exponentially distributed with mean β^{-1} we have

$$\begin{aligned} E_c(V(X|V=c)) &= \int_0^\infty V(X|V=c)\beta e^{-\beta c}dc \\ &= \int_0^\infty \frac{e^{2\alpha c}-2\alpha c e^{\alpha c}-1}{\alpha^2}\beta e^{-\beta c}dc \\ &= \frac{\beta}{\alpha^2}\left(\int_0^\infty e^{-(\beta-2\alpha)c}dc - 2\alpha\int_0^\infty c\, e^{-(\beta-\alpha)c}dc - \int_0^\infty e^{-\beta c}dc\right) \end{aligned}$$

If $\beta\leq 2\alpha$, then $E_c(V(X|V=c))=\infty$. If, on the other hand, $\beta>2\alpha$, then

$$\begin{aligned} E_c(V(X|V=c)) &= \frac{\beta}{\alpha^2}\left(\frac{1}{\beta-2\alpha} - 2\alpha\left(-\left.\frac{1+(\beta-\alpha)c}{(\beta-\alpha)^2}e^{-(\beta-\alpha)c}\right|_0^\infty\right) - \frac{1}{\beta}\right) \\ &= \frac{\beta}{\alpha^2}\left(\frac{1}{\beta-2\alpha} - \frac{2\alpha}{(\beta-\alpha)^2} - \frac{1}{\beta}\right). \end{aligned}$$

(Chap 2, Ex. 26 c (cont'd))

Hence,

$$E_c(V(X|V=c)) = \begin{cases} \frac{2\alpha}{(\beta-2\alpha)(\beta-\alpha)^2} & \text{when } \beta > 2\alpha, \\ \infty & \text{when } \beta \leq 2\alpha. \end{cases} \qquad (11)$$

Still under assumption of an exponentially distributed V with mean β^{-1}, we derive the second term on the right-hand side of (7):

$$\begin{aligned} V_c(E(X|V=c)) &= \int_0^\infty (E(X|V=c) - E(X))^2 \beta e^{-\beta c} dc \\ &= \int_0^\infty \left(\frac{e^{\alpha c}-1}{\alpha} - \frac{1}{\beta-\alpha}\right)^2 \beta e^{-\beta c} dc \qquad [\text{by } (5) \,\&\, (6)] \\ &= \frac{\beta}{\alpha^2}\int_0^\infty (e^{-(\beta-2\alpha)c} - 2e^{-(\beta-\alpha)c} + e^{-\beta c}) dc \\ &- \frac{2\beta}{\alpha(\beta-\alpha)}\int_0^\infty (e^{-(\beta-\alpha)c} - e^{-\beta c}) dc \\ &+ \frac{\beta}{(\beta-\alpha)^2}\int_0^\infty e^{-\beta c} dc. \end{aligned}$$

If $\beta \leq 2\alpha$, then $V_c(E(X|V=c)) = \infty$. Otherwise

$$V_c(E(X|V=c)) = \frac{\beta}{\alpha^2}\left(\frac{1}{\beta-2\alpha} - \frac{2}{\beta-\alpha} + \frac{1}{\beta}\right) - \frac{2\beta}{\alpha(\beta-\alpha)}\left(\frac{1}{\beta-\alpha} - \frac{1}{\beta}\right) + \frac{1}{(\beta-\alpha)^2}.$$

Hence,

$$V_c(E(X|V=c)) = \begin{cases} \frac{\beta}{(\beta-2\alpha)(\beta-\alpha)^2} & \text{when } \beta > 2\alpha, \\ \infty & \text{when } \beta \leq 2\alpha. \end{cases} \qquad (12)$$

By adding Equations (11) and (12) according to the decomposition formula, Eq. 7, we finally obtain, for the case of exponentially distributed characteristic gaps:

$$V(X) = \begin{cases} \frac{\beta+2\alpha}{\beta-2\alpha} \times \frac{1}{(\beta-\alpha)^2} & \text{when } \beta > 2\alpha, \\ \infty & \text{when } \beta \leq 2\alpha. \end{cases} \qquad (13)$$

□

Chapter 3, Exercise 1

'A single-server queueing system...'

a By Equation (1.1),

$$P_0 = [1 + \sum_{k=1}^{\infty} \frac{\lambda_0\lambda_1\cdots\lambda_{k-1}}{\mu_1\mu_2\cdots\mu_k}]^{-1} = [1 + \sum_{k=1}^{\infty} \frac{(\lambda/\mu)^k}{k!}]^{-1} = e^{-\lambda/\mu},$$

and for $j = 1, 2, \ldots$

$$P_j = \frac{\lambda_0\lambda_1\cdots\lambda_{j-1}}{\mu_1\mu_2\cdots\mu_j} P_0 = \frac{(\lambda/\mu)^j}{j!} e^{-\lambda/\mu}.$$

Since $\mu^{-1} = \tau$,

$$P_j = \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau} \qquad (j = 0, 1, \ldots).$$

By Equation (2.6), for $j = 0, 1, \ldots,$

$$\Pi_j = \lambda_j P_j / \sum_{k=0}^{\infty} \lambda_k P_k = \frac{\lambda}{j+1}\frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau} / \sum_{k=0}^{\infty} \frac{\lambda}{k+1}\frac{(\lambda\tau)^k}{k!} e^{-\lambda\tau} = \frac{\frac{(\lambda\tau)^{j+1}}{(j+1)!} e^{-\lambda\tau}}{\sum_{k=0}^{\infty} \frac{(\lambda\tau)^k}{k!} e^{-\lambda\tau} - e^{-\lambda\tau}}.$$

Hence,

$$\Pi_j = (1 - e^{-\lambda\tau})^{-1} P_{j+1} \quad (j = 0, 1, \ldots).$$

As $s = 1$, by (1.9) the carried load is

$$a' = \sum_{j=1}^{\infty} P_j = 1 - P_0 = 1 - e^{-\lambda\tau}.$$

Obviously, the mean arrival rate is $\bar{\lambda} = \Sigma_{j=0}^{\infty} \lambda_j P_j$, so

$$\bar{\lambda} = \sum_{j=0}^{\infty} \frac{\lambda}{j+1}\frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau} = \tau^{-1}(1 - e^{-\lambda\tau}).$$

It follows that the offered load is $a = \bar{\lambda}\tau = 1 - e^{-\lambda\tau}$, and $a = a'$.

b Here the arrival rate is λ, but the effective arrival rate – concerning arrivals effecting a change of state – in state j is $\lambda_j = \lambda[1 - j/(j+1)] = \lambda/(j+1)$. Also, $\mu_j = \mu$. Thus, the queueing system can be modeled as a birth-and-death process with the same parameters as the model of part (a). Consequently, $\{P_j\}$ is as in part (a). Furthermore, since the arrival process is Poisson, $\Pi_j = P_j$. We conclude that

(Chap. 3, Ex. 1 b)

$$\Pi_j = P_j = \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau} \qquad (j = 0, 1, \ldots).$$

As in part (a),

$$a' = 1 - e^{-\lambda\tau}.$$

However, the offered load is

$$a = \lambda\tau.$$

Letting P denote the probability that an arbitrary arrival does not receive service,

$$P = \sum_{j=0}^{\infty} \frac{j}{j+1} \Pi_j = \sum_{j=0}^{\infty} \left(1 - \frac{1}{j+1}\right) P_j = 1 - \frac{1}{\lambda\tau} \sum_{j=0}^{\infty} \frac{(\lambda\tau)^{j+1}}{(j+1)!} e^{-\lambda\tau}$$

$$= 1 - \frac{1 - e^{-\lambda\tau}}{\lambda\tau} = 1 - \frac{a'}{a} = \frac{a - a'}{a}.$$

$\boxed{c}$ As in the models of parts (a) and (b), $\lambda_j/\mu_{j+1} = \lambda/[(j+1)\mu]$, so the state distribution $\{P_j\}$ is the same in those cases. Furthermore, since the arrival process is Poisson, $\Pi_j = P_j$. Hence,

$$\Pi_j = P_j = \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau} \qquad (j = 0, 1, \ldots).$$

Also, as in parts (a) and (b), the carried load is

$$a' = 1 - e^{-\lambda\tau}.$$

The birth-and-death process will not be affected by preemption coupled with service in reverse order of arrival. In this case, a customer who arrives in state j will be served at rate μ_{j+1} when in service, and his mean service time, which is not affected by preemption, is μ_{j+1}^{-1}. Thus the overall mean service time equals

$$\overline{\mu} = \sum_{j=0}^{\infty} \Pi_j \mu_{j+1}^{-1} = \sum_{j=0}^{\infty} \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau} \frac{1}{(j+1)\mu} = \frac{\mu^{-1}}{\lambda\tau} \sum_{j=0}^{\infty} \frac{(\lambda\tau)^{j+1}}{(j+1)!} e^{-\lambda\tau} = \frac{1}{\lambda}(1 - e^{-\lambda\tau})$$

The offered load is therefore $a = \lambda\overline{\mu} = 1 - e^{-\lambda\tau}$. Hence, $a = a'$, of course.

$\square$

Chapter 3, Exercise 2

'Customers arrive at a two-chair shoe-shine stand...'

$\boxed{a}$ $\lambda = 10,\ \mu = 10,\ s = 1,\ k = 1$

The corresponding birth-and-death model has: $\lambda_0 = \lambda_1 = \lambda = 10$, $\lambda_2 = 0$; $\mu_0 = 0$, $\mu_1 = \mu_2 = \mu = 10$. By (1.1),

$$P_0 = \left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2}\right)^{-1}, \quad P_1 = \frac{\lambda_0}{\mu_1}P_0, \quad P_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}P_0.$$

Hence,

$$(P_0, P_1, P_2) = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right).$$

$\boxed{b}$ The mean number of customers served per hour is

$$\Delta = \mu_1 P_1 + \mu_2 P_2 = \frac{10}{3} + \frac{10}{3} = \underline{6.67}$$

$\boxed{c}$ $\lambda = 10,\ \mu = 10,\ s = 2,\ k = 0$

The corresponding birth-and-death model has: $\lambda_0 = \lambda_1 = \lambda = 10$, $\lambda_2 = 0$; $\mu_0 = 0$, $\mu_1 = \mu = 10$, $\mu_2 = 2\mu = 20$. Applying the above formulas we find

$$(P_0, P_1, P_2) = \left(\tfrac{4}{10}, \tfrac{4}{10}, \tfrac{2}{10}\right)$$

and

$$\Delta = \underline{8.00}.$$

Chapter 3, Exercise 3

'Derive (3.5) from the definition (1.9) and the probabilities (3.3)!'

$$a' = \sum_{j=1}^{s} j P_j = \frac{\sum_{j=1}^{s} j a^j/j!}{\sum_{k=0}^{s} a^k/k!} = a\,\frac{\sum_{j=1}^{s} a^{j-1}/(j-1)!}{\sum_{k=0}^{s} a^k/k!}$$

$$= a\,\frac{\sum_{j=0}^{s-1} a^j/j!}{\sum_{k=0}^{s} a^k/k!} = a\left[1 - \frac{a^s/s!}{\sum_{k=0}^{s} a^k/k!}\right]. \qquad [a = \lambda/\mu]$$

By (3.4), then

$$a' = a[1 - B(s,a)] \qquad (3.5) \qquad \square$$

Chapter 3, Exercise 4

'Consider an Erlang loss system with 10 servers.'

The solution requires the evaluation of $B(s,a)$. Figures A-1 and A-2 of Appendix A provide the answers. Alternatively one can use a table of the cumulative Poisson distribution, since

$$B(s,a) = \frac{\frac{a^s}{s!} e^{-a}}{\sum_{k=0}^{s} \frac{a^k}{k!} e^{-a}},$$

and both numerator and denominator may be read off or easily calculated from a table with cumulative Poisson probabilities. We find

$$B(10,4.5) = \frac{0.0104}{0.9933} = 0.0105$$

(and $B(10,4.0) = 0.0053$). Accordingly we accept $a = 4.5$ as an approximate solution of $B(10,a) = 0.01$. We also find

$$B(16,9.0) = 0.0110, \qquad B(17,9.0) = 0.0058.$$

Thus, a doubling of the offered load does not necessitate a doubling of the number of servers, from 10 to 20, in order to prevent service degradation. Only 7 servers need be added to the system.

Chapter 3, Exercise 5

'An entrepreneur offers services...'

The offered load is $a = \lambda\tau = 4 \times 1 = 4.0$. The hourly profit at operating cost c equals $H(s;c) = \lambda[1 - B(s,a)]\, 2.5 - sc = 10[1 - B(s,4.0)] - sc$. Hence,

s	1	2	3	4	5	6	7	8	9	10
$B(s,4.0)$	.800	.615	.451	.311	.199	.117	.063	.030	.013	.005
$H(s;1.0)$	1.00	1.85	2.49	2.89	3.01	2.83	2.37	1.70	0.87	-0.05

Thus, at $c = 1.0$, the optimal number of servers is 5, and the corresponding profit rate equals 3.01. The break-even point for c is $c_0 = 2.0$: with $s = 1$ the entrepreneur will just break even; with $s > 1$ he will lose. □

Chapter 3, Exercise 6

'Show that $B(s,a) = a\,B(s-1,a)/[s + a\,B(s-1,a)]$.'

By (3.4), for $s \geq 1$,

$$B(s,a) = \frac{a^s/s!}{\sum_{k=0}^{s} a^k/k!} = \frac{a}{s}\,\frac{a^{s-1}/(s-1)!}{\sum_{k=0}^{s-1} a^k/k!}\,\frac{\sum_{k=0}^{s-1} a^k/k!}{\sum_{k=0}^{s} a^k/k!}$$

$$= \frac{a}{s} B(s-1,a)[1 - B(s,a)],$$

where $B(0,a) \equiv 1$. Solving for $B(s,a)$ we derive

$$B(s,a) = \frac{a\,B(s-1,a)}{s + a\,B(s-1,a)} \qquad (s = 1, 2, \ldots).$$

Chapter 3, Exercise 7

'Consider an Erlang loss system with retrials.'

No comment.

Chapter 3, Exercise 8

'Consider an equilibrium s-server Erlang loss system ...'

In the Erlang loss system the event {next arrival is blocked} will occur if and only if (a) the observer finds all servers busy, and (b) next arrival occurs before next service completion. Obviously, event (a) has probability $B(s,a)$ $(= P_s = \Pi_s)$. Given (a), event (b) has probability $\lambda/(\lambda + s\mu)$, by Eq. (5.23) of Chapter 2, as time to next arrival and time to next service completion are independent exponential variables with parameters λ and $s\mu$, respectively. Hence

$$p = B(s,a)\frac{\lambda}{\lambda + s\mu} = \frac{a}{a+s} B(s,a).$$

The reason p is not equal to $B(s,a)$, as one might naively think, is that "next arrival" is not an arbitrary arrival.

□

Chapter 3, Exercise 9

'The Erlang loss system as a semi-Markov process.'

We consider the s-server Erlang loss system with exponential service times, and let λ = arrival rate and μ = service rate.

a Clearly,

$$p_{ij} = \begin{cases} 0 & (|i-j| \neq 1), \\ \lambda/(\lambda+i\mu) & (0 \leq i \leq s-1,\ j=i+1), \\ i\mu/(\lambda+i\mu) & (1 \leq i \leq s-1,\ j=i-1), \\ 1 & (i=s,\ j=s-1). \end{cases} \tag{6}$$

b Clearly,

$$m_i = \begin{cases} 1/(\lambda+i\mu) & (0 \leq i \leq s-1), \\ 1/s\mu & (i=s). \end{cases} \tag{7}$$

c Substitution of Eq. (6) into Eq. (3) yields

$$\begin{aligned} P_0^* &= \frac{\mu}{\lambda+\mu} P_1^* & (s>1), \\ P_i^* &= \frac{\lambda}{\lambda+(i-1)\mu} P_{i-1}^* + \frac{(i+1)\mu}{\lambda+(i+1)\mu} P_{i+1}^* & (s>2,\ 1 \leq i \leq s-2), \\ P_{s-1}^* &= \frac{\lambda}{\lambda+(s-2)\mu} + 1 \cdot P_s^* & (s>1), \\ P_s^* &= \frac{\lambda}{\lambda+(s-1)\mu} P_{s-1}^* . \end{aligned}$$

By recursive solution we obtain

$$P_i^* = \begin{cases} \frac{\lambda+i\mu}{\lambda} \frac{(\lambda/\mu)^i}{i!} P_0^* & (0 \leq i \leq s-1), \\ \frac{(\lambda/\mu)^{s-1}}{(s-1)!} P_0^* & (i=s). \end{cases} \tag{8}$$

Now, by (7) and (8),

$$m_i P_i^* = \frac{1}{\lambda} \frac{(\lambda/\mu)^i}{i!} P_0^* \qquad (0 \leq i \leq s). \tag{9}$$

Inserting this expression into Eq. (5), with $k=s$, we derive Eq. (33):

$$P_j = \frac{(\lambda/\mu)^j/j!}{\sum_{i=0}^{s} (\lambda/\mu)^i/i!} \qquad (j=0,1,\dots,s) \qquad \square$$

Chapter 3, Exercise 10

'Two independent Poisson streams of traffic...'

Let the high priority stream parameters be λ_1 and τ_1, where $\lambda_1 = 20$ and $\tau_1 = 0.2$, and let the low priority stream parameters be λ_2 and τ_2. The two service time distributions may be general. For $s = 10$, the average overflow rate of high priority customers is known to be $\bar{\lambda}_1 = 2$. We wish to determine $a_2 = \lambda_2\tau_2$.

The primary group serves two independent streams of Poisson traffic on a BCC basis. Therefore, as argued in the text, the primary system is an Erlang loss system with arrival rate $\lambda = \lambda_1 + \lambda_2$ and a mixed service time distribution with the mean $\tau = (\lambda_1/\lambda)\tau_1 + (\lambda_2/\lambda)\tau_2$. The total offered load is $a = \lambda\tau = \lambda_1\tau_1 + \lambda_2\tau_2 = 20 \times 0.2 + \lambda_2\tau_2 = 4 + a_2$. The percentage overflow of high priority customers clearly is $\bar{\lambda}_1/\lambda_1 = 2/20 \doteq 0.10$. The same percentage will overflow from each stream arriving at the primary group, so $B(s,a) = 0.10$. That is, $B(10,a) = 0.10$. Solving by use of the graph in Appendix A-1 we find $a = 7.5$ ($B(10,7.5) = 0.0995$). Hence, $a_2 = a - a_1 = 7.5 - 4.0$. Thus,

$$a_2 = 3.5.$$

$\boxed{a}$ Denote by λ_2^*, a_2^* and a^* the new values of λ_2, a_2 and a. We have $\lambda_2^* = 2\lambda_2$. Hence, $a_2^* = \lambda_2^*\tau_2 = 2\lambda_2\tau_2 = 2a_2 = 7$, whereby $a^* = a_1 + a_2^* = 4 + 7 = 11$. It follows that the new overflow rate (average) of high priority customers will be

$$\bar{\lambda}_1^* = \lambda_1 B(s, a^*) = 20\, B(10, 11) = 20 \cdot 0.260 = 5.2.$$

The factor of increase is

$$\frac{\bar{\lambda}_1^*}{\bar{\lambda}_1} = \frac{5.2}{2.0} = 2.6.$$

$\boxed{b}$ It is not permissible to design the backup group by use of Erlang's loss formula which assumes Poisson traffic. The overflow traffic is not Poisson. Disregarding this fact will lead to underestimation of the loss on the backup group, one would think. $\square$

Chapter 3, Exercise 11

'Prove equation (3.12)'.

For all $t > 0$,

$$0 \leq t[1 - H(t)] = \int_t^\infty t\,dH(x) \leq \int_t^\infty x\,dH(x).$$

Now, $\mu^{-1} = \int_0^\infty x\,dH(x) < \infty$ implies $\lim_{t\to\infty} \int_t^\infty x\,dH(x) = 0$. Hence, taking limits we obtain

$$0 \leq \lim_{t\to\infty} t[1 - H(t)] \leq \lim_{t\to\infty} \int_t^\infty x\,dH(x) = 0.$$

Thus,

$$\lim_{t\to\infty} t[1 - H(t)] = 0. \qquad (3.12)$$

Chapter 3, Exercise 12

'Blocked customers held.'

[a] Each customer stays in the system (queue + service) for a time T that follows the sojourn time distribution H(x). Thus the queueing system may be modeled as an infinite server queue where the sojourn time is interpreted as a service time. It follows that $\{P_j(t)\}$ is the Poisson distribution

$$P_j(t) = \frac{[\lambda t p(t)]^j}{j!} e^{-\lambda t p(t)} \qquad (j = 0, 1, \ldots). \qquad (3.11)$$

with

$$p(t) = 1 - H(t) + \int_0^t \frac{x}{t}\,dH(x), \qquad (3.9)$$

where now H(x) is the sojourn time distribution function.

[b] Assume that T has the exponential distribution with mean μ^{-1}. Customers may defect before reaching the server. For a customer who does enter service, the remaining sojourn time (= service time) will, by the Markov property, also be exponentially distributed with mean μ^{-1}.

(Chap. 3, Ex. 12 c)

$\boxed{c}$ As in part (b), let T follow the exponential distribution. Let δ denote the mean defection rate (from queue). In state $j > s$ the defection rate is $(j-s)\mu$. Hence

$$\delta = \sum_{j=s+1}^{\infty} (j-s)\mu P_j .$$

By (3.8),

$$P_j = \frac{(\lambda/\mu)^j}{j!} e^{-\frac{\lambda}{\mu}} \quad (j = 0, 1, 2, \ldots).$$

μ^{-1} is both mean sojourn time and mean service time (in the normal sense), so $a = \lambda/\mu$ is the offered load. Thus

$$\begin{aligned} q = \frac{\delta}{\lambda} &= \frac{1}{\lambda} \sum_{j=s+1}^{\infty} (j-s)\mu \frac{a^j}{j!} e^{-a} \\ &= \sum_{j=s+1}^{\infty} \frac{a^{j-1}}{(j-1)!} e^{-a} - \frac{s}{a} \sum_{j=s+1}^{\infty} \frac{a^j}{j!} e^{-a} \\ &= \sum_{j=s}^{\infty} \frac{a^j}{j!} e^{-a} - \frac{s}{a} \sum_{j=s+1}^{\infty} \frac{a^j}{j!} e^{-a}. \end{aligned}$$

That is,

$$q = P(s,a) - \frac{s}{a} P(s+1,a).$$

Per unit time $\lambda[1-q]$ $(= \lambda - \delta)$ will enter service. The mean service time equals μ^{-1}. It follows that the carried load equals $a' = \lambda[1-q]\mu^{-1} = a[1-q]$. Thus, $q = 1 - a'/a$.

Chapter 3, Exercise 13

'Suppose that a company with a private telephone network...'

Let s_1 = number of flat rate trunks, s_2 = number of measured rate trunks. Assume an ordered hunt such that a call will be carried by a flat rate trunk whenever possible. Evidently, this policy will minimize the relevant costs. The priority within the two classes of trunks is immaterial.

The associated hourly cost is

$$H(s_1, s_2) = 14 s_1 + 30 \sum_{j=s_1+1}^{s_1+s_2} \tilde{P}_j ,$$

(Chap. 3, Ex. 13)

where

$$\tilde{p}_j = a[B(j-1,a) - B(j,a)], \qquad (3.18)$$

with $a = 2$ erlangs.

If $14 > 30\tilde{p}_1$, let $j^* = 0$. If $14 \leq 30\tilde{p}_1$, let j^* be the maximal j such that $14 \leq 30\tilde{p}_j$. By (3.18), $\tilde{p}_2 = 2(0.6667 - 0.4000) = 0.5333$ and $\tilde{p}_3 = 2(0.4000 - 0.2105) = 0.3790$. Hence, $14 < 30\tilde{p}_2 = 16.00$, but $14 > 30\tilde{p}_3 = 11.37$. As $\tilde{p}_1 > \tilde{p}_2 > \cdots$, obviously $j^* = 2$.

Considering the cost function $H(s_1,s_2)$ and the relations $\tilde{p}_1 > \tilde{p}_2 > \cdots$,

$$s_1^* = min(j^*, s) \qquad (*)$$

is the optimal number of flat rate trunks out of a total of s $(= s_1 + s_2)$ trunks.

It is a requirement that $B(s_1+s_2, 2) \leq 0.02$. We have $B(5,2) = 0.0367$ and $B(6,2) = 0.0121$, and since the cost structure does not explicitly account for blocking costs, $s_1 + s_2 = 6$ is the optimal number of trunks. Hence, by (*),

$$s_1^* = min(2,6) = \underline{2}$$

is the optimal number of trunks, and the associated cost is

$$\begin{aligned} H(2,4) &= 14 \cdot 2 + 30(\tilde{p}_3 + \tilde{p}_4 + \tilde{p}_5 + \tilde{p}_6) \\ &= 28 + 30a[B(2,a) - B(6,a)] \qquad [\text{by } (3.18)] \\ &= 28 + 60(0.4000 - 0.0121) \\ &= \underline{51.27}. \end{aligned}$$

Given $s_1 + s_2 = 6$, the direct approach is to calculate $H(s_1, 6-s_1)$ $(= 14s_1 + 60(B(s_1,2) - B(6,2)))$ for $s_1 = 0,1,\ldots,6$. The result is:

s_1	0	1	2	3	4	5	6
$H(s_1,6-s_1)$	59.27	53.28	51.27	53.40	60.99	71.48	84.00

Again, $s_1^* = 2$.

□

Chapter 3, Exercise 14

'Prove that in an Erlang loss system with ordered hunt...'

Suppose there are s servers. Consider an arbitrary customer; denote by A_j the event that on arrival he finds the first j servers busy, and let E_j denote the event that the customer will be served by server j, meaning that the first $j-1$ servers are busy, whereas server j is free. Obviously, $A_j \subset A_{j-1}$ and $E_j = A_{j-1} - A_j$. Hence, $P\{E_j\} = P\{A_{j-1}\} - P\{A_j\}$ $(j=1,\ldots,s)$. With ordered hunt the first j servers, $j \leq s$, function as an Erlang loss system, so $P\{A_j\} = B(j,a)$. Thus,

$$P\{E_j\} = B(j-1,a) - B(j,a) \qquad (j=1,\ldots,s).$$

By (3.18),

$$\frac{\tilde{p}_j}{a} = B(j-1,a) - B(j,a) \qquad (j=1,\ldots,s).$$

It follows that

$$P\{E_j\} = \frac{\tilde{p}_j}{a} \qquad (j=1,\ldots,s).$$

Chapter 3, Exercise 15

'Prove that the variance v of the Erlang loss distribution...'

We shall demonstrate that the variance of the state variable J with distribution

$$P\{J=j\} = P_j = \frac{a^j/j!}{\sum_{k=0}^{s} a^k/k!} \qquad (j=0,1,\ldots,s) \qquad (3.3)$$

may be expressed as $V(J) = v = a'(1-\tilde{p}_s)$.

First we prove the formula in the simple case $s=1$. By (3.5),

$$E(J) = \sum_{j=0}^{1} j P_j = a' = a - aB(1,a).$$

J is a zero-one variable, so that $E(J^2) = E(J)$. Hence

$$E(J^2) = a - aB(1,a).$$

(Chap. 3, Ex. 15)

Hence,

$$V(J) = E(J^2) - E^2(J) = a[1-B(1,a)](1-a[1-B(1,a)]).$$

By (3.5), $a' = a[1-B(1,a)]$, and by (3.18), $\tilde{p}_1 = a[1-B(1,a)]$. Thus

$$V(J) = v = a'(1-\tilde{p}_s) \qquad (s=1).$$

Now consider the case $s \geq 2$. To begin, we express the variance $V(J)$ in terms of s, a and $B(s,a)$:

$$E(J) = \Sigma_{j=0}^{s} j P_j = a' = a - a\,B(s,a). \qquad [\text{by } (3.5)]$$

$$E(J(J-1)) = \Sigma_{j=0}^{s} j(j-1)P_j = \Sigma_{j=2}^{s} j(j-1)P_j = \frac{a^2 \Sigma_{j=2}^{s} a^{j-2}/(j-2)!}{\Sigma_{j=0}^{s} a^j/j!}$$

$$= a^2\Big(1-(1+\tfrac{s}{a})\frac{a^s/s!}{\Sigma_{j=0}^{s} a^j/j!}\Big) = a^2 - a^2B(s,a) - as\,B(s,a).$$

$$E(J^2) = E(J(J-1)) + E(J) = a^2 - a^2B(s,a) - as\,B(s,a) + a - a\,B(s,a).$$

$$E^2(J) = (a - a\,B(s,a))^2 = a^2 - 2a^2B(s,a) + a^2B^2(s,a).$$

Hence,

$$V(J) = E(J^2) - E^2(J) = a - a\,B(s,a) - as\,B(s,a) + a^2B(s,a) - a^2B^2(s,a),$$

which can be rewritten

$$V(J) = a[1-B(s,a)]\Big(1 - \frac{s\,B(s,a)}{1-B(s,a)} + a\,B(s,a)\Big).$$

If the equation of Exercise 6 is solved w.r.t. $B(s-1,a)$ we find

$$\frac{s\,B(s,a)}{1-B(s,a)} = a\,B(s-1,a).$$

Thus,

$$V(J) = a[1-B(s,a)]\big(1 - a[B(s-1,a) - B(s,a)]\big).$$

Finally, using Equations (3.5) and (3.18) we obtain

$$V(J) = v = a'(1-\tilde{p}_s) \qquad (s \geq 1).$$

$\square$

Chapter 3, Exercise 16

'a. Show that, for every integer $s > a$, $C(s,a) = \ldots$'

$\boxed{a}$ A rewriting of (4.8) gives

$$C(s,a) = \frac{\frac{s}{s-a}\frac{a^s}{s!}}{\sum_{k=0}^{s}\frac{a^k}{k!} + \frac{a}{s-a}\frac{a^s}{s!}} \qquad (s > a).$$

Dividing numerator and denominator by $\sum_{k=0}^{s} a^k/k!$ and introducing $B(s,a) = (a^s/s!)/\sum_{k=0}^{s} a^k/k!$ we easily derive

$$C(s,a) = \frac{s\,B(s,a)}{s - a(1-B(s,a))} \qquad (s > a). \qquad (1)$$

$\boxed{b}$ Another rewriting of (4.8) gives

$$C(s,a) = \frac{\frac{a}{s-a}\frac{a^{s-1}}{(s-1)!}}{\sum_{k=0}^{s-1}\frac{a^k}{k!} + \frac{a}{s-a}\frac{a^{s-1}}{(s-1)!}} \quad (s > a).$$

Dividing numerator and denominator by $\sum_{k=0}^{s-1} a^k/k!$ and introducing $B(s-1,a) = (a^{s-1}/(s-1)!)/\sum_{k=0}^{s-1} a^k/k!$ we obtain

$$C(s,a) = \frac{1}{1 + (s-a)[a\,B(s-1,a)]^{-1}} \qquad (s > a). \qquad (2)$$

$\boxed{c}$ By (1), for $s-1 > a$, that is, for $s > a+1$,

$$C(s-1,a) = \frac{(s-1)\,B(s-1,a)}{(s-1) - a(1-B(s-1,a))} \qquad (s > a+1).$$

Solving for $B(s-1,a)$ leads to

$$B(s-1,a) = \frac{(s-1-a)\,C(s-1,a)}{s-1-aC(s-1,a)} \qquad (s > a+1).$$

Insertion of the above expression into (2) results in

$$C(s,a) = \frac{1}{1 + (\frac{s-a}{a})\frac{s-1-aC(s-1,a)}{(s-1-a)C(s-1,a)}} \qquad (s > a+1). \qquad (3)$$

□

Chapter 3, Exercise 17

'Review and reconsider Exercises 4 and 5 of Chapter 1.'

The equilibrium state probabilities $\{P_j\}$ derived for the delay system in Exercise 4 and for the loss-delay system in Exercise 5, hold for Poisson arrivals and exponential service times. This may be proved rigorously by modeling the systems as birth-and-death processes, and then applying Eq. (1.1).

For the loss-delay system of Exercise 5 of Chapter 1, let s = number of servers, n = waiting room size, λ = arrival rate, μ = service rate, offered load $a = \lambda/\mu$. It was found that

$$P_j = \begin{cases} \dfrac{a^j}{j!} P_0 & (j = 1, 2, \ldots, s-1), \\ \dfrac{a^j}{s!\, s^{j-s}} P_0 & (j = s, \ldots, s+n), \end{cases} \qquad (*)$$

where $P_0 = \left(\sum_{k=0}^{s-1} a^k/k! + [a^s/s!] \sum_{i=0}^{n} (a/s)^i \right)^{-1}$, or,

$$P_0 = \left(\sum_{k=0}^{s-1} \frac{a^k}{k!} + \frac{a^s}{s!} \, \frac{1 - (\frac{a}{s})^{n+1}}{1 - \frac{a}{s}} \right)^{-1}. \qquad (**)$$

Denote by P_L, P_W, P_S, the equilibrium probabilities of being lost (denied service), having to wait in queue, and getting served immediately. Note, $\Pi_j = P_j$, since the arrival process is Poisson. Hence,

$$P_L = \Pi_{s+n} = P_{s+n},$$

$$P_W = \sum_{j=s}^{s+n-1} \Pi_j = \sum_{j=s}^{s+n-1} P_j \qquad (n \geqq 1),$$

$$P_S = \sum_{j=0}^{s-1} \Pi_j = \sum_{j=0}^{s-1} P_j.$$

By $(*)$,

$$P_L = \frac{a^{s+n}}{s!\, s^n} P_0,$$

$$P_W = \frac{a^s}{s!} \, \frac{1 - (a/s)^n}{1 - (a/s)} P_0 \qquad (n \geqq 1),$$

$$P_S = \sum_{j=0}^{s-1} \frac{a^j}{j!} P_0,$$

with P_0 given by $(**)$. $\square$

Chapter 3, Exercise 18

'Is the analysis leading to (4.12) valid for...'

The answer is no. The reason is that the mean idle period, which is the mean residual interarrival time at the end of the busy period, is not, in general, equal to the mean interarrival time λ^{-1} as in the case of Poisson arrivals, i.e. exponentially distributed interarrival times.

Chapter 3, Exercise 19

'Consider again the premise of Exercise 13. Now, however,...'

The subject is an Erlang delay system with $s=4$ trunks and $a = 2$. Let s_1 = number of flat-rate trunks, and let $s - s_1 = 4 - s_1$ = number of measured-rate trunks. s_1 must be set to minimize the ordered hunt hourly cost, assuming that flat-rate trunks have priority,

$$H(s_1) = 14 s_1 + 30 \sum_{j=s_1+1}^{4} p_j , \qquad (*)$$

where

$$p_j = \tilde{p}_j [1 - \rho C(s,a)] + \rho C(s,a), \qquad (4.16)$$

and

$$\tilde{p}_j = a[B(j-1,a) - B(j,a)]. \qquad (3.18)$$

Here, $\rho = a/s = 0.5$, and $C(s,a) = C(4,2) = 0.1739$ according to tables of the Erlang delay formula (4.8), see Fig. A-3, Appendix A. $\tilde{p}_j$ is determined using tables of the Erlang loss formula (3.4). We find

j	1	2	3	4
$\tilde{p}_j$	.6667	.5334	.3790	.2306
p_j	.6957	.5740	.4330	.2975

Substitution of the p_j's into (*) yields

s_1	0	1	2	3	4
$H(s_1)$	60.01	53.14	49.92	50.93	56.00

Best choice therefore is $s_1^* = 2$, and $H(s_1^*) = 49.92$. □

Chapter 3, Exercise 20

'Prove that for an s-server Erlang delay system...'

Special version of theorem

Let $\{P_j^\circ\}$ be the equilibrium state probabilities of an Erlang loss system, and let $\{P_j^*\}$ be the equilibrium state probabilities of an Erlang delay system. Suppose the systems have the same number of servers s and identical parameters λ and μ. In the delay system, let Q_j^* be defined as the conditional probability of state j, given $j \leq s$, that is, $Q_j^* = P_j^*/\sum_{k=0}^{s} P_k^*$ $(j = 0, 1, \ldots, s)$. Then $P_j^\circ = Q_j^*$ $(j = 0, 1, \ldots, s)$.

Proof. By (3.3),

$$P_j^\circ = \frac{(\lambda/\mu)^j/j!}{\sum_{k=0}^{s} (\lambda/\mu)^k/k!} \qquad (j = 0, 1, \ldots, s).$$

By (4.4),

$$P_j^* = \frac{(\lambda/\mu)^j}{j!} P_0^* \qquad (j = 0, 1, \ldots, s).$$

Assume that an equilibrium distribution exists, so that $P_0^* > 0$. Then

$$Q_j^* = \frac{P_j^*}{\sum_{k=0}^{s} P_k^*} = \frac{(\lambda/\mu)^j/j!}{\sum_{k=0}^{s} (\lambda/\mu)^k/k!} \qquad (j = 0, 1, \ldots, s).$$

Thus $P_j^\circ = Q_j^*$ for $j = 0, 1, \ldots, s$.

General version of theorem

Consider two birth-and-death processes with parameters $(\{\lambda_j^\circ\}, \{\mu_j^\circ\})$ and $(\{\lambda_j^*\}, \{\mu_j^*\})$. Assume equilibrium state distributions $\{P_j^\circ\}$ and $\{P_j^*\}$ exist and $P_0^\circ > 0$, $P_0^* > 0$. Assume $\lambda_j^\circ = \lambda_j^* = \lambda_j$ for $j = 0, \ldots, s-1$ and $\mu_j^\circ = \mu_j^* = \mu_j$ for $j = 1, \ldots, s$, for some $s \geq 1$. Let $Q_j^\circ = P_j^\circ/\sum_{k=0}^{s} P_k^\circ$ and $Q_j^* = P_j^*/\sum_{k=0}^{s} P_k^*$ $(j = 0, 1, \ldots, s)$ be the conditional probability of state j, given $j \leq s$. Then $Q_j^\circ = Q_j^*$ for $j = 0, 1, \ldots, s$.

Proof. The result follows easily from the fact that $P_j^\circ/P_0^\circ = P_j^*/P_0^* = (\lambda_0\lambda_1\cdots\lambda_{j-1})/(\mu_1\mu_2\cdots\mu_j)$ for $j = 1, \ldots, s$. Observe, in the special case above $Q_j^\circ = P_j^\circ$. $\square$

Chapter 3, Exercise 21

'Reconsider Ex. 14 with "E. loss system" replaced by "E. delay system."

It will be shown that the statement made in Exercise 14 holds true also with "Erlang loss system" replaced by "Erlang delay system."

We consider an Erlang delay system with s servers and ordered hunt. Let X be an arbitrary customer. Let k be the state of the system when X arrives. Let $D = \{X \text{ is delayed}\}$, $\bar{D} = \{X \text{ is not delayed}\}$, $E_j = \{X \text{ is served by server } j\}$. Then,

$$P\{E_j\} = P\{E_j, D\} + P\{E_j, \bar{D}\}.$$

First we calculate $P\{E_j, D\}$. Write $P\{E_j, D\} = P\{E_j | D\} P\{D\}$. Given Poisson arrivals, $P\{D\} = \sum_{k=s}^{\infty} P_k = C(s,a)$, and given exponential service times, $P\{E_j | D\} = 1/s$. Thus

$$P\{E_j, D\} = \frac{1}{s} C(s,a).$$

Next we calculate $P\{E_j, \bar{D}\}$. Observe, $\{E_j, \bar{D}\}$ is equivalent to $\{k \leq s, E_j, \bar{D}\}$. With Poisson traffic, $P\{k \leq s\} = \sum_{k=0}^{s} P_k = 1 - \rho C(s,a)$, and conditional on $k \leq s$ the probability of service by server j without delay is $\tilde{p}_j / a$, according to Exercise 14, since the system functions like an Erlang loss system when $k \leq s$. Hence,

$$\begin{aligned} P\{E_j, \bar{D}\} &= P\{k \leq s, E_j, \bar{D}\} = P\{E_j, \bar{D} | k \leq s\} P\{k \leq s\}, \\ &= \frac{\tilde{p}_j}{a} [1 - \rho C(s,a)]. \end{aligned}$$

It follows that

$$P\{E_j\} = \frac{\tilde{p}_j [1 - \rho C(s,a)] + \rho C(s,a)}{a}$$

where $\tilde{p}_j = a[B(j-1,a) - B(j,a)]$. By (4.16) the numerator equals the load p_j carried by the j'th ordered server. Hence,

$$P\{E_j\} = \frac{p_j}{a} \quad (j = 1, 2, \ldots, s).$$

This result might have been easily derived by employing Little's theorem, $L = \lambda W$ (see Sec. 5.2), by which $p_j = \lambda P\{E_j\} \mu^{-1}$. $\square$

Chapter 3, Exercise 22

'Repeat Exercise 5 with "Erlang loss system" replaced by...'

We consider an Erlang delay system with $\lambda = 4$ and $\mu = 1$. Then the offered load is $a = \lambda\tau = \lambda\mu^{-1} = 4$. The objective function is $H(s;c) = \lambda 2.5 - \lambda P\{W > 0.5\} \cdot 10.0 - sc$. Thus

$$H(s;c) = 10 - 40\, P\{W > 0.5\} - sc.$$

By (4.25), $P\{W > t\} = C(s,a)\, e^{-(s\mu-\lambda)t}$. Thus

$$P\{W > 0.5\} = C(s,4)\, e^{-(s-4)0.5}.$$

It follows that

	5	6	7	8	9	10
$C(s,4)$	.5541	.2848	.1351	.0590	.0238	.0088
$e^{-(s-4)0.5}$	.6065	.3679	.2231	.1353	.0821	.0498
$P\{W>0.5\}$	.3361	.1048	.0301	.0080	.0020	.0004
$H(s;1.0)$	−8.44	−0.19	1.80	1.68	0.92	−0.01

Thus, at $c = 1.00$, the optimal number of servers is 7, and the corresponding profit rate equals 1.80. The break-even point for c is $c_0 = 1.00 + 1.80/7 = 1.26$. Given this operating cost, the entrepreneur will break even for $s = 7$, but will have a negative profit rate for $s \neq 7$.

In case the entrepreneur may select any customer from the queue, the profit will be maximized, for any s, if the customer selected is the one who has waited the longest, but less than ½ hr.

Chapter 3, Exercise 23

'Consider a 10-server Erlang delay system that handles...'

BCD	s	λ	μ^{-1}	$a = \lambda\mu^{-1}$	$C(s,a)$	$E(W\mid W>0) = \mu^{-1}/(s-a)$
case 0	10	λ_0	μ_0^{-1}	6	.1013	$W_0 = \mu_0^{-1}/4$
case 1	10	$(1+\frac{1}{3})\lambda_0$	μ_0^{-1}	8	.4092	$2W_0$
case 2	10	λ_0	$(1+\frac{1}{3})\mu_0^{-1}$	8	.4092	$2(1+\frac{1}{3})W_0$

(Chap. 3, Ex. 23)

Note, by (4.26), $E(W|W>0) = 1/[(1-\rho)s\mu]$, where $\rho = a/s$. Thus, $E(W|W>0) = \mu^{-1}/(s-a)$.

The lesson is that $C(s,a)$ depends on s and a, whereas the conditional mean wait $E(W|W>0)$ depends on s, a and μ. The response to a 1/3 increase in λ is a $100(0.4092-0.1013)/0.1013 = 304\%$ increase in $C(s,a)$ and a 100 % increase in $E(W|W>0)$. A 1/3 increase in μ^{-1}, resulting in the same a, also leads to a 304 % increase in $C(s,a)$, but the increase in $E(W|W>0)$ will be 167 %.

Chapter 3, Exercise 24

'In an Erlang delay system with service in order of arrival...'

By (4.24), $P\{W>t|W>0\} = e^{-(1-\rho)s\mu t}$, and by (4.26), $E(W|W>0) = \frac{1}{(1-\rho)s\mu}$. Hence,

$$P\{W > E(W|W>0)|W>0\} = e^{-(1-\rho)s\mu\frac{1}{(1-\rho)s\mu}} = e^{-1} = 0.3679.$$

Chapter 3, Exercise 25

'Consider a telephone system in which the central office...'

In an Erlang delay system with service in order of arrival the waiting time distribution for blocked customers is the exponential distribution $P\{W>t|W>0\} = e^{-(1-\rho)s\mu t}$, see (4.24). Hence, if a customer has waited 30 sec., his remaining waiting time will still be exponentially distributed with mean $[(1-\rho)s\mu]^{-1}$.

One thing the customer should not do after waiting 30 sec. is to put down the receiver and try again immediately. If he does that and waits until he gets through to the server, he will increase the waiting time by an expected 30ρ secs due to those customers who, thanks to his rash act, got ahead of him in the waiting line.

A better choice is to hang up and make another call $T>0$ secs. later, waiting until served. The associated expected waiting time will converge to $C(s,a)/[(1-\rho)s\mu]$ as $T\to\infty$. As the limiting value is less than $1/[(1-\rho)s\mu]$, the customer may be better off, everything considered, calling later. □

Chapter 3, Exercise 26

'Show that in the Erlang delay system...'

$$E(W^2) = [1-C(s,a)]\,E(W^2|W=0) + C(s,a)\,E(W^2|W>0)$$
$$= C(s,a)\,E(W^2|W>0).$$

By (4.24), with order-of-arrival service the waiting time for blocked customers will be exponentially distributed with parameter $(1-\rho)s\mu$. Hence, $E(W^2|W>0) = 2/[(1-\rho)s\mu]^2$, so that

$$E(W^2) = \frac{2\,C(s,a)}{(s\mu)^2(1-\rho)^2}.$$

By (4.27),

$$E^2(W) = \frac{C^2(s,a)}{(s\mu)^2(1-\rho)^2}.$$

The variance is derived by substitution of these two expressions into $V(W) = E(W^2) - E^2(W)$. The result is

$$V(W) = \frac{1-(1-C(s,a))^2}{(s\mu)^2(1-\rho)^2}.$$

Chapter 3, Exercise 27

'Let W be the waiting time and T the sojourn time...'

$s=1$. Hence, by (4.4) and (4.5), $P_j = (1-a)a^j$. With Poisson arrivals $\Pi_j = P_j$, so the probability that an arbitrary customer finds j present in the system is

$$\Pi_j = (1-a)a^j \quad (j=0,1,\ldots). \tag{1}$$

The probability that he will observe j in the queue, given that the server is occupied is $P\{Q=j|W>0\} = \Pi_{j+1}/\sum_{k=1}^{\infty}\Pi_k = (1-a)a^{j+1}/a$. Hence, see also (4.23),

$$P\{Q=j|W>0\} = (1-a)a^j \quad (j=0,1,\ldots). \tag{2}$$

Now assume order-of-arrival service. The sojourn time will be the sum of $j+1$ exponential service times where the probability distribution of j (and therefore of $j+1$) is given by (1). The conditional waiting time is the sum of $j+1$ exponential

(Chap. 3, Ex. 27)

service times where the probability distribution of j (and therefore of $j+1$) is given by (2).

The two probability distributions (1) and (2), are identical. Consequently, the sojourn time and the conditional waiting time follow the same distribution in this case, namely

$$P\{T>t\} = P\{W>t \mid W>0\} = e^{-(1-a)\mu t}$$

according to eq. (4.24).

Chapter 3, Exercise 28

'a. Consider an Erlang delay system, and denote by L...'

Suppose $a<s$. For convenience, let L_q and W_q denote the mean queue length and mean waiting time, resp., and let L_s and W_s denote the mean number of customers in the system and mean sojourn time, resp.

[a] By (4.4) and (4.7),

$$L_q = \sum_{j=s}^{\infty}(j-s)P_j = \sum_{j=s}^{\infty}(j-s)\frac{a^j}{s!\,s^{j-s}}P_0$$
$$= \frac{a^s}{s!(1-a/s)}P_0\sum_{k=0}^{\infty}k\left(\tfrac{a}{s}\right)^k\left(1-\tfrac{a}{s}\right) = C(s,a)\sum_{k=0}^{\infty}k\left(\tfrac{a}{s}\right)^k\left(1-\tfrac{a}{s}\right).$$

The mean of a geometric distribution with parameter $p = 1-\frac{a}{s}$ is $(1-p)/p = \frac{a}{s}/(1-\frac{a}{s})$. Hence, $\sum_{k=0}^{\infty} k(\frac{a}{s})^k(1-\frac{a}{s}) = \frac{a}{s}/(1-\frac{a}{s})$. Substitution of this expression, and a rewriting, yield

$$L_q = \lambda\frac{C(s,a)}{(1-\rho)s\mu}.$$

Finally, by (4.27),

$$L_q = \lambda W_q.$$

[b] Clearly $L_s = L_q + \frac{\lambda}{\mu}$ and $W_s = W_q + \frac{1}{\mu}$. Hence, using the relation $L_q = \lambda W_q$ it follows that

$$L_s = \lambda W_s.$$

□

Chapter 3, Exercise 29

'Prove that in an Erlang delay system with order-of-arrival service...'

Let P be the probability that a blocked customer will still be in the queue when next arrival takes place. Let r_j be the probability that a blocked customer, who joins the queue when $Q=j$ customers are waiting, will still be in the queue at next arrival epoch. By the theorem of total probability,

$$P = \sum_{j=0}^{\infty} r_j P\{Q=j|W>0\}.$$

The arrival rate is λ and, as long as all servers are busy, the service completion rate is $s\mu$. Therefore, by (5.23) of Chapter 2, $s\mu/(s\mu+\lambda)$ is the probability, in all-busy states, that next event will be a service completion rather than an arrival. Since, with service in order of arrival, the blocked customer will get into service before next arrival if and only if at least $j+1$ service completions occur before any arrival,

$$r_j = 1 - \left(\frac{s\mu}{s\mu+\lambda}\right)^{j+1} = 1 - \frac{1}{(1+\rho)^{j+1}}.$$

By (4.23), $P\{Q=j|W>0\} = (1-\rho)\rho^j$ if $\rho<1$. Hence, if $\rho<1$,

$$P = \sum_{j=0}^{\infty}\left(1-\frac{1}{(1+\rho)^{j+1}}\right)(1-\rho)\rho^j = 1 - \frac{1-\rho}{1+\rho}\sum_{j=0}^{\infty}\left(\frac{\rho}{1+\rho}\right)^j = 1 - \frac{1-\rho}{1+\rho}\,\frac{1}{1-[\rho/(1+\rho)]}.$$

Hence, $P=\rho$ as asserted.

Chapter 3, Exercise 30

'Let N be the number of customers found by an arrival...'

Evidently, eq. (4.19), $P\{W>t|W>0\} = \sum_{j=0}^{\infty} P\{W>t|N=s+j\}P\{Q=j|W>0\}$, holds for any Erlang delay system regardless of queue discipline. By definition, $P\{Q=j|W>0\} = P\{N=s+j\}/\sum_{k=0}^{\infty} P\{N=s+k\}$. For nonbiased queue disciplines $\{N(t)\}$ is a birth-and-death process, independent of the discipline. Hence $P\{N=k\}$, and therefore $P\{Q=j|W>0\}$, are the same for all nonbiased q.d. By (4.23), for order-of-arrival service $P\{Q=j|W>0\} = (1-\rho)\rho^j$ $(j=0,1,\ldots)$. It follows that for all nonbiased q.d. we have $P\{Q=j|W>0\} = (1-\rho)\rho^j$ $(j=0,1,\ldots)$. Substitution into (4.19) shows that

$$P\{W>t|W>0\} = (1-\rho)\sum_{j=0}^{\infty} \rho^j P\{W>t|N=s+j\},$$

for all nonbiased queue disciplines. $\square$

Chapter 3, Exercise 31

'Consider the differential-difference equations...

$$\frac{d}{dt} F_j(t) = c F_{j-1}(t) - c F_j(t) \qquad [t \geqq 0;\ j = 0,1,\ldots;\ F_{-1}(t) = 0] \tag{1}$$

where c is an arbitrary constant. Define

$$F(x,t) = \sum_{j=0}^{\infty} F_j(t)\, x^j. \tag{2}$$

a

$$\frac{d}{dt} F_j(t)\, x^j = c F_{j-1}(t)\, x^j - c F_j(t)\, x^j \qquad (j = 0,1,\ldots),$$

$$\sum_{j=0}^{\infty} \frac{d}{dt} F_j(t)\, x^j = c x \sum_{j=1}^{\infty} F_{j-1}(t)\, x^{j-1} - c \sum_{j=0}^{\infty} F_j(t)\, x^j,$$

$$\frac{d}{dt} \sum_{j=0}^{\infty} F_j(t)\, x^j = c x \sum_{j=0}^{\infty} F_j(t)\, x^j - c \sum_{j=0}^{\infty} F_j(t)\, x^j,$$

$$\frac{\partial}{\partial t} F(x,t) = c(x-1) F(x,t). \tag{3}$$

Hence, $F(x,t) = k(x)\, e^{-(1-x)ct}$, by which

$$F(x,t) = F(x,0)\, e^{-(1-x)ct}. \tag{4}$$

b In the case of a Poisson process Eq. (1) holds with $c = \lambda$ and $F_j(t) = P_j(t) = P\{N(t) = j\}$, according to Eq. (2.5) of Chapter 2. As $F_0(0) = 1$, clearly $F(x,0) = 1$, so that in this case $F(x,t) = e^{-(1-x)ct} = e^{-\lambda t(1-x)}$. By Eq. (4.3) of Chapter 2, this is the generating function of a Poisson distribution with parameter λt. It follows that the probability of j arrivals in $[0,t]$ equals

$$P_j(t) = \frac{(\lambda t)^j}{j!}\, e^{-\lambda t} \qquad (j = 0,1,\ldots).$$

c As long as there are at least s customers in the system, the departure process is Poisson with parameter $s\mu$. Let $\tilde{P}_i(t)$ denote the probability of i departures within $[0,t]$, assuming that all servers are busy. By the usual argument,

$$\tilde{P}_i(t+h) = \tilde{P}_i(t)[1 - hs\mu] + \tilde{P}_{i-1}(t)\, hs\mu + o(h) \qquad (i = 0,1,\ldots).$$

(Chap. 3, Ex. 31 c)

Hence, with $\tilde{P}_{-1}(t) = 0$,

$$\sum_{i=0}^{j} \tilde{P}_i(t+h) = [1 - hs\mu] \sum_{i=0}^{j} \tilde{P}_i(t) + hs\mu \sum_{i=0}^{j-1} \tilde{P}_i(t) + o(h) \qquad (j = 0, 1, \ldots)$$

Evidently, $W_j(t) = \sum_{i=0}^{j} \tilde{P}_i(t)$, so that

$$W_j(t+h) = [1 - hs\mu] W_j(t) + hs\mu W_{j-1}(t) + o(h) \qquad (j = 0, 1, \ldots),$$

where $W_{-1}(t) = 0$. Hence,

$$\frac{d}{dt} W_j(t) = s\mu W_{j-1}(t) - s\mu W_j(t) \quad [t \geq 0;\ j = 0, 1, \ldots;\ W_{-1}(t) = 0] \qquad (5)$$

$\boxed{d}$ Eq. (5) has the same form as Eq. (1). Consequently, if we define

$$W(x,t) = \sum_{j=0}^{\infty} W_j(t) x^j, \qquad (6)$$

then, by (2) and (4),

$$W(x,t) = W(x,0)\, e^{-(1-x)s\mu t}. \qquad (7)$$

$\boxed{e}$ $W_j(0) = P\{W > 0 \mid N = j + s\} = 1$ for all j. Hence, for $x < 1$,

$$W(x,0) = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}. \qquad (8)$$

$\boxed{f}$

$$\begin{aligned} P\{W > t \mid W > 0\} &= (1-\rho) \sum_{j=0}^{\infty} P\{W > t \mid N = s + j\} \rho^j && [\text{by Ex. 30c}] \\ &= (1-\rho) \sum_{j=0}^{\infty} W_j(t) \rho^j && [\text{by def. of } W_j(t)] \\ &= (1-\rho) W(\rho, t). && [\text{by def. of } W(x,t)] \qquad (9) \end{aligned}$$

$\boxed{g}$ Equations (7), (8), (9) together yield

$$\begin{aligned} P\{W > t \mid W > 0\} &= (1-\rho) \frac{1}{1-\rho} e^{-(1-\rho)s\mu t} \\ &= e^{-(1-\rho)s\mu t}. \end{aligned} \qquad (10)$$

(Chap. 3, Ex. 31 h)

$\boxed{h}$ By Equations (6), (7) and (8),

$$\sum_{j=0}^{\infty} W_j(t)\, x^j = \frac{1}{1-x}\, e^{-s\mu t} e^{s\mu t x}.$$

Thus,

$$\sum_{j=0}^{\infty} W_j(t)\, x^j = \Big(\sum_{i=0}^{\infty} x^i\Big) e^{-s\mu t}\Big(\sum_{i=0}^{\infty} \frac{(s\mu t)^i}{i!} x^i\Big),$$

$$\sum_{j=0}^{\infty} W_j(t)\, x^j = \sum_{j=0}^{\infty}\Big(\sum_{k=0}^{j} \frac{(s\mu t)^k}{k!} e^{-s\mu t}\Big) x^j .$$

Equating coefficients of x^j on left- and right-hand sides yields

$$W_j(t) = \sum_{k=0}^{j} \frac{(s\mu t)^k}{k!} e^{-s\mu t}. \qquad (11)$$

Chapter 3, Exercise 32

'Service in random order.' – cf Ex. 28 and 36 of Chap. 5

$$W_j(t) = P\{W > t \mid N = s + j\}$$

$\boxed{a}$ Let the test customer arrive at $t = 0$. During the time interval $[0, h]$ one of the following mutually exclusive events will occur: (1) The test customer departs from queue, (2) A customer arrives; (3) A customer other than the test customer departs from queue, (4) Neither arrival nor departure from queue (system) take place; (5) Two or more arrivals or departures occur.

Event 1 precludes the possibility that the test customer will be present in the queue at time $h+t$, and event 5 has probability $o(h)$. Disregarding terms of order $o(h)$, events 2, 3 and 4 have probability λh, $(j/j+1)s\mu h$ and $1-(\lambda+s\mu)h$, respectively. Hence, by the theorem of total probability,

$$W_j(h+t) = \lambda h\, W_{j+1}(t) + \frac{j}{j+1} s\mu h\, W_{j-1}(t) + [1-(\lambda+s\mu)h]\, W_j(t) + o(h), \qquad (1)$$
$$[j = 0, 1, \ldots ;\ W_{-1}(t) = 0].$$

Hence,

$$\frac{d}{dt} W_j(t) = \lambda W_{j+1}(t) + \frac{j}{j+1} s\mu W_{j-1}(t) - (\lambda+s\mu) W_j(t) \quad [j = 0, 1, \ldots;\ W_{-1}(t) = 0], \quad (2)$$

where $W_j(0) = 1 \quad (j = 0, 1, \ldots)$.

(Chap. 3, Ex. 32 b)

$\boxed{b}$ Define

$$W_j^{(\nu)} = \frac{d^\nu}{dt^\nu} W_j(t)\Big|_{t=0} \quad (j=0,1,\dots;\ \nu=0,1,\dots). \tag{3}$$

In particular, $W_j^{(0)} = W_j(0) = 1$.

Suppose that $W_j(t)$ has the Maclaurin series representation

$$W_j(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} W_j^{(\nu)} \quad (j=0,1,\dots). \tag{4}$$

According to Exercise 30,

$$P\{W>t \mid W>0\} = (1-\rho)\sum_{j=0}^{\infty} \rho^j W_j(t)$$

By (4),

$$\begin{aligned} P\{W>t \mid W>0\} &= (1-\rho)\sum_{j=0}^{\infty} \rho^j \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} W_j^{(\nu)} \\ &= (1-\rho)\sum_{j=0}^{\infty} \rho^j \Big[1 + \sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu!} W_j^{(\nu)}\Big] \end{aligned}$$

Use of $\Sigma_{j=0}^{\infty} \rho^j = (1-\rho)^{-1}$, and a change of the order of summation yield

$$P\{W>t \mid W>0\} = 1 + (1-\rho)\sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu!} \sum_{j=0}^{\infty} \rho^j W_j^{(\nu)}. \tag{5}$$

$\boxed{c}$ Repeated differentiation of Equation (2) gives

$$\begin{aligned} \frac{d^\nu}{dt^\nu} W_j(t) &= \lambda \frac{d^{\nu-1}}{dt^{\nu-1}} W_{j+1}(t) + \frac{j}{j+1} s\mu \frac{d^{\nu-1}}{dt^{\nu-1}} W_{j-1}(t) \\ &\quad - (\lambda + s\mu) \frac{d^{\nu-1}}{dt^{\nu-1}} W_j(t) \qquad [j=0,1,\dots,\ \nu=1,2,\dots] \end{aligned}$$

Setting $t=0$ we obtain

$$W_0^{(\nu)} = \lambda W_1^{(\nu-1)} - (\lambda+s\mu) W_0^{(\nu-1)} \qquad (\nu=1,2,\dots),$$

$$W_j^{(\nu)} = \lambda W_{j+1}^{(\nu-1)} + \frac{j}{j+1} s\mu W_{j-1}^{(\nu-1)} - (\lambda+s\mu) W_j^{(\nu-1)} \quad \begin{pmatrix} j=1,2,\dots; \\ \nu=1,2,\dots \end{pmatrix}.$$

First we solve for $\nu=1$. Recalling that $W_j^{(0)}=1$ for all j, we easily derive

$$W_j^{(1)} = -\frac{s\mu}{j+1} \qquad (j=0,1,\dots). \tag{$*$}$$

(Chap. 3, Ex. 32 c)

Next we solve for $\nu = 2$, making use of (*). The result is

$$W_j^{(2)} = \begin{cases} (s\mu)^2[1+\frac{\rho}{2}] & (j=0), \\ (s\mu)^2\frac{\rho}{(j+1)(j+2)} & (j \geq 1). \end{cases} \qquad (**)$$

By (*),

$$\sum_{j=0}^{\infty}\rho^j W_j^{(1)} = -s\mu\sum_{j=0}^{\infty}\frac{\rho^j}{j+1} = -s\mu\frac{1}{\rho}\sum_{j=1}^{\infty}\frac{\rho^j}{j}.$$

For $0<\rho<1$, $\Sigma_{j=1}^{\infty}\rho^j/j = -\ln(1-\rho) = \ln\frac{1}{1-\rho}$. Hence,

$$\sum_{j=0}^{\infty}\rho^j W_j^{(1)} = -s\mu\frac{1}{\rho}\ln\frac{1}{1-\rho}.$$

By (**),

$$\sum_{j=0}^{\infty}\rho^j W_j^{(2)} = (s\mu)^2[1+\sum_{j=0}^{\infty}\frac{\rho^{j+1}}{(j+1)(j+2)}] = (s\mu)^2[1+\frac{1}{\rho}\sum_{j=1}^{\infty}\frac{\rho^{j+1}}{j(j+1)}].$$

Now let

$$S(\rho) = \sum_{j=1}^{\infty}\frac{\rho^{j+1}}{j(j+1)}.$$

Considering $S(\rho)$ as a function of ρ, differentiation results in $dS(\rho)/d\rho = \Sigma_{j=1}^{\infty}\rho^j/j = -\ln(1-\rho)$. Therefore, reversing the process, $S(\rho) = c + \int(-\ln(1-\rho))d\rho = c + (1-\rho)\ln(1-\rho) - (1-\rho)$. From $S(0)=0$ we derive $c = 1$. Thus

$$\sum_{j=1}^{\infty}\frac{\rho^{j+1}}{j(j+1)} = \rho - (1-\rho)\ln\frac{1}{1-\rho}.$$

it follows that

$$\sum_{j=0}^{\infty}\rho^j W_j^{(2)} = (s\mu)^2[2-\frac{1-\rho}{\rho}\ln\frac{1}{1-\rho}].$$

Finally, substitution of the found expressions for $\Sigma_{j=0}^{\infty}\rho^j W_j^{(1)}$ and $\Sigma_{j=0}^{\infty}\rho^j W_j^{(2)}$ into Equation (5) gives

$$P\{W>t|W>0\} = 1 - s\mu t\frac{1-\rho}{\rho}\ln\frac{1}{1-\rho} + \frac{(s\mu t)^2}{2!}(1-\rho)[2-\frac{1-\rho}{\rho}\ln\frac{1}{1-\rho}] - + \cdots \qquad (6)$$

□

Chapter 3, Exercise 33

'Let $B(t)$ be the distribution function of the busy period...'

It is clear that when service is in reverse order of arrival, then, for all $N \geq 1$, the waiting time is a busy period initiated by the presently served customer (whose remaining time in service is exponentially distributed) and including all arrivals later than the test customer until he is permitted to enter service. (The same holds true for a GI/M/s system for $N \geq s$.) Hence, for an arbitrary customer, $P\{W \leq t \mid W>0\} = B(t)$. It follows that the mean waiting time for waiting customers is the mean of the busy period, that is, $E(W \mid W>0) = b = \tau/(1-\rho)$, by (4.12).

Chapter 3, Exercise 34

'Show that $\lim \Pi_j[n] = P_j$, ...'

By (7.7) and $\hat{a} = \gamma/\mu$,

$$\Pi_j[n] = \frac{\binom{n-1}{j}\left(\frac{\gamma}{\mu}\right)^j}{\sum_{k=0}^{s}\binom{n-1}{k}\left(\frac{\gamma}{\mu}\right)^k} \qquad (j = 0, 1, \ldots, s).$$

For $j = 0, 1, \ldots, s$,

$$\lim_{\substack{n\to\infty \\ \gamma\to 0 \\ n\gamma=\lambda}} \binom{n-1}{j}\left(\frac{\gamma}{\mu}\right)^j = \lim_{n\to\infty} \binom{n-1}{j}\left(\frac{\lambda}{\mu n}\right)^j$$

$$= \left[(\lambda/\mu)^j/j!\right] \lim_{n\to\infty}\left(\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{n-j}{n}\right)$$

$$= (\lambda/\mu)^j/j!$$

Hence,

$$\lim_{\substack{n\to\infty \\ \gamma\to 0 \\ n\gamma=\lambda}} \Pi_j[n] = \frac{\lim\binom{n-1}{j}\left(\frac{\gamma}{\mu}\right)^j}{\sum_{k=0}^{s}\lim\binom{n-1}{k}\left(\frac{\gamma}{\mu}\right)^k} = \frac{(\lambda/\mu)^j/j!}{\sum_{k=0}^{s}(\lambda/\mu)^k/k!} = P_j,$$

where, by Equation (3.3), P_j is the statistical equilibrium probability of j busy servers in the Erlang loss system. $\square$

Chapter 3, Exercise 35

'Four sources share access to two servers.'

n = number of sources; $s = 2$; $\gamma^{-1} = 27$ min.; $\mu^{-1} = 3$ min.; $\hat{a} = \gamma/\mu = \frac{1}{9}$; blocked customers cleared.

The blocking probability is given by Engset's formula, that is Eq. (7.7) for $j = s$:

$$\Pi_s[n] = \frac{\binom{n-1}{s}\hat{a}^s}{\sum_{k=0}^{s}\binom{n-1}{k}\hat{a}^k}.$$

Since

$$\Pi_2[4] = \frac{3(\frac{1}{9})^2}{1 + 3(\frac{1}{9}) + 3(\frac{1}{9})^2} = \frac{1}{37} = 0.0270,$$

and

$$\Pi_2[5] = \frac{6(\frac{1}{9})^2}{1 + 4(\frac{1}{9}) + 6(\frac{1}{9})^2} = \frac{2}{41} = 0.0488,$$

the effect of going from four to five sources is a percent increase in the probability of blocking equal to

$$P = 100\,[(\Pi_2[5]/\Pi_2[4]) - 1] = 80\,\%.$$

To calculate the expected number of requests for service per hour, say r, we go through the following steps:

(i) $P_s[n] = \binom{n}{s}\hat{a}^s / \sum_{k=0}^{s}\binom{n}{k}\hat{a}^k$ (7.3),

(ii) $a' = a^*(1 - (1 - \frac{s}{n})P_s[n])$, where $a^* = n\hat{a}/(1+\hat{a})$ (7.8),

(iii) $a = a'/(1 - \Pi_s[n])$ (7.9),

(iv) $r = 60 \cdot a/\mu^{-1} = 20 \cdot a$.

For $n = 4$ and $n = 5$ we find

	$P_2[n]$	a'	a	r
$n = 4$	$\frac{2}{41} = 0.0488$	$\frac{16}{41} = 0.3902$	$\frac{148}{369} = 0.4011$	8.02
$n = 5$	$\frac{5}{68} = 0.0735$	$\frac{65}{136} = 0.4779$	$\frac{205}{408} = 0.5025$	10.05

The lower bounds for r are 8 and 10, respectively. □

Chapter 3, Exercise 36

'Verify equation (8.10)'

By Equations (6.8), (8.3), (8.6), (8.7) and (8.8),

$$
\begin{aligned}
P\{W>t\} &= \sum_{j=0}^{n-s-1} P\{W>t \mid N=s+j\}\, P\{N=s+j\} \\
&= \sum_{j=0}^{n-s-1} \left(e^{-s\mu t} \sum_{i=0}^{j} \frac{(s\mu t)^i}{i!} \right) P_{s+j}[n-1] \\
&= e^{-s\mu t} \sum_{j=0}^{n-s-1} \left(\sum_{i=0}^{j} \frac{(s\mu t)^i}{i!} \right) \frac{(n-1)!\, \hat{a}^{s+j}}{(n-1-s-j)!\, s!\, s^j} P_0[n-1] \\
&= c\, e^{-\phi(t)} \sum_{j=0}^{n-s-1} \sum_{i=0}^{j} \frac{\left(\frac{\hat{a}}{s}\right)^{-(n-1-s-j)}}{(n-1-s-j)!} \frac{(s\mu t)^i}{i!},
\end{aligned}
$$

where

$$\phi(t) = \frac{s\mu}{\gamma} + s\mu t$$

and

$$c = \Pi_0[n] \frac{(n-1)!\, \hat{a}^s}{s!} \left(\frac{\hat{a}}{s}\right)^{n-s-1} e^{s\mu/\gamma}$$

Thus

$$P\{W>t\} = c\, e^{-\phi(t)} \sum_{j=0}^{n-s-1} \sum_{i=0}^{j} \frac{(s\mu/\gamma)^{n-1-s-j}}{(n-1-s-j)!} \frac{(s\mu t)^i}{i!}.$$

By the substitution $k = n-1-s-j$,

$$P\{W>t\} = c\, e^{-\phi(t)} \sum_{k=0}^{n-s-1} \sum_{i=0}^{n-s-1-k} \frac{(s\mu/\gamma)^k}{k!} \frac{(s\mu t)^i}{i!}.$$

Defining $x = s\mu/\gamma$ and $y = s\mu t$ the doublesum may be written

$$
\begin{aligned}
\sum_{k=0}^{n-s-1} \sum_{i=0}^{n-s-1-k} \frac{x^k}{k!} \frac{y^i}{i!} &= \frac{x^0 y^0}{0!\,0!} + \left(\frac{x^0 y^1}{0!\,1!} + \frac{x^1 y^0}{1!\,0!} \right) + \left(\frac{x^0 y^2}{0!\,2!} + \frac{x^1 y^1}{1!\,1!} + \frac{x^2 y^0}{2!\,0!} \right) \\
&\quad + \cdots + \left(\frac{x^0 y^{n-s-1}}{0!\,(n-s-1)!} + \cdots + \frac{x^{n-s-1} y^0}{(n-s-1)!\,0!} \right)
\end{aligned}
$$

By the binomial formula, $\sum_{m=0}^{j} \frac{x^m}{m!} \frac{y^{j-m}}{(j-m)!} = \frac{(x+y)^j}{j!}$. Hence

$$\sum_{k=0}^{n-s-1} \sum_{i=0}^{n-s-1-k} \frac{x^k}{k!} \frac{y^i}{i!} = \sum_{j=0}^{n-s-1} \frac{(x+y)^j}{j!}.$$

We conclude that

$$P\{W>t\} = c \sum_{j=0}^{n-s-1} \frac{[\phi(t)]^j}{j!} e^{-\phi(t)} \qquad (8.10) \qquad \square$$

Chapter 3, Exercise 37

'Verify equation (8.18) by direct calculation'

As $a' = a$ by (8.16), then

$$\sum_{j=1}^{n} j P_j[n] = \left(\sum_{j=1}^{s} j P_j[n] + \sum_{j=s+1}^{n} s P_j[n]\right) + \sum_{j=s+1}^{n} (j-s) P_j[n]$$
$$= a + \sum_{k=0}^{n-s-1} (k+1) P_{s+k+1}[n] \qquad (1)$$

By (8.3),
$$\frac{P_{s+k+1}[n]}{P_{s+k}[n-1]} = \frac{n\hat{a}}{s} \frac{P_0[n]}{P_0[n-1]}. \qquad (2)$$

By (1) and (2),
$$\sum_{j=1}^{n} j P_j[n] = a + n\hat{a} \frac{P_0[n]}{P_0[n-1]} \frac{1}{\mu^{-1}} \sum_{k=0}^{n-s-1} \frac{k+1}{s\mu} P_{s+k}[n-1]. \qquad (3)$$

$P_{s+k}[n-1] = P\{N = s+k\}$ by (8.7) and $(k+1)/s\mu = E(W|N = s+k)$. Substitution into (3) and application of $E(W) = \Sigma_{k=0}^{n-s-1} E(W|N=s+k)P\{N=s+k\}$, see (8.13), gives
$$\sum_{j=1}^{n} j P_j[n] = a + n\hat{a} \frac{P_0[n]}{P_0[n-1]} \frac{E(W)}{\mu^{-1}}. \qquad (4)$$

By (8.4),
$$\frac{P_0[n]}{P_0[n-1]} = \sum_{j=0}^{s-1} \binom{n-1}{j} \hat{a}^j P_0[n] + \sum_{j=s}^{n-1} \frac{(n-1)!}{(n-1-j)!\,s!\,s^{j-s}} \hat{a}^j P_0[n]$$
$$= \frac{1}{n} \left(\sum_{j=0}^{s-1} (n-j)\binom{n}{j} \hat{a}^j P_0[n] + \sum_{j=s}^{n} (n-j) \frac{n!}{(n-j)!\,s!\,s^{j-s}} \hat{a}^j P_0[n] \right).$$

By (8.3) this is seen to equal
$$\frac{P_0[n]}{P_0[n-1]} = \frac{1}{n} \sum_{j=0}^{n} (n-j) P_j[n] = \frac{1}{n}\left(n - \sum_{j=0}^{n} j P_j[n]\right)$$

Substituting $n - \Sigma_{j=0}^{n} j P_j[n] = a/\hat{a}$ from (8.17) we finally obtain
$$\frac{P_0[n]}{P_0[n-1]} = \frac{a}{n\hat{a}}. \qquad (5)$$

By (4) and (5),
$$\sum_{j=1}^{n} j P_j[n] = a\left(1 + \frac{E(W)}{\mu^{-1}}\right) \qquad (8.18)$$

$\square$

Chapter 3, Exercise 38

'Reconsider Exercise 35, but instead of...'

n = number of sources, $s = 2$, $\gamma^{-1} = 27$ min., $\mu^{-1} = 3$ min., $\hat{a} = \gamma/\mu = \frac{1}{9}$; blocked customers delayed.

To begin, we calculate the state distribution $\{P_j[n]\}_j$, for $n = 3, 4, 5$, using Equations (8.3) and (8.4).

$P_j[n]$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$n=3$	.7286	.2429	.0270	.0015	–	–
$n=4$	.6548	.2910	.0485	.0054	.0003	–
$n=5$	.5875	.3264	.0725	.0121	.0013	.0001

a Blocking probability.

$$P_B = \sum_{j=s}^{n-1} \Pi_j[n] = \sum_{j=s}^{n-1} P_j[n-1].$$

$n=4$: $\quad P_B = \sum_{j=2}^{3} P_j[3] = 0.0270 + 0.0015 = \underline{0.0285}$

$n=5$: $\quad P_B = \sum_{j=2}^{4} P_j[4] = 0.0485 + 0.0054 + 0.0003 = \underline{0.0542}$

The percent increase in blocking probability is

$$P = 100\left[\left(\sum_{j=2}^{4} \Pi_j[5] \Big/ \sum_{j=2}^{3} \Pi_j[4]\right) - 1\right] = 90\,\%$$

b Requests for service per hour.

$$r = 60\,a/\mu^{-1} = 20\,a.$$

Since $a = a'$ for a BCD system,

$$a = a' = \sum_{j=0}^{s-1} j P_j[n] + s \sum_{j=s}^{n} P_j[n] = P_1[n] + 2(1 - P_0[n] - P_1[n]) = 2 - 2P_0[n] - P_1[n].$$

$n=4$: $\quad a = a' = 2 - 2\cdot 0.6548 - 0.2910 = 0.3994$, $\quad r = 20a = \underline{7.988}$

$n=5$: $\quad a = a' = 2 - 2\cdot 0.5875 - 0.3264 = 0.4986$, $\quad r = 20a = \underline{9.972}$

Upper bounds for r are 8 and 10, respectively.

(Chap. 3, Ex. 38 c)

$\boxed{c}$ Server occupancy.

$$\rho = a'/s .$$

$n = 4$: $\quad \rho = 0.3994/2 = \underline{0.1997}$

$n = 5$: $\quad \rho = 0.4986/2 = \underline{0.2493}$

$\boxed{d}$ Mean waiting time

By Eqs. (8.7), (8.9) and (8.13), the mean waiting time in seconds is

$$E(W) = \frac{60}{s\mu}\sum_{i=s}^{n-1}(i-s+1)\,P_i[n-1].$$

$n = 4$: $\quad E(W) = 90(P_2[3] + 2P_3[3]) = 90(0.0270 + 2\cdot 0.0015) = \underline{2.7}$ sec

$n = 5$: $\quad E(W) = 90(P_2[4] + 2P_3[4] + 3P_4[4]) = 90(0.0485 + 2\cdot 0.0054 + 3\cdot 0.0003) = \underline{5.4}$ sec

$\boxed{e}$ $P\{W > 45 \text{ sec.}\}$

By Eqs. (8.7), (8.10), (8.11) and (8.12),

$$P\{W > t\} = c_0\, e^{-s\mu t}\sum_{j=0}^{n-s-1}\frac{[\phi(t)]^j}{j!}, \qquad (8.10a)$$

where

$$\phi(t) = \frac{s\mu}{\gamma} + s\mu t \qquad (8.11)$$

and

$$c_0 = P_0[n-1]\,\frac{(n-1)!\,\hat{a}^s}{s!}\left(\frac{\hat{a}}{s}\right)^{n-s-1}, \qquad (8.12a)$$

with t measured in minutes.

For $t = 3/4$ min., $e^{-s\mu t} = e^{-1/2} = 0.6065$ and $\phi(t) = 18.5$. Also, $\hat{a} = 1/9$ and $\hat{a}/s = 1/18$.

$n = 4$: $\quad c_0 = P_0[3]\,\frac{3!}{2!\,9^2}\,\frac{1}{18} = \frac{0.7286}{486} = 0.001499,$

$P\{W > 3/4\} = c_0 e^{-1/2}(1 + 18.5) = 0.001499\cdot 0.6065\cdot 19.5 = \underline{0.0177}.$

$n = 5$: $\quad c_0 = P_0[4]\,\frac{4!}{2!\,9^2}\,\frac{1}{18^2} = \frac{0.6548}{2187} = 0.000299,$

$P\{W > 3/4\} = c_0 e^{-1/2}(1 + 18.5 + \frac{18.5^2}{2!}) = 0.000299\cdot 0.6065\cdot 190.6 = \underline{0.0346}.$

(Chap. 3, Ex. 38 f)

$\boxed{f}$ Proportion of time a source is idle

Evidently,

$$f = \frac{\gamma^{-1}}{\gamma^{-1} + E(W)/60 + \mu^{-1}} \qquad \left(= \frac{n - \sum_{j=1}^{n} j P_j[n]}{n} \right).$$

$n = 4$: $\quad f = 27/(27 + 2.7/60 + 3) = \underline{0.899}$.

$n = 5$: $\quad f = 27/(27 + 5.4/60 + 3) = \underline{0.897}$.

Compare with upper bound 0.9.

Chapter 3, Exercise 39

'Using Equations (6.3) and (8.17), show that $a\Pi_j[n] = (n-j)\hat{a}P_j[n]$...'

Assume a BCD system with quasi-random input generated by n sources and with exponential service times. By Eq. (6.3),

$$\Pi_j[n] = \frac{(n-j)P_j[n]}{\sum_{k=0}^{n-1}(n-k)P_k[n]} \qquad (j = 0, 1, \ldots, n-1). \tag{6.3}$$

Clearly, $\Pi_n[n] = 0$. Thus (6.3) is valid also for $j = n$. Furthermore, extending the summation to include $k = n$ does not affect the value of the denominator. Hence, by (6.3),

$$\Pi_j[n] = \frac{(n-j)P_j[n]}{n - \sum_{k=1}^{n} kP_k[n]} \qquad (j = 0, 1, \ldots, n). \tag{$*$}$$

Now, by Eq. (8.17),

$$n - \sum_{k=1}^{n} kP_k[n] = \frac{a}{\hat{a}}.$$

Inserting this expression into $(*)$ we obtain

$$a\Pi_j[n] = (n-j)\hat{a}P_j[n] \qquad (j = 0, 1, \ldots, n).$$

□

Chapter 3, Exercise 40

'Consider a single-server queueing system with quasirandom...'

The queueing process under consideration is a birth-and-death process with $\lambda_j = (n-j)\gamma$ and $\mu_j = j\mu$ for $j = 0, 1, \ldots, n$. Use of Eq. (1.1) results in the equilibrium state probabilities

$$P_j[n] = \binom{n}{j} \hat{a}^j P_0[n] \qquad (j = 0, 1, \ldots, n),$$

with $\hat{a} = \gamma/\mu$. As $1 = \sum_{j=0}^{n} P_j[n] = (1+\hat{a})^n P_0[n]$,

$$P_j[n] = \frac{\binom{n}{j}\hat{a}^j}{(1+\hat{a})^n} \qquad (j = 0, 1, \ldots, n). \qquad (*)$$

Since we deal with a queue with quasirandom input and blocked customers delayed, it is true that $\Pi_j[n] = P_j[n-1]$ for all $j = 0, 1, \ldots, n-1$. Hence, by (*),

$$\Pi_j[n] = \frac{\binom{n-1}{j}\hat{a}^j}{(1+\hat{a})^{n-1}} \qquad (j = 0, 1, \ldots, n-1).$$

Chapter 3, Exercise 41

'Queue with feedback'.

The arrival rate of new customers to the system is λ. The effective departure rate (from system) per customer in service is $(1-p)\mu$. Thus the queueing process is a birth-and-death process with the parameters $\lambda_n = \lambda$ for all n, and $\mu_n = n(1-p)\mu$ for $0 \leq n \leq s-1$, $\mu_n = s(1-p)\mu$ for $n \geq s$. Offered load is $a = \lambda/[(1-p)\mu]$.

The state of the system behaves precisely as in an ordinary BCD queue with parameters s, λ and $(1-p)\mu$. Also, $\Pi_j = P_j$ due to Poisson arrivals, where $\{\Pi_j\}$ is the arrival distribution for new customers. The equilibrium probability that a new arrival finds all servers busy equals

$$C(s,a) = \sum_{j=s}^{\infty} \Pi_j = \sum_{j=s}^{\infty} P_j \qquad (a < s),$$

with $C(s,a)$ given by Erlang's delay formula, Eq. (4.8). $\square$

Chapter 3, Exercise 42

'A single server serves customers of two priority classes...'

Poisson arrivals and exponential service times are assumed for both customer classes. The parameters are λ_1 and μ_1 for the high priority class, λ_2 and μ_2 for the low priority class. In parts (a)-(e) preemptive-repeat priority discipline will be assumed.

$\boxed{a}$ By Eq. (5.23) of Chapter 2, the probability of preemption for a class 2 customer who has just entered or reentered service equals $\lambda_1/(\lambda_1+\mu_2)$. Hence, the number N of preemptions experienced by a class 2 customer has the geometric distribution

$$P\{N=k\} = \left(\frac{\lambda_1}{\lambda_1+\mu_2}\right)^k \frac{\mu_2}{\lambda_1+\mu_2} \quad (k=0,1,\ldots) \qquad (1)$$

$\boxed{b}$ The accumulated service time of a class 2 customer is not affected by preemptions (which in effect only interrupt the service), given exponential service time and preemptive-repeat rule. Letting S denote the total time an arbitrary class 2 customer occupies the server, we have

$$P\{S \leqq t\} = 1 - e^{-\mu_2 t}, \qquad (2)$$

just as if there were no preemptions allowed.

$\boxed{c}$ Let T denote the extended service time, composed of the actual service time S and the sum $\Sigma_{j=1}^{N} X_j$ of the N time intervals during which the customer is preempted from service:

$$T = S + \sum_{j=1}^{N} X_j . \qquad (3)$$

Since N and $\{X_j\}$ are independent, by part (b) of Exercise 4 of Chapter 2,

$$E(T) = E(S) + E(N)E(X), \qquad (4)$$

(Chap. 3, Ex. 42 c)

where E(X) denotes the common mean of $X_1, X_2, \dots$. Now,

$$E(S) = \frac{1}{\mu_2}, \tag{5}$$

$$E(N) = \frac{\lambda_1/(\lambda_1+\mu_2)}{\mu_2/(\lambda_1+\mu_2)} = \frac{\lambda_1}{\mu_2}, \tag{6}$$

$$E(X) = \frac{\mu_1^{-1}}{1-(\lambda_1/\mu_1)} = \frac{1}{\mu_1-\lambda_1}. \quad (\mu_1 > \lambda_1). \tag{7}$$

Eq. (5) follows from Eq. (2). Eq. (6) follows from Eq. (1) since a variable with the geometric distribution $P\{N=k\} = q^k p$ has the mean q/p. See also Chapter 2, Exercise 21 a ($E(M) = \lambda\tau$). Eq. (7) follows from the observation that each X_j is a busy period in a single-server queue with only class 1 customers. Thus Eq. (4.12) applies with $\tau = \mu_1^{-1}$ and $a = \lambda_1/\mu_1$.

Substitution of (5), (6) and (7) into (4) yields

$$E(T) = \frac{\mu_1}{\mu_2(\mu_1-\lambda_1)} = \frac{\mu_2^{-1}}{1-a_1} \quad (\mu_1 > \lambda_1). \tag{8}$$

$\boxed{d}$ The service of high-priority customers is in no way affected by the presence of low-priority customers. Therefore, the waiting time W_1 of an arbitrary class 1 customer will have the distribution given by Eq. (4.25). Hence, by Exercise 4 g of Chapter 1,

$$P\{W_1 > t\} = C(1, \tfrac{\lambda_1}{\mu_1})\, e^{-(\mu_1-\lambda_1)t} = \frac{\lambda_1}{\mu_1} e^{-(\mu_1-\lambda_1)t} \tag{9}$$

$\boxed{e}$ Conditions for bounded delays:

$$\begin{aligned} &\text{High-priority customers}: \quad \frac{\lambda_1}{\mu_1} < 1, \\ &\text{Low-priority customers}: \quad \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} < 1 \end{aligned} \tag{10}$$

$\boxed{f}$ Under the exponential service time assumption, the remaining service time at preemption will be exponentially distributed with mean μ_2^{-1}. Hence, an assumption of preemptive-resume priority discipline does not change the results in parts a - e.

□

Chapter 3, Exercise 43

'Priority reservation.'

Let λ_E = eastbound traffic call rate, λ_W = westbound traffic call rate, μ = service rate. Hence the offered loads are $a_1 = \lambda_E/\mu$ and $a_2 = \lambda_W/\mu$, respectively.

[a] Suppose $1 \leqq n \leqq s-1$. A westbound call will be cleared in arrival state $j \geq s-n$, whereas an eastbound call will be cleared only if $j = s$.

The queueing process can be modeled as a birth-and-death process with $\lambda_j = \lambda_E + \lambda_W$ for $j = 0, 1, \ldots, s-n-1$; $\lambda_j = \lambda_E$ for $j = s-n, \ldots, s-1$; $\mu_j = j\mu$ for $j = 0, 1, \ldots, s$. By Eq. (3.15) of Chapter 2 then

$$
\begin{aligned}
(\lambda_E + \lambda_W) P_0 &= 1 \cdot \mu P_1 \\
&\vdots \\
(\lambda_E + \lambda_W) P_{s-n-1} &= (s-n)\mu P_{s-n} \\
\lambda_E P_{s-n} &= (s-n+1)\mu P_{s-n+1} \\
&\vdots \\
\lambda_E P_{s-1} &= s\mu P_s .
\end{aligned}
$$

[b] Recursive solution of the above state equations give

$$
P_j = \begin{cases} \dfrac{(a_1+a_2)^j}{j!} P_0 & (j = 1, 2, \ldots, s-n), \\[2ex] \left(\dfrac{a_1+a_2}{a_1}\right)^{s-n} \dfrac{a_1^j}{j!} P_0 & (j = s-n+1, \ldots, s). \end{cases}
$$

As usual, P_0 is found by use of the condition $\sum_{j=0}^{s} P_j = 1$.

[c] Loss on eastbound traffic $= P_s$.

Loss on westbound traffic $= \sum_{j=s-n}^{s} P_j$. □

Chapter 3, Exercise 44

In order to minimize its telephone bill ...

a Equilibrium state probabilities for flat-rate queue.

The equilibrium state probabilities $\{P_j\}$ for the flat-rate queueing system can be found from the following equilibrium state equations:

$$
\begin{aligned}
(\lambda_1+\lambda_2)P_0 &= 1\cdot\mu P_1 \\
&\vdots \\
(\lambda_1+\lambda_2)P_{s-1} &= s\mu P_s \\
\lambda_2 P_s &= s\mu P_{s+1} \\
&\vdots \\
\lambda_2 P_{s+i} &= s\mu P_{s+i+1} \\
&\vdots
\end{aligned}
$$

By recursive solution,

$$
P_j = \begin{cases} \dfrac{(a_1+a_2)^j}{j!}P_0 & (j = 1, 2, \ldots, s-1), \\ \dfrac{(a_1+a_2)^s}{s!}\left(\dfrac{a_2}{s}\right)^{j-s}P_0 & (j = s, s+1, \ldots), \end{cases}
$$

and

$$
P_0 = \left[\sum_{k=0}^{s-1}\frac{(a_1+a_2)^k}{k!} + \frac{(a_1+a_2)^s}{s!}\,\frac{1}{1-a_2/s}\right]^{-1},
$$

with $a_1=\lambda_1/\mu$ and $a_2=\lambda_2/\mu$, where $a_2<s$. If $a_2 \geq s$, then $P_j=0$ for all j.

b The blocking probability $B(s) = \sum_{j=s}^{\infty}\Pi_j = \sum_{j=s}^{\infty}P_j$.

$$
B(s) = \frac{\dfrac{(a_1+a_2)^s}{s!}\,\dfrac{1}{1-a_2/s}}{\sum_{k=0}^{s-1}\dfrac{(a_1+a_2)^k}{k!} + \dfrac{(a_1+a_2)^s}{s!}\cdot\dfrac{1}{1-a_2/s}} \qquad (a_2 < s).
$$

Observe that calculation of $B(s)$ is facilitated by the formula

$$B(s) = s\,B(s,a_1+a_2)/(s-a_2(1-B(s,a_1+a_2))),$$

as is easily verified, and the recurrence of Exercise 6 of Chapter 3.

$\boxed{c}$ The overall cost per minute, $c(s)$

Cost parameters:
c = cost per minute of a flat-rate trunk,
r_0 = cost of a toll call for the first minute or fraction thereof,
r = cost of a toll call for each additional minute or fraction thereof.
Letting M denote the random number of 1-minute intervals beyond the initial 1-minute interval, obviously

$$c(s) = cs + \lambda_1 B(s)[r_0 + E(M)r],$$

since $\lambda_1 B(s)$ is the average overflow rate of high-priority customers requesting service from the flat-rate trunks, and $r_0 + E(M)r$ is the mean cost of a toll call.

Now, given exponential service time with mean μ^{-1}, the probability of holding the line for at least 1 more minute equals $e^{-\mu}$ at the start of each 1-minute interval. Therefore, $P\{M=k\} = (e^{-\mu})^k(1-e^{-\mu})$ for $k = 0,1,\ldots$. Hence $E(M) = e^{-\mu}/(1-e^{-\mu})$, and

$$c(s) = cs + \lambda_1 [r_0 + \frac{r}{e^{\mu}-1}] B(s).$$

$\boxed{d}$ Mean waiting time for low-priority calls, $E(W_2)$

Let W_2 = waiting time of an arbitrary low-priority customer. Observe, $P_{s+j} = B(s)(a_2/s)^j(1-a_2/s)$ for $j = 0,1,\ldots$, and, for N equal to the arrival state of the customer, $E(W_2|N=s+j) = (j+1)(s\mu)^{-1}$. Hence, for $a_2 < s$, since $\Pi_{s+j} = P_{s+j}$,

$$E(W_2) = \sum_{j=0}^{\infty} E(W_2|N=s+j)\Pi_{s+j} = B(s)(s\mu)^{-1}\sum_{j=0}^{\infty}(j+1)(\frac{a_2}{s})^j(1-\frac{a_2}{s})$$

$$= B(s)(s\mu)^{-1}(1-\frac{a_2}{s})^{-1} = \frac{B(s)}{s\mu-\lambda_2} \quad \text{[analogous with Eq. (4.27)]}$$

$\boxed{e}$ Occupancy of flat-rate trunks, ρ.

Clearly, $\rho = 1$ if $a_2 \geq s$. In case $a_2 < s$, the carried load on the flat-rate server group is $a' = a_1[1-B(s)] + a_2$. Hence,

$$\rho = \frac{a'}{s} = \frac{a_1[1-B(s)]+a_2}{s} \qquad (a_2 < s).$$

$\square$

Chapter 3, Exercise 45

'Time-varying Poisson input'.

a First assume that $\lambda(t)$ is continuous and differentiable for all t. By the reasoning used for derivation of Eq. (2.5) of Chapter 2 we find

$$\frac{d}{dt}P_j(t) = \lambda(t)P_{j-1}(t) - \lambda(t)P_j(t) \qquad [j=0,1,\dots;\ P_{-1}(t)=0],$$

the initial condition being $P_0(0)=1$. Solution by recurrence starting with $j=0$ yields the Poisson distribution

$$P_j(t) = \frac{(\Lambda(t))^j}{j!}e^{-\Lambda(t)} \qquad (j=0,1,..) \qquad (1)$$

where

$$\Lambda(t) = \int_0^t \lambda(x)dx. \qquad (2)$$

The equations can be shown to hold also in the case of a piecewise continuous and differentiable $\lambda(t)$. This may be done by utilizing the additivity property of the Poisson distribution.

b In the infinite server queue a customer who arrives at time $x<t$ will still be in service at time t with probability $1-H(t-x)$. Hence, counting only arrivals that will be in the system at time t, the effective arrival rate at $x<t$ equals $\lambda(x) = \lambda[1-H(t-x)]$. The corresponding counting process is a Poisson process with time-varying rate. By (1) and (2) the number of customers in the system (= in service) at t will have the Poisson distribution with mean

$$\begin{aligned}\Lambda(t) &= \int_0^t \lambda[1-H(t-x)]dx = \lambda\int_0^t[1-H(x)]dx \\ &= \lambda[t(1-H(t)) + \int_0^t x\,dH(x)] \\ &= \lambda t\,p(t),\end{aligned}$$

where $p(t) = 1-H(t)+\int_0^t \frac{x}{t}dH(x)$. This proves Equation (3.11).

Finally, we observe that also Eq. (4.26) of Chapter 2 may be proved in a similar way by appeal to the notion of a time-varying Poisson process. □

Chapter 3, Exercise 46

'Transient analysis of the single-server Erlang delay model.'

a For the M/M/1 system, clearly

$$P_0(t+h) = P_0(t)[1-\lambda h] + P_1(t)\mu h + o(h),$$

$$P_j(t+h) = P_{j-1}(t)\lambda h + P_j(t)[1-(\lambda+\mu)h] + P_{j+1}(t)\mu h + o(h) \qquad (j = 1,2,\dots).$$

Hence,

$$\frac{d}{dt}P_0(t) = -\lambda P_0(t) + \mu P_1(t),$$

$$\frac{d}{dt}P_j(t) = \lambda P_{j-1}(t) - (\lambda+\mu)P_j(t) + \mu P_{j+1}(t) \qquad (j = 1,2,\dots).$$

Choosing μ^{-1} as the time unit, then $\mu = 1$ and $\lambda = \lambda/\mu = a$, so that the above equation system becomes

$$\frac{d}{dt}P_0(t) = -a P_0(t) + P_1(t), \tag{1}$$

$$\frac{d}{dt}P_j(t) = a P_{j-1}(t) - (1+a)P_j(t) + P_{j+1}(t) \qquad (j = 1,2,\dots). \tag{2}$$

b Consider the auxiliary system of equations

$$\frac{d}{dt}\hat{P}_j(t) = a\hat{P}_{j-1}(t) - (1+a)\hat{P}_j(t) + \hat{P}_{j+1}(t) \qquad (j = 0,\pm1,\pm2,\dots), \tag{3}$$

$$\hat{P}_0(t) = a\hat{P}_{-1}(t). \tag{4}$$

(3) and (4) together imply

$$\frac{d}{dt}\hat{P}_0(t) = -a\hat{P}_0(t) + \hat{P}_1(t), \tag{1a}$$

and by (3),

$$\frac{d}{dt}\hat{P}_j(t) = a\hat{P}_{j-1}(t) - (1+a)\hat{P}_j(t) + \hat{P}_{j+1}(t) \qquad (j = 1,2,\dots) \tag{2a}$$

Thus, if $\hat{P}_j(t)$ $(j = 0,\pm1,\pm2,\dots)$ is a solution to Eqs. (3) and (4), then $\hat{P}_j(t)$ $(j = 0,1,2,\dots)$ will be a solution to Equations (1a) and (2a). As (1a) and (2a) are formally identical to (1) and (2), we conclude that if $\hat{P}_j(t)$ for $j = 0,\pm1,\pm2,\dots$ solves (3) and (4), then $P_j(t) = \hat{P}_j(t)$, for $j = 0,1,2,\dots$ and all t, will also solve (1) and (2).

(Chap. 3, Ex. 46 c)

$\boxed{c}$ Let $\hat{P}(z,t)$ denote the generating function

$$\hat{P}(z,t) = \sum_{j=-\infty}^{\infty} \hat{P}_j(t)\, z^j. \tag{6}$$

Multiplication of Eq. (3) by z^j and summation for all j result in

$$\sum_{j=-\infty}^{\infty} \frac{d}{dt}\hat{P}_j(t)\, z^j = az\sum_{j=-\infty}^{\infty}\hat{P}_{j-1}(t)\,z^{j-1} - (1+a)\sum_{j=-\infty}^{\infty}\hat{P}_j(t)\,z^j + z^{-1}\sum_{j=-\infty}^{\infty}\hat{P}_{j+1}(t)\,z^{j+1},$$

or,

$$\frac{d}{dt}\sum_{j=-\infty}^{\infty}\hat{P}_j(t)\,z^j = [az - (1+a) + z^{-1}]\sum_{j=-\infty}^{\infty}\hat{P}_j(t)\,z^j,$$

which, by (6), is the same as

$$\frac{d}{dt}\hat{P}(z,t) = [az - (1+a) + z^{-1}]\,\hat{P}(z,t), \tag{7}$$

whose general solution is

$$\hat{P}(z,t) = G(z)\, e^{[-(1+a)t + (az+z^{-1})t]}, \tag{8}$$

where $G(z)$ is any function of z.

$\boxed{d}$ Eq. (8) may be rewritten as

$$\hat{P}(z,t) = G(z)\, e^{-(1+a)t}\, e^{\frac{1}{2}[2a^{1/2}t][(a^{1/2}z) + (a^{1/2}z)^{-1}]}.$$

We shall use the fact that

$$e^{\frac{1}{2}y(x+x^{-1})} = \sum_{k=-\infty}^{\infty} I_k(y)\, x^k, \tag{9}$$

where $I_k(y)$ are the modified Bessel functions. Now, setting $y = 2a^{1/2}t$ and $x = a^{1/2}z$ it is seen immediately that

$$\hat{P}(z,t) = G(z)\, e^{-(1+a)t}\sum_{k=-\infty}^{\infty} I_k(2a^{1/2}t)\, a^{k/2} z^k. \tag{10}$$

(Chap. 3, Ex. 46 e)

$\boxed{e}$ Suppose $G(z)$ has the expansion

$$G(z) = \sum_{j=-\infty}^{\infty} c_{-j}\, z^j. \tag{11}$$

Insertion into (10) and collection of terms by powers of z lead to

$$\hat{P}(z,t) = \sum_{j=-\infty}^{\infty} \Big(e^{-(1+a)t} \sum_{k=-\infty}^{\infty} c_k\, a^{\frac{1}{2}(j+k)} I_{j+k}(2a^{1/2}t) \Big)\, z^j.$$

Comparison with Eq. (6) shows that

$$\hat{P}_j(t) = e^{-(1+a)t} \sum_{k=-\infty}^{\infty} c_k\, a^{\frac{1}{2}(j+k)} I_{j+k}(2a^{1/2}t). \tag{12}$$

$\boxed{f}$ For $y=0$ Eq. (9) specializes to

$$1 = \sum_{k=-\infty}^{\infty} I_k(0)\, x^k,$$

whereby $I_0(0) = 1$ and $I_k(0) = 0$ for $k \neq 0$. By (12) then

$$\hat{P}_j(0) = \sum_{k=-\infty}^{\infty} c_k\, a^{\frac{1}{2}(j+k)} I_{j+k}(0) = c_{-j}.$$

Let i = initial state, so that $P_i(0) = 1$. Then (provided $\hat{P}_j(t) = P_j(t)$ for $j = 0, 1, \ldots$) $c_{-i} = 1$, and $c_k = 0$ for $k \leq 0$ but $k \neq -i$. Thus Eq. (12) can be written

$$\hat{P}_j(t) = e^{-(1+a)t}\Big[a^{\frac{1}{2}(j-i)} I_{j-i}(2a^{1/2}t) + \sum_{k=1}^{\infty} c_k\, a^{\frac{1}{2}(j+k)} I_{j+k}(2a^{1/2}t) \Big]. \tag{13}$$

$\boxed{g}$ By Eq. (13),

$$\hat{P}_0(t) = e^{-(1+a)t}\Big[a^{-\frac{1}{2}i} I_{-i}(2a^{1/2}t) + \sum_{k=1}^{\infty} c_k\, a^{\frac{1}{2}k} I_k(2a^{1/2}t) \Big],$$

$$\hat{P}_{-1}(t) = e^{-(1+a)t}\Big[a^{-\frac{1}{2}-\frac{1}{2}i} I_{-(i+1)}(2a^{1/2}t) + \sum_{k=1}^{\infty} c_k\, a^{-\frac{1}{2}+\frac{1}{2}k} I_{k-1}(2a^{1/2}t) \Big]$$

From these expressions and Eq. (4), $\hat{P}_0(t) = a\hat{P}_{-1}(t)$, we obtain

$$a^{-\frac{1}{2}i} I_{-i} + \sum_{k=1}^{\infty} d_k I_k = a^{-\frac{1}{2}i+\frac{1}{2}} I_{-(i+1)} + a^{1/2} \sum_{k=1}^{\infty} d_k I_{k-1}, \tag{14}$$

where $d_k = c_k a^{\frac{1}{2}k}$ and $I_k = I_k(2a^{1/2}t)$.

(Chap. 3, Ex. 46 h)

[h] Since $I_{-k}(y) = I_k(y)$, Eq. (14) can be written

$$a^{-\frac{1}{2}i} I_i + \sum_{k=1}^{\infty} d_k I_k = a^{-\frac{1}{2}i+\frac{1}{2}} I_{i+1} + a^{\frac{1}{2}} \sum_{k=1}^{\infty} d_k I_{k-1}.$$

This equation should hold for all t and hence for all arguments $2a^{1/2}t$ of the I_r's $(r=0,1,\ldots)$. Consequently, the coefficient of each I_r must equal zero. For example, for $i=2$ this requirement leads to the following set of equations,

$$\begin{aligned} 0 &= a^{\frac{1}{2}} d_1 \\ d_1 &= a^{\frac{1}{2}} d_2 \\ a^{-\frac{1}{2}i} + d_2 &= a^{\frac{1}{2}} d_3 \\ d_3 &= a^{\frac{1}{2}} d_4 + a^{-\frac{1}{2}i+\frac{1}{2}} \\ d_4 &= a^{\frac{1}{2}} d_5 \\ d_5 &= a^{\frac{1}{2}} d_6 \\ &\vdots \end{aligned}$$

For arbitrary initial state i the solution is

$$d_k = 0 \qquad (k=1,2,\ldots,i) \qquad [\text{void if } i=0]$$

$$d_{i+1} = a^{-\frac{1}{2}i-\frac{1}{2}}$$

$$d_{i+1+m} = a^{-\frac{1}{2}i-\frac{1}{2}m-\frac{1}{2}}(1-a) \qquad (m=1,2,\ldots)$$

[i] Substituting the found values of $d_k = c_k a^{\frac{1}{2}k}$ into (13), at the same time replacing $\tilde{P}_j(t)$ with $P_j(t)$ (the initialization permits this), we get

$$P_j(t) = e^{-(1+a)t}\Big[a^{\frac{1}{2}(j-i)} I_{j-i}(2a^{1/2}t) + a^{\frac{1}{2}(j-i)-\frac{1}{2}} I_{j+i+1}(2a^{1/2}t) + (1-a)\sum_{m=1}^{\infty} a^{\frac{1}{2}(j-i)-\frac{1}{2}m-\frac{1}{2}} I_{j+i+1+m}(2a^{1/2}t)\Big].$$

Simplification results in

$$P_j(t) = a^{\frac{1}{2}(j-i)} e^{-(1+a)t}\Big[I_{j-i}(2a^{1/2}t) + a^{-\frac{1}{2}} I_{j+i+1}(2a^{1/2}t) + (1-a)\sum_{k=2}^{\infty} a^{-k/2} I_{j+i+k}(2a^{1/2}t)\Big]. \tag{15}$$

[j] Equation (15) holds for all values of the carried load a. □

Chapter 4, Exercise 1

'Finite-source systems with nonidentical sources.'

a Arguing as in Section 2 of Chapter 3 we find

$$b_2 = \frac{\gamma_2 P(1,0)}{\gamma_2 P(0,0) + \gamma_2 P(1,0)} ,$$

expressing that the probability that source 2 is blocked equals the number of blocked source 2 - calls per unit time divided by the total number of source 2 - calls per unit time. Hence

$$b_2 = \frac{P(1,0)}{P(0,0) + P(1,0)} . \tag{1}$$

b With source 2 inactive, the system is in fact a one server, one source system, and the sole equilibrium state equation is $\gamma_1 P_0 = \mu_1 P_1$. Given $P_0 + P_1 = 1$, we find $P_1 = (\gamma_1/\mu_1)/(1+\gamma_1/\mu_1)$. b_2' is defined as the probability that source 2 at a randomly chosen point in time finds the server occupied. Clearly, $b_2' = P_1$. That is, whether blocked customer cleared or delayed (!),

$$b_2' = \frac{\gamma_1/\mu_1}{1 + (\gamma_1/\mu_1)} . \tag{2}$$

c Blocked customers cleared.

First we calculate source 2's blocking probability b_2. The conservation-of-flow equations when both sources are active are

$$(\gamma_1 + \gamma_2) P(0,0) = \mu_1 P(1,0) + \mu_2 P(0,1)$$

$$\mu_1 P(1,0) = \gamma_1 P(0,0)$$

$$\mu_2 P(0,1) = \gamma_2 P(0,0)$$

We need only $P(1,0)$ in terms of $P(0,0)$. The middle equation gives us $P(1,0) = (\gamma_1/\mu_1) P(0,0)$. By (1) then

$$b_2 = \frac{(\gamma_1/\mu_1) P(0,0)}{P(0,0) + (\gamma_1/\mu_1) P(0,0)} = \frac{\gamma_1/\mu_1}{1 + (\gamma_1/\mu_1)} . \tag{3}$$

A comparison with Eq. (2) shows that in the BCC case $b_2 = b_2'$.

(Chap. 4, Ex. 1)

$\boxed{d}$ Blocked customers delayed.

Again we calculate source 2's blocking probability b_2. The conservation-of-flow equations are those found in Section 4.1 above. Omitting one equation we have

$$(\gamma_2 + \mu_1) P(1,0) = \gamma_1 P(0,0) + \mu_2 P(2,1),$$

$$(\gamma_1 + \mu_2) P(0,1) = \gamma_2 P(0,0) + \mu_1 P(1,2),$$

$$\mu_1 P(1,2) = \gamma_2 P(1,0),$$

$$\mu_2 P(2,1) = \gamma_1 P(0,1).$$

Substituting the last two equations into the first two, and then eliminating $P(0,1)$ and solving for $P(1,0)$ we derive

$$P(1,0) = \frac{\gamma_1(\mu_2 + \gamma_1 + \gamma_2)}{\mu_1\mu_2 + \mu_1\gamma_1 + \mu_2\gamma_2} P(0,0).$$

Substitution of this expression into Eq. (1) gives

$$b_2 = \frac{\gamma_1[\mu_2 + \gamma_1 + \gamma_2]}{\mu_1\mu_2 + \mu_1\gamma_1 + \mu_2\gamma_2 + \gamma_1[\mu_2 + \gamma_1 + \gamma_2]},$$

or,

$$b_2 = \frac{(\gamma_1/\mu_1)[\mu_2 + \gamma_1 + \gamma_2]}{[\mu_2 + \gamma_1 + (\mu_2/\mu_1)\gamma_2] + (\gamma_1/\mu_1)[\mu_2 + \gamma_1 + \gamma_2]}. \tag{4}$$

We shall prove that $b_2 = b_2'$ if and only if $\mu_1 = \mu_2$. First assume $\mu_1 = \mu_2$. Then Eq. (4) reduces to $b_2 = (\gamma_1/\mu_1)/(1 + \gamma_1/\mu_1) = b_2'$, by Eq. (2). Conversely, assume $b_2 = b_2'$. By Equations (2) and (4) this implies

$$\frac{\mu_2 + \gamma_1 + \gamma_2}{\mu_1\mu_2 + \mu_1\gamma_1 + \mu_2\gamma_2 + \gamma_1\mu_2 + \gamma_1^2 + \gamma_1\gamma_2} = \frac{1}{\mu_1 + \gamma_1} \qquad [b_2 = b_2']$$

it follows easily that $\mu_1 = \mu_2$. We conclude that in this particular BCD model with nonidentical sources, the arriving customer's 2-source distribution and his observer's 1-source distribution are the same if and only if $\mu_1 = \mu_2$. □

Chapter 4, Exercise 2

'a. Three cities A, B, and C, are interconnected by two trunk groups;'

In every case, let λ_i, μ_i, j_i denote arrival rate, completion rate, and number of calls in progress, respectively, for city connection no. i $(i=1,2,3)$, where $i=1$ refers to A−B, $i=2$ refers to B−C, $i=3$ refers to A−C. Let $P(j_1,j_2,j_3)$ be the equilibrium state probability of state (j_1,j_2,j_3). Always, it is understood that $j_1, j_2, j_3 \geq 0$.

$\boxed{a}$ In case (a) the equilibrium state equations are:

$$\begin{aligned}&(\lambda_1+\lambda_2+\lambda_3+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \\ &\lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3)+\lambda_3 P(j_1,j_2,j_3-1) \\ &+(j_1+1)\mu_1 P(j_1+1,j_2,j_3)+(j_2+1)\mu_2 P(j_1,j_2+1,j_3)+(j_3+1)\mu_3 P(j_1,j_2,j_3+1)\end{aligned} \qquad \begin{pmatrix} j_1+j_3<s_1 \\ j_2+j_3<s_2\end{pmatrix}$$

$$\begin{aligned}&(\lambda_1+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \\ &\lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3)+\lambda_3 P(j_1,j_2,j_3-1) \\ &+(j_1+1)\mu_1 P(j_1+1,j_2,j_3)\end{aligned} \qquad \begin{pmatrix} j_1+j_3<s_1 \\ j_2+j_3=s_2\end{pmatrix}$$

$$\begin{aligned}&(\lambda_2+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \\ &\lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3)+\lambda_3 P(j_1,j_2,j_3-1) \\ &+(j_2+1)\mu_2 P(j_1,j_2+1,j_3)\end{aligned} \qquad \begin{pmatrix} j_1+j_3=s_1 \\ j_2+j_3<s_2\end{pmatrix}$$

$$\begin{aligned}&(j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \\ &\lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3)+\lambda_3 P(j_1,j_2,j_3-1)\end{aligned} \qquad \begin{pmatrix} j_1+j_3=s_1 \\ j_2+j_3=s_2\end{pmatrix}$$

If $s_1=\infty$ and $s_2=\infty$, then j_1, j_2 and j_3 are independent Poisson variables, and

$$P(j_1,j_2,j_3) = \frac{(\lambda_1/\mu_1)^{j_1}}{j_1!}\,\frac{(\lambda_2/\mu_2)^{j_2}}{j_2!}\,\frac{(\lambda_3/\mu_3)^{j_3}}{j_3!}\cdot c \qquad [s_1=\infty,\ s_2=\infty].$$

Because of a correspondence between terms on left- and right-hand sides of all the equilibrium state equations in case (a) it is clear that also in the present case the solution has the form

$$P(j_1,j_2,j_3) = \frac{(\lambda_1/\mu_1)^{j_1}}{j_1!}\,\frac{(\lambda_2/\mu_2)^{j_2}}{j_2!}\,\frac{(\lambda_3/\mu_3)^{j_3}}{j_3!}\cdot c \qquad \begin{pmatrix} 0\leq j_1+j_3\leq s_1, \\ 0\leq j_2+j_3\leq s_2\end{pmatrix}.$$

c is found from the normalization equation $\sum P(j_1,j_2,j_3)=1$.

(Chap. 4, Ex. 2 b)

$\boxed{b}$ In case (b) the equilibrium state equations are:

$$(\lambda_1+\lambda_2+\lambda_3+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3) + \lambda_3 P(j_1,j_2,j_3-1)+(j_1+1)\mu_1 P(j_1+1,j_2,j_3) + (j_2+1)\mu_2 P(j_1,j_2+1,j_3)+(j_3+1)\mu_3 P(j_1,j_2,j_3+1) \qquad \begin{pmatrix} j_1+\max(0,j_3-s)<s_1, \\ j_2+\max(0,j_3-s)<s_2 \end{pmatrix}$$

$$(\lambda_1+\lambda_3+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3) + \lambda_3 P(j_1,j_2,j_3-1)+(j_1+1)\mu_1 P(j_1+1,j_2,j_3) + (j_3+1)\mu_3 P(j_1,j_2,j_3+1) \qquad (j_1<s_1,\ j_2=s_2,\ j_3<s)$$

$$(\lambda_2+\lambda_3+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3) + \lambda_3 P(j_1,j_2,j_3-1)+(j_2+1)\mu_2 P(j_1,j_2+1,j_3) + (j_3+1)\mu_3 P(j_1,j_2,j_3+1) \qquad (j_1=s_1,\ j_2<s_2,\ j_3<s)$$

$$(\lambda_1+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3) + \lambda_3 P(j_1,j_2,j_3-1)+(j_1+1)\mu_1 P(j_1+1,j_2,j_3) \qquad \begin{pmatrix} j_1+(j_3-s)<s_1, \\ j_2+(j_3-s)=s_2, \\ j_3\geq s \end{pmatrix}$$

$$(\lambda_2+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3) + \lambda_3 P(j_1,j_2,j_3-1)+(j_2+1)\mu_2 P(j_1,j_2+1,j_3) \qquad \begin{pmatrix} j_1+(j_3-s)=s_1, \\ j_2+(j_3-s)<s_2, \\ j_3\geq s \end{pmatrix}$$

$$(\lambda_3+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3) + \lambda_3 P(j_1,j_2,j_3-1)+(j_3+1)\mu_3 P(j_1,j_2,j_3+1) \qquad (j_1=s_1,\ j_2=s_2,\ j_3<s)$$

$$(j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3) + \lambda_3 P(j_1,j_2,j_3-1). \qquad \begin{pmatrix} j_1+(j_3-s)=s_1, \\ j_2+(j_3-s)=s_2 \\ j_3\geq s \end{pmatrix}$$

The solution is of the same type as in case (a), namely

$$P(j_1,j_2,j_3) = \frac{(\lambda_1/\mu_1)^{j_1}}{j_1!}\times\frac{(\lambda_2/\mu_2)^{j_2}}{j_2!}\times\frac{(\lambda_3/\mu_3)^{j_3}}{j_3!}c \qquad \begin{pmatrix} 0\leq j_1+\max(0,j_3-s)\leq s_1, \\ 0\leq j_2+\max(0,j_3-s)\leq s_2 \end{pmatrix}$$

(Chap. 4, Ex. 2 c)

$\boxed{c}$ Without a switching capability the state description must be (j_1, j_2, j_3', j_3'') where j_3' and j_3'' denote directly and indirectly connected calls between A and C. This complicates the equilibrium state equations somewhat, but worse, the decomposition property is lost, so that the solution method above is not applicable.

$\boxed{d}$ In case d, denote by s_3 the number of trunks directly connecting A and C. Observe that $j_1 > s_1 \Rightarrow \{j_2 < s_2, j_3 < s_3\}$ with similar implications of $j_2 > s_2$ and $j_3 > s_3$. The equilibrium state equations are:

$$\begin{aligned}
&(\lambda_1+\lambda_2+\lambda_3+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \\
&\quad \lambda_1 P(j_1-1,j_2,j_3) + \lambda_2 P(j_1,j_2-1,j_3) \\
&\quad + \lambda_3 P(j_1,j_2,j_3-1) + (j_1+1)\mu_1 P(j_1+1,j_2,j_3) \\
&\quad + (j_2+1)\mu_2 P(j_1,j_2+1,j_3) + (j_3+1)\mu_3 P(j_1,j_2,j_3+1)
\end{aligned}
\qquad
\begin{aligned}
&\begin{pmatrix} j_1+\max(0,j_3-s_3)<s_1, \\ j_2+\max(0,j_3-s_3)<s_2 \end{pmatrix}, \\
\text{or } &\begin{pmatrix} j_1+\max(0,j_2-s_2)<s_1, \\ j_3+\max(0,j_2-s_2)<s_3 \end{pmatrix}, \\
\text{or } &\begin{pmatrix} j_2+\max(0,j_1-s_1)<s_2, \\ j_3+\max(0,j_1-s_1)<s_3 \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
&(\lambda_1+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \\
&\quad \lambda_1 P(j_1-1,j_2,j_3) + \lambda_2 P(j_1,j_2-1,j_3) \\
&\quad + \lambda_3 P(j_1,j_2,j_3-1) + (j_1+1)\mu_1 P(j_1+1,j_2,j_3)
\end{aligned}
\qquad
\begin{aligned}
&(j_1+(j_3-s_3)<s_1,\ j_2+(j_3-s_3)=s_2,\ j_3\geq s_3), \\
\text{or } &(j_1+(j_2-s_2)<s_1,\ j_2\geq s_2,\ j_3+(j_2-s_2)=s_3).
\end{aligned}$$

$$\begin{aligned}
&(\lambda_2+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \\
&\quad \lambda_1 P(j_1-1,j_2,j_3) + \lambda_2 P(j_1,j_2-1,j_3) \\
&\quad + \lambda_3 P(j_1,j_2,j_3-1) + (j_2+1)\mu_2 P(j_1,j_2+1,j_3)
\end{aligned}
\qquad
\begin{aligned}
&(j_1+(j_3-s_3)=s_1,\ j_2+(j_3-s_3)<s_2,\ j_3\geq s_3), \\
\text{or } &(j_1\geq s_1,\ j_2+(j_1-s_1)<s_2,\ j_3+(j_1-s_1)=s_3).
\end{aligned}$$

$$\begin{aligned}
&(\lambda_3+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \\
&\quad \lambda_1 P(j_1-1,j_2,j_3) + \lambda_2 P(j_1,j_2-1,j_3) \\
&\quad + \lambda_3 P(j_1,j_2,j_3-1) + (j_3+1)\mu_3 P(j_1,j_2,j_3+1)
\end{aligned}
\qquad
\begin{aligned}
&(j_1+(j_2-s_2)=s_1,\ j_2\geq s_2,\ j_3+(j_2-s_2)<s_3), \\
\text{or } &(j_1\geq s_1,\ j_2+(j_1-s_1)=s_2,\ j_3+(j_1-s_1)<s_3).
\end{aligned}$$

$$\begin{aligned}
&(j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = \\
&\quad \lambda_1 P(j_1-1,j_2,j_3) + \lambda_2 P(j_1,j_2-1,j_3) \\
&\quad + \lambda_3 P(j_1,j_2,j_3-1)
\end{aligned}
\qquad
\begin{aligned}
&(j_1\geq s_1,\ j_2+(j_1-s_1)=s_2,\ j_3+(j_1-s_1)=s_3), \\
\text{or } &(j_1+(j_2-s_2)=s_1,\ j_2\geq s_2,\ j_3+(j_2-s_2)=s_3), \\
\text{or } &(j_1+(j_3-s_3)=s_1,\ j_2+(j_3-s_3)=s_2,\ j_3\geq s_3).
\end{aligned}$$

Again, the solution is of the same type as in case (a), namely

$$P(j_1,j_2,j_3) = \frac{(\lambda_1/\mu_1)^{j_1}}{j_1!}\,\frac{(\lambda_2/\mu_2)^{j_2}}{j_2!}\,\frac{(\lambda_3/\mu_3)^{j_3}}{j_3!}\,c$$

for all feasible combinations (j_1,j_2,j_3). As usual, c is found from the condition $\Sigma P(j_1,j_2,j_3) = 1$. $\square$

Chapter 4, Exercise 3

'A group of s trunks serves two types of Poisson traffic on a BCC..'

Let j_1 be the number of ordinary calls and j_2 the number of wideband calls in progress. Then j_1+kj_2 is the total number of trunks that are held. The equilibrium state equations are as follows, assuming $j_1 \geq 0$ and $j_2 \geq 0$,

$$(\lambda_1+\lambda_2+j_1\mu_1+j_2\mu_2)P(j_1,j_2) = \lambda_1 P(j_1-1,j_2)+\lambda_2 P(j_1,j_2-1) + (j_1+1)\mu_1 P(j_1+1,j_2)+(j_2+1)\mu_2 P(j_1,j_2+1) \qquad (0 \leq j_1+kj_2 \leq s-k)$$

$$(\lambda_1+j_1\mu_1+j_2\mu_2)P(j_1,j_2) = \lambda_1 P(j_1-1,j_2)+\lambda_2 P(j_1,j_2-1) + (j_1+1)\mu_1 P(j_1+1,j_2) \qquad (s-k < j_1+kj_2 < s)$$

$$(j_1\mu_1+j_2\mu_2)P(j_1,j_2) = \lambda_1 P(j_1-1,j_2)+\lambda_2 P(j_1,j_2-1). \qquad (j_1+kj_2 = s)$$

By the same considerations as before, it is seen that the solution is given by

$$P(j_1,j_2) = \frac{(\lambda_1/\mu_1)^{j_1}}{j_1!}\,\frac{(\lambda_2/\mu_2)^{j_2}}{j_2!}\,C \qquad (0 \leq j_1+kj_2 \leq s),$$

where

$$C = \left[\sum_{0 \leq j_1+kj_2 \leq s} \frac{(\lambda_1/\mu_1)^{j_1}}{j_1!}\,\frac{(\lambda_2/\mu_2)^{j_2}}{j_2!}\right]^{-1}.$$

Denote by $j_2^* = [\frac{s}{k}]$ the highest possible number of wideband calls in progress. Let P_1 be the probability that an ordinary call is lost, and let P_2 be the probability that a wideband call is lost. Then, clearly,

$$P_1 = P\{j_1+kj_2 = s\} = \sum_{j_2=0}^{j_2^*} P(s-kj_2,j_2),$$

and

$$P_2 = P\{s-k+1 \leq j_1+kj_2 \leq s\} = \sum_{j_2=0}^{j_2^*-1}\ \sum_{j_1=s-j_2k-(k-1)}^{s-j_2k} P(j_1,j_2) + \sum_{j_1=0}^{s-j_2^*k} P(j_1,j_2^*).$$ $\square$

Chapter 4, Exercise 4

'A group of s servers handles n types of customers.'

Let $I(x) = 0$ if $x \leq 0$, $I(x) = 1$ if $x > 0$. With this notation the equilibrium state equations can be written

$$\left(\sum_{i=1}^{n} \lambda_i I(k_i - j_i) + \sum_{i=1}^{n} j_i \mu_i\right) P(j_1, j_2, \ldots, j_n) = \qquad \begin{pmatrix} 0 \leq j_i \leq k_i,\ i = 1, \ldots, n; \\ 0 \leq \Sigma_{i=1}^{n} j_i < s \end{pmatrix}$$

$$\sum_{i=1}^{n} \lambda_i P(\ldots, j_{i-1}, j_i - 1, j_{i+1}, \ldots)$$

$$+ \sum_{i=1}^{n} (j_i + 1)\mu_i I(k_i - j_i) P(\ldots, j_{i-1}, j_i + 1, j_{i+1}, \ldots),$$

$$\left(\sum_{i=1}^{n} j_i \mu_i\right) P(j_1, j_2, \ldots, j_n) = \sum_{i=1}^{n} \lambda_i P(\ldots, j_{i-1}, j_i - 1, j_{i+1}, \ldots) \qquad \begin{pmatrix} 0 \leq j_i \leq k_i, \\ i = 1, \ldots, n; \\ \Sigma_{i=1}^{n} j_i = s \end{pmatrix},$$

where j_i denotes the number of customers of type i, and $j_i \geq 0$.

The correspondence between LHS and RHS terms such as $\lambda_i I(k_i - j_i) P(j_1, j_2, \ldots, j_n)$ and $(j_i+1)\mu_i I(k_i - j_i) P(\ldots, j_{i-1}, j_i + 1, j_{i+1})$ once more indicates a solution of the form

$$P(j_1, j_2, \ldots, j_n) = \frac{(\lambda_1/\mu_1)^{j_1}}{j_1!} \frac{(\lambda_2/\mu_2)^{j_2}}{j_2!} \cdots \frac{(\lambda_n/\mu_n)^{j_n}}{j_n!} \times c \qquad \begin{pmatrix} 0 \leq j_i \leq k_i,\ i = 1, \ldots, n; \\ 0 \leq \Sigma_{i=1}^{n} j_i \leq s \end{pmatrix}$$

where, as usual, c is determined from $\Sigma P(j_1, j_2, \ldots, j_n)$.

Let P_0 be the equilibrium probability that all servers are busy, and let P_i be the probability that $j_i = k_i$ while not all the servers are busy. Obviously,

$$P_0 = \sum_{(j_1, \ldots, j_n) \in S_0} P(j_1, j_2, \ldots, j_n), \tag{1}$$

where

$$S_0 = \{(j_1, j_2, \ldots, j_n) : 0 \leq j_i \leq k_i,\ i = 1, \ldots, n; \sum_{i=1}^{n} j_i = s\},$$

and

$$P_i = \sum_{(j_1, \ldots, j_n) \in S_i} P(j_1, j_2, \ldots, j_n), \tag{2}$$

where

$$S_i = \{(j_1, j_2, \ldots, j_n) : j_i = k_i;\ 0 \leq j_r \leq k_r,\ r \neq i;\ \sum_{i=1}^{n} j_i < s\}.$$

By the assumption of Poisson arrival streams, the probability $\hat{P}_i$ that a customer of type i will be blocked equals

$$\hat{P}_i = P_0 + P_i. \tag{3}$$

□

Chapter 4, Exercise 5

'A group of s servers handles two types of customers on a BCC basis.'

[a] The equilibrium state equations are, for $j_1, j_2 \geq 0$,

$$(\lambda + (n-j_2)\gamma + j_1\mu_1 + j_2\mu_2) P_n(j_1, j_2) = \lambda P_n(j_1 - 1, j_2) + (n - j_2 + 1)\gamma P_n(j_1, j_2 - 1) + (j_1+1)\mu_1 P_n(j_1+1, j_2) + (j_2+1)\mu_2 P_n(j_1, j_2+1) \qquad \begin{pmatrix} 0 \leq j_1 + j_2 < s, \\ j_2 < \min(s,n) \end{pmatrix}$$

$$(\lambda + j_1\mu_1 + j_2\mu_2) P_n(j_1, j_2) = \lambda P_n(j_1 - 1, j_2) + (n - j_2 + 1)\gamma P_n(j_1, j_2 - 1) + (j_1+1)\mu_1 P_n(j_1+1, j_2) \qquad \begin{pmatrix} 0 \leq j_1 + j_2 < s, \\ j_2 = \min(s,n) \end{pmatrix}$$

$$(j_1\mu_1 + j_2\mu_2) P_n(j_1, j_2) = \lambda P_n(j_1 - 1, j_2) + (n - j_2 + 1)\gamma P_n(j_1, j_2 - 1). \qquad \begin{pmatrix} j_1 + j_2 = s, \\ j_2 \leq \min(s,n) \end{pmatrix}$$

In case $s = \infty$, the equilibrium states j_1 and j_2 are independent, $\lambda P^{(1)}(j_1) = (j_1+1)\mu_1 P^{(1)}(j_1+1)$ and $(n-j_2)\gamma P^{(2)}(j_2) = (j_2+1)\mu_2 P^{(2)}(j_2+1)$, by which $P^{(1)}(j_1) = [(\lambda/\mu_1)^{j_1}/j_1!]\, c^{(1)}$ and $P^{(2)}(j_2) = \binom{n}{j_2}(\gamma/\mu_2)^{j_2} c^{(2)}$, $j_2 \leq n$. This suggests that in the present case where $s < \infty$ we will have the product solution

$$P_n(j_1, j_2) = \frac{(\lambda/\mu_1)^{j_1}}{j_1!}\binom{n}{j_2}\left(\frac{\gamma}{\mu_2}\right)^{j_2} c_n \qquad \begin{pmatrix} 0 \leq j_1 + j_2 \leq s, \\ j_2 \leq \min(s,n) \end{pmatrix} \qquad (1)$$

for all feasible (j_1, j_2), where c_n is determined from the condition $\sum_{j_2=0}^{\min(s,n)} \sum_{j_1=0}^{s-j_2} P_n(j_1, j_2) = 1$. Equation (1) is verified by noting that the equilibrium state equations can be decomposed into equations of the two types $\lambda P_n(j_1, j_2) = (j_1+1)\mu_1 P_n(j_1+1, j_2)$ and $(n-j_2)\gamma P_n(j_1, j_2) = (j_2+1)\mu_2 P_n(j_1, j_2+1)$ which are both satisfied by (1).

[b] Customers of type 1 arrive in a Poisson stream. Therefore $\Pi'_n(j_1, j_2; t) = P_n(j_1, j_2; t)$ and

$$\Pi'_n(j_1, j_2) = P_n(j_1, j_2) \qquad \begin{pmatrix} 0 \leq j_1 + j_2 \leq s, \\ j_2 \leq \min(s,n) \end{pmatrix} \qquad (2)$$

By an analogy to Eq. (2.6) of Chapter 3 the equilibrium probability that a customer of type 2 will arrive in state (j_1, j_2) is

(Chap. 4, Ex. 5 b)

$$\Pi_n^2(j_1,j_2) = \frac{(n-j_2)\gamma P_n(j_1,j_2)}{\sum_{k_2=0}^{\min(s,n)} \sum_{k_1=0}^{s-k_2} (n-k_2)\gamma P_n(k_1,k_2)} \qquad \begin{pmatrix} 0 \le j_1+j_2 \le s, \\ j_2 \le \min(s,n) \end{pmatrix}$$

By (1), then,

$$\Pi_n^2(j_1,j_2) = \frac{(n-j_2)\frac{(\lambda/\mu_1)^{j_1}}{j_1!}\binom{n}{j_2}\left(\frac{\gamma}{\mu_2}\right)^{j_2}}{\sum_{k_2=0}^{\min(s,n)} \sum_{k_1=0}^{s-k_2} (n-k_2)\frac{(\lambda/\mu_1)^{k_1}}{k_1!}\binom{n}{k_2}\left(\frac{\gamma}{\mu_2}\right)^{k_2}} \qquad \begin{pmatrix} 0 \le j_1+j_2 \le s, \\ j_2 \le \min(s,n) \end{pmatrix} \quad (*)$$

We shall deal with the two cases $n \le s$ and $n > s$ in turn.

$\underline{n \le s.}$ Here $\min(s,n) = n$. Obviously, $\Pi_n^2(j_1,n) = P_{n-1}(j_1,n) = 0$. For $j_2 \le n-1 = \min(s,n-1)$, Eq. (*) gives, after rewriting,

$$\Pi_n^2(j_1,j_2) = \frac{\frac{(\lambda/\mu_1)^{j_1}}{j_1!}\binom{n-1}{j_2}\left(\frac{\gamma}{\mu_2}\right)^{j_2}}{\sum_{k_2=0}^{\min(s,n-1)} \sum_{k_1=0}^{s-k_2} \frac{(\lambda/\mu_1)^{k_1}}{k_1!}\binom{n-1}{k_2}\left(\frac{\gamma}{\mu_2}\right)^{k_2}} \qquad \begin{pmatrix} 0 \le j_1+j_2 \le s, \\ j_2 \le \min(s,n-1) \end{pmatrix}$$

Comparison with Eq. (1) shows that $\Pi_n^2(j_1,j_2) = P_{n-1}(j_1,j_2)$ for $0 \le j_1+j_2 \le s$ and $j_2 \le \min(s,n-1)$. Thus we have shown that if $n \le s$, then $\Pi_n^2(j_1,j_2) = P_{n-1}(j_1,j_2)$ for $0 \le j_1+j_2 \le s$ and $j_2 \le \min(s,n)$.

$\underline{n > s.}$ Here $\min(s,n) = \min(s,n-1)$ $(=s)$ and $n-j_2 > 0$ for all feasible j_2. By (*) it follows that, again,

$$\Pi_n^2(j_1,j_2) = \frac{\frac{(\lambda/\mu_1)^{j_1}}{j_1!}\binom{n-1}{j_2}\left(\frac{\gamma}{\mu_2}\right)^{j_2}}{\sum_{k_2=0}^{\min(s,n-1)} \sum_{k_1=0}^{s-k_2} \frac{(\lambda/\mu_1)^{k_1}}{k_1!}\binom{n-1}{k_2}\left(\frac{\gamma}{\mu_2}\right)^{k_2}} \qquad \begin{pmatrix} 0 \le j_1+j_2 \le s \\ j_2 \le \min(s,n-1) \end{pmatrix}$$

Comparison with Eq. (1) shows that also if $n > s$, then $\Pi_n^2(j_1,j_2) = P_{n-1}(j_1,j_2)$ for $0 \le j_1+j_2 \le s$ and $j_2 \le \min(s,n)$. Note that, despite appearances, the range of j_2 is not the same for $\Pi_n^2(j_1,j_2)$ when $n \le s$ and when $n > s$, since in the latter case we have just made the substitution $\min(s,n) = \min(s,n-1)$.

We conclude that, for any $n \ge 1$,

$$\Pi_n^2(j_1,j_2) = P_{n-1}(j_1,j_2) \qquad \begin{pmatrix} 0 \le j_1+j_2 \le s, \\ j_2 \le \min(s,n) \end{pmatrix} \qquad (3)$$

□

Chapter 4, Exercise 6

'Calls arrive according to a Poisson process ...'

For $j_1, j_2 \geq 0$, the equilibrium state equations are

$$\left(\frac{n_1-j_1}{n_1+n_2-j_1-j_2}\lambda + \frac{n_2-j_2}{n_1+n_2-j_1-j_2}\lambda + j_1\mu + j_2\mu\right) P(j_1,j_2) = \qquad (j_1<s_1,\ j_2<s_2)$$
$$\frac{n_1-j_1+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1-1,j_2) + \frac{n_2-j_2+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1,j_2-1)$$
$$+ (j_1+1)\mu P(j_1+1,j_2) + (j_2+1)\mu P(j_1,j_2+1)$$

$$\left(\frac{n_2-j_2}{n_1+n_2-j_1-j_2}\lambda + j_1\mu + j_2\mu\right) P(j_1,j_2) = \qquad (j_1=s_1,\ j_2<s_2)$$
$$\frac{n_1-j_1+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1-1,j_2) + \frac{n_2-j_2+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1,j_2-1)$$
$$+ (j_2+1)\mu P(j_1,j_2+1)$$

$$\left(\frac{n_1-j_1}{n_1+n_2-j_1-j_2}\lambda + j_1\mu + j_2\mu\right) P(j_1,j_2) = \qquad (j_1<s_1,\ j_2=s_2)$$
$$\frac{n_1-j_1+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1-1,j_2) + \frac{n_2-j_2+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1,j_2-1)$$
$$+ (j_1+1)\mu P(j_1+1,j_2)$$

$$\left(j_1\mu + j_2\mu\right) P(j_1,j_2) = \qquad (j_1=s_1,\ j_2=s_2)$$
$$\frac{n_1-j_1+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1-1,j_2) + \frac{n_2-j_2+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1,j_2-1).$$

Clearly, a solution to the equations

$$\frac{n_1-j_1}{n_1+n_2-j_1-j_2}\lambda P(j_1,j_2) = (j_1+1)\mu P(j_1+1,j_2) \qquad \begin{pmatrix} 0 \leq j_1 < s_1, \\ 0 \leq j_2 \leq s_2 \end{pmatrix}, \qquad (*)$$

$$\frac{n_2-j_2}{n_1+n_2-j_1-j_2}\lambda P(j_1,j_2) = (j_2+1)\mu P(j_1,j_2+1) \qquad \begin{pmatrix} 0 \leq j_1 \leq s_1, \\ 0 \leq j_2 < s_2 \end{pmatrix}, \qquad (**)$$

will also be a solution to the equilibrium state equations above.

(Chap. 4, Ex. 6)

By recursive solution, (*) yields

$$P(j_1,0) = \frac{n_1!(n_1+n_2-j_1)!}{j_1!\,(n_1+n_2)!\,(n_1-j_1)!}\left(\frac{\lambda}{\mu}\right)^{j_1} P(0,0) \qquad (j_1 = 0,1,\ldots,s_1),$$

and (**) yields

$$P(j_1,j_2) = \frac{n_2!(n_1+n_2-j_1-j_2)!}{j_2!(n_1+n_2-j_1)!\,(n_2-j_2)!}\left(\frac{\lambda}{\mu}\right)^{j_2} P(j_1,0) \quad \begin{pmatrix} j_1 = 0,1,\ldots,s_1, \\ j_2 = 0,1,\ldots,s_2 \end{pmatrix}.$$

Combination of the two equations and the substitution $\frac{\lambda}{\mu} = a$ result in

$$P(j_1,j_2) = \frac{\binom{n_1}{j_1}\binom{n_2}{j_2}}{\binom{n_1+n_2}{j_1+j_2}} \times \frac{a^{j_1+j_2}}{(j_1+j_2)!} P(0,0) \qquad \begin{pmatrix} 0 \le j_1 \le s_1 \\ 0 \le j_2 \le s_2 \end{pmatrix}.$$

When $(n_1,n_2) = (s_1,s_2)$ no call is lost if any trunk is idle, and every call is lost if all trunks are occupied. By (3.3) of Chapter 3 then $P_{j_1+j_2} = [a^{j_1+j_2}/(j_1+j_2)!]\, P_0$. Furthermore, in this case, an arriving customer will with equal probability be directed to every idle trunk. Consequently, the j_1+j_2 occupied trunks are actually drawn at random from the n_1+n_2 trunks. By the hypergeometric distribution $P(j_1|j_1+j_2) = \binom{n_1}{j_1}\binom{n_2}{j_2}/\binom{n_1+n_2}{j_1+j_2}$. It follows that, for $(n_1,n_2) = (s_1,s_2)$, $P(j_1,j_2) = P(j_1|j_1+j_2)\cdot P_{j_1+j_2}$ in agreement with the derived formula for $P(j_1,j_2)$. This may suggest that the formula holds also for $n_1 \ge s_1$ and $n_2 \ge s_2$, as it does.

Chapter 4, Exercise 7

'Customers arrive according to a Poisson process ...'

a By the theorem of total probability,

$$\sum_{x_1+\cdots+x_s=j} \tilde{P}(x_1,\ldots,x_s) = P_j \qquad (j = 0,1,\ldots,s)$$

for any $\{\mu_i\}$. If $\mu_1 = \ldots = \mu_s = \mu$, then the model specializes to the Erlang loss model, with P_j given by Eq. (3.3) of Chapter 3.

(Chap. 4, Ex. 7a)

With random server selection and $\mu_1 = \cdots = \mu_s = \mu$, clearly all combinations of j busy servers have equal probability. Since the number of combinations is $\binom{s}{j}$,

$$\widetilde{P}(x_1, \ldots, x_s) = \binom{s}{j}^{-1} P_j \qquad (x_1 + \ldots + x_s = j;\ \mu_1 = \cdots \mu_s = \mu).$$

$\boxed{b}$ For $x_i \in \{0,1\}$, the equilibrium state equations may be written:

$$\left(\lambda \frac{1}{s-j} \sum_{i=1}^{s} (1-x_i) + \sum_{i=1}^{s} x_i \mu_i\right) \widetilde{P}(x_1, \ldots, x_s) = \qquad \left(0 \leq \sum_{i=1}^{s} x_i = j < s\right)$$
$$\lambda \frac{1}{s-j+1} \left(\widetilde{P}(x_1 - 1, x_2, \ldots, x_s) + \widetilde{P}(x_1, x_2 - 1, \ldots, x_s) + \cdots + \widetilde{P}(x_1, \ldots, x_{s-1}, x_s - 1)\right)$$
$$+ (x_1+1)\mu_1 \widetilde{P}(x_1+1, x_2, \ldots, x_s) + \cdots + (x_i+1)\mu_i \widetilde{P}(x_1, \ldots, x_i+1, \ldots, x_s) + \cdots + (x_s+1)\mu_s \widetilde{P}(x_1, \ldots, x_{s-1}, x_s+1),$$

$$\left(\sum_{i=1}^{s} x_i \mu_i\right) \widetilde{P}(x_1, \ldots, x_s) = \qquad \left(\sum_{i=1}^{s} x_i = s\right)$$
$$\lambda \left(\widetilde{P}(x_1 - 1, x_2, \ldots, x_s) + \widetilde{P}(x_1, x_2 - 1, \ldots, x_s) + \cdots + \widetilde{P}(x_1, \ldots, x_{s-1}, x_s - 1)\right)$$

Once more, there is a pairwise correspondence between LHS and RHS terms. It is seen that the equilibrium state equations will be satisfied by $\widetilde{P}(x_1, \ldots, x_s)$ satisfying

$$\lambda \frac{1}{s-j} \widetilde{P}(x_1, \ldots, x_k, \ldots, x_s) = (x_k+1)\mu_k \widetilde{P}(x_1, \ldots, x_k+1, \ldots, x_s) \quad \left(x_k = 0,\ 0 \leq \sum_{i=1}^{s} x_i = j < s\right) \quad (*)$$

Let $\widetilde{P}_0 = \widetilde{P}(0,0,\ldots,0)$. By $(*)$, $\widetilde{P}(x_1, \ldots, x_k, \ldots, x_s) = \frac{1}{s} \frac{\lambda}{\mu_k} \widetilde{P}_0$ for $x_k = 1$ and $\Sigma_{i=1}^{s} x_i = j = 1$ (i.e. $x_i = 0$ for $i \neq k$). This finding can be expressed:

$$\widetilde{P}(x_1, \ldots, x_s) = \frac{(s-1)!}{s!} \prod_{i=1}^{s} \left(\frac{\lambda}{\mu_i}\right)^{x_i} \widetilde{P}_0 \qquad \left(\sum_{i=1}^{s} x_i = j = 1,\ x_i \in \{0,1\}\right).$$

By recursion, $(*)$ yields

$$\widetilde{P}(x_1, \ldots, x_s) = \frac{(s-j)!}{s!} \prod_{i=1}^{s} \left(\frac{\lambda}{\mu_i}\right)^{x_i} \widetilde{P}_0 \qquad \left(\sum_{i=1}^{s} x_i = j\ (j = 1, 2, \ldots, s),\ x_i \in \{0,1\}\right)$$

Notice, the formula also holds for $\sum_{i=1}^{s} x_i = j = 0$. Finally, rewriting this formula we obtain

$$\widetilde{P}(x_1, \ldots, x_s) = \binom{s}{\Sigma x_i}^{-1} \frac{\prod_{i=1}^{s} \left(\frac{\lambda}{\mu_i}\right)^{x_i}}{(\Sigma x_i)!} \widetilde{P}_0 \qquad (x_i \in \{0,1\},\ i = 1, \ldots, s)$$

As usual, $\widetilde{P}_0$ is found by the normalization condition. $\square$

Chapter 4, Exercise 8

'Network of queues.'

a) Let λ_i denote the mean arrival rate at Q_i. Obviously,

$$\lambda_1 = \lambda, \quad \lambda_2 = p_{12}\lambda, \quad \lambda_3 = p_{13}\lambda, \quad \lambda_4 = (1-p_{12}p_2)\lambda.$$

In the following it will be assumed that $\lambda_i/\mu_i < s_i$ for all i.

The arrival process at Q_1 is Poisson. By Burke's theorem, then, the equilibrium output from Q_1 is Poisson. The assignment of this output by lottery leads to a decomposition into independent Poisson streams with rates $\lambda_2 = p_{12}\lambda$ and $\lambda_3 = p_{13}\lambda$, respectively. We also note that the sum of two independent Poisson streams is Poisson. It can be concluded that the input to every queue is Poisson, so that each queue functions as an Erlang delay system with equilibrium state probabilities given by (4.3) and (4.4) of Chapter 3. Thus, for $i = 1,2,3,4$,

$$P_i(j_i) = \begin{cases} c_i \dfrac{(\lambda_i/\mu_i)^{j_i}}{j_i!} & (j_i = 0,\ldots,s_i-1), \\[2ex] c_i \dfrac{(\lambda_i/\mu_i)}{s_i!\, s_i^{\,j_i-s_i}} & (j_i = s_i, s_i+1,\ldots). \end{cases}$$

Furthermore, as a consequence of Burke's theorem, the states are independent, that is

$$P(j_1,j_2,j_3,j_4) = P_1(j_1)P_2(j_2)P_3(j_3)P_4(j_4).$$

b) With feedback from Q_2 to Q_1, the mean arrival rate at Q_1 is

$$\lambda_1^* = \lambda + (p_{12}p_2)\lambda + (p_{12}p_2)^2\lambda + \cdots$$

in this particular case, therefore, the mean arrival rates are

$$\lambda_1^* = \frac{\lambda}{1-p_{12}p_2}, \quad \lambda_2^* = \frac{p_{12}\lambda}{1-p_{12}p_2}, \quad \lambda_3^* = \frac{p_{13}\lambda}{1-p_{12}p_2}, \quad \lambda_4^* = \lambda.$$

Let $\mu_i(j)$ be defined as in (2.12). Then the equilibrium state equations are:

$$\begin{aligned}
&(\lambda + \mu_1(j_1) + \mu_2(j_2) + \mu_3(j_3) + \mu_4(j_4))\,P(j_1,j_2,j_3,j_4) = \\
&\quad \lambda P(j_1-1,j_2,j_3,j_4) + \mu_1(j_1+1)p_{12}P(j_1+1,j_2-1,j_3,j_4) + \mu_1(j_1+1)p_{13}P(j_1+1,j_2,j_3-1,j_4) \\
&\quad + \mu_2(j_2+1)p_2P(j_1-1,j_2+1,j_3,j_4) + \mu_2(j_2+1)p_{24}P(j_1,j_2+1,j_3,j_4-1) \\
&\quad + \mu_3(j_3+1)P(j_1,j_2,j_3+1,j_4-1) + \mu_4(j_4+1)P(j_1,j_2,j_3,j_4+1) \qquad (j_1,j_2,j_3,j_4 \geq 0)
\end{aligned}$$

(Chap. 4, Ex. 8 b)

Consider the following five equations obtained by pairing terms on LHS and RHS of the equilibrium state equations,

$$\lambda P(j_1, j_2, j_3, j_4) = \mu_4(j_4+1) P(j_1, j_2, j_3, j_4+1), \tag{1}$$

$$\mu_1(j_1) P(j_1, j_2, j_3, j_4) = \lambda P(j_1-1, j_2, j_3, j_4) + \mu_2(j_2+1) p_2 P(j_1-1, j_2+1, j_3, j_4), \tag{2}$$

$$\mu_2(j_2) P(j_1, j_2, j_3, j_4) = \mu_1(j_1+1) p_{12} P(j_1+1, j_2-1, j_3, j_4), \tag{3}$$

$$\mu_3(j_3) P(j_1, j_2, j_3, j_4) = \mu_1(j_1+1) p_{13} P(j_1+1, j_2, j_3-1, j_4), \tag{4}$$

$$\mu_4(j_4) P(j_1, j_2, j_3, j_4) = \mu_2(j_2+1) p_{24} P(j_1, j_2+1, j_3, j_4-1) + \mu_3(j_3+1) P(j_1, j_2, j_3+1, j_4-1). \tag{5}$$

It is easily seen that a solution to (1)-(5) will also be a solution to the equilibrium state equations.

By (3), $\mu_2(j_2+1) P(j_1-1, j_2+1, j_3, j_4) = \mu_1(j_1) p_{12} P(j_1, j_2, j_3, j_4)$. Substitution into (2) and rewriting lead to

$$\frac{\lambda}{1-p_{12}p_2} P(j_1, j_2, j_3, j_4) = \mu_1(j_1+1) P(j_1+1, j_2, j_3, j_4). \tag{*}$$

A rewriting of (3) gives

$$\mu_2(j_2) = \mu_1(j_1+1) p_{12} \frac{P(j_1+1, j_2, j_3, j_4)}{P(j_1, j_2, j_3, j_4)} \frac{P(j_1+1, j_2-1, j_3, j_4)}{P(j_1+1, j_2, j_3, j_4)},$$

which, by (*), simplifies to

$$\mu_2(j_2) = \frac{p_{12}\lambda}{1-p_{12}p_2} \frac{P(j_1+1, j_2-1, j_3, j_4)}{P(j_1+1, j_2, j_3, j_4)}.$$

Hence,

$$\frac{p_{12}\lambda}{1-p_{12}p_2} P(j_1, j_2, j_3, j_4) = \mu_2(j_2+1) P(j_1, j_2+1, j_3, j_4). \tag{**}$$

Similarly, by (4) and (*),

$$\mu_3(j_3) = \frac{p_{13}\lambda}{1-p_{12}p_2} \frac{P(j_1+1, j_2, j_3-1, j_4)}{P(j_1+1, j_2, j_3, j_4)},$$

whereby

$$\frac{p_{13}\lambda}{1-p_{12}p_2} P(j_1, j_2, j_3, j_4) = \mu_3(j_3+1) P(j_1, j_2, j_3+1, j_4). \tag{***}$$

(Chap. 4, Ex. 8 b (cont'd))

Equations (*), (**), (***), have been derived from and are equivalent to Equations (2), (3), (4). Eq. (1) has the desired form, and we shall keep it the way it is. The final equation, (5), is redundant. To see this, combine (1) and (5) into

$$\lambda P(j_1, j_2, j_3, j_4) = \mu_2(j_2+1) p_{24} P(j_1, j_2+1, j_3, j_4) + \mu_3(j_3+1) P(j_1, j_2, j_3+1, j_4).$$

By (**) and (***), the right-hand side equals

$$\left[\frac{p_{12}\, p_{24}}{1 - p_{12} p_2} + \frac{p_{13}}{1 - p_{12} p_2}\right] \lambda P(j_1, j_2, j_3, j_4) = \lambda P(j_1, j_2, j_3, j_4)$$

since $p_{12} p_{24} + p_{13} = 1 - p_{12} p_2$. This proves redundancy.

We conclude that (1)-(5) are equivalent to the following system of equations,

$$\lambda_1^* P(j_1, j_2, j_3, j_4) = \mu_1(j_1+1) P(j_1+1, j_2, j_3, j_4), \tag{6}$$

$$\lambda_2^* P(j_1, j_2, j_3, j_4) = \mu_2(j_2+1) P(j_1, j_2+1, j_3, j_4), \tag{7}$$

$$\lambda_3^* P(j_1, j_2, j_3, j_4) = \mu_3(j_3+1) P(j_1, j_2, j_3+1, j_4), \tag{8}$$

$$\lambda_4^* P(j_1, j_2, j_3, j_4) = \mu_4(j_4+1) P(j_1, j_2, j_3, j_4+1). \tag{9}$$

where λ_i^* $(i = 1,2,3,4)$ is the mean arrival rate at Q_i.

Recursive solution of Eq. (6), for example, gives, for fixed j_2, j_3, j_4,

$$P(j_1, j_2, j_3, j_4) = \begin{cases} P(0, j_2, j_3, j_4)\, \dfrac{(\lambda_1^*/\mu_1)^{j_1}}{j_1!} & (j_1 = 0, \ldots, s_1 - 1), \\ P(0, j_2, j_3, j_4)\, \dfrac{(\lambda_1^*/\mu_1)^{j_1}}{s_1!\, s_1^{j_1 - s_1}} & (j_1 = s_1, s_1+1, \ldots). \end{cases}$$

The marginal probability of j_1, $P_1(j_1)$, is found by summation over all possible j_2, j_3, j_4. In general we find

$$P_i(j_i) = \begin{cases} P_i(0)\, \dfrac{(\lambda_i^*/\mu_i)^{j_i}}{j_i!} & (j_i = 0, \ldots, s_i - 1), \\ P_i(0)\, \dfrac{(\lambda_i^*/\mu_i)^{j_i}}{s_i!\, s_i^{j_i - s_i}} & (j_i = s_i, s_i+1, \ldots). \end{cases}$$

In addition it may be shown that the condition for independence holds:

$$P(j_1, j_2, j_3, j_4) = P_1(j_1) P_2(j_2) P_3(j_3) P_4(j_4) \quad (j_i \geq 0,\ i = 1,2,3,4) \qquad \square$$

Chapter 4, Exercise 9

'Closed networks of queues.'

$\boxed{a}$ As before, let $\mu_i(j_i) = j_i\mu_i$ if $j_i \leq s_i$, $\mu_i(j_i) = s_i\mu_i$ if $j_i > s_i$. The equilibrium state equations are as follows,

$$(\mu_1(j_1) + \mu_2(j_2) + \cdots + \mu_m(j_m))P(j_1, j_2, \ldots, j_m) =$$
$$\mu_1(j_1+1)P(j_1+1, j_2-1, j_3, \ldots, j_m) + \mu_2(j_2+1)P(j_1, j_2+1, j_3-1, \ldots, j_m)$$
$$+ \cdots + \mu_m(j_m+1)P(j_1-1, j_2, \ldots, j_m+1) \qquad (j_i \geq 0 \text{ for } i = 1, \ldots, m;\ \Sigma j_i = n).$$

Consider the following m equations extracted from the equilibrium state equations,

$$\mu_1(j_1)P(j_1, j_2, \ldots, j_m) = \mu_m(j_m+1)P(j_1-1, j_2, \ldots, j_m+1), \qquad (1')$$
$$\mu_2(j_2)P(j_1, j_2, \ldots, j_m) = \mu_1(j_1+1)P(j_1+1, j_2-1, \ldots, j_m), \qquad (2')$$
$$\mu_3(j_3)P(j_1, j_2, \ldots, j_m) = \mu_2(j_2+1)P(j_1, j_2+1, j_3-1, \ldots, j_m), \qquad (3')$$
$$\vdots$$
$$\mu_{m-1}(j_{m-1})P(j_1, j_2, \ldots, j_m) = \mu_{m-2}(j_{m-2}+1)P(\ldots, j_{m-2}+1, j_{m-1}-1, j_m), \qquad ((m-1)')$$
$$\mu_m(j_m)P(j_1, j_2, \ldots, j_m) = \mu_{m-1}(j_{m-1}+1)P(j_1, \ldots, j_{m-1}+1, j_m-1). \qquad (m')$$

It is clear that a solution to these equations will also be a solution to the equilibrium state equations. Anyone of the equations can be omitted. We choose to discard (1).

Now write

$$P(j_1, j_2, \ldots, j_m) = P(n, 0, \ldots, 0) \times \frac{P(j_1, n-j_1, 0, \ldots, 0)}{P(n, 0, \ldots, 0, 0, 0)} \times \frac{P(j_1, j_2, n-j_1-j_2, 0, \ldots, 0)}{P(j_1, n-j_1, 0, \ldots, 0, 0, 0)}$$
$$\times \cdots \times \frac{P(j_1, j_2, \ldots, j_m)}{P(j_1, j_2, \ldots, n-j_1-j_2-\cdots-j_{m-2}, 0)}.$$

By a similar rewriting and the application of Eq. (2'), factor no. 2 on the right can be expressed

$$\frac{P(j_1, n-j_1, 0, \ldots, 0)}{P(n, 0, \ldots, 0, 0, 0)} = \frac{P(n-1, 1, 0, \ldots, 0)}{P(n, 0, \ldots, 0, 0)} \times \frac{P(n-2, 2, 0, \ldots, 0)}{P(n-1, 1, 0, \ldots, 0)} \times \cdots \times \frac{P(j_1, n-j_1, 0, \ldots, 0)}{P(j_1+1, n-j_1-1, 0, \ldots, 0)}$$
$$= \frac{\mu_1(n)}{\mu_2(1)} \times \frac{\mu_1(n-1)}{\mu_2(2)} \times \cdots \times \frac{\mu_1(j_1+1)}{\mu_2(n-j_1)}. \qquad (j_1 < n)$$

(Chap. 4, Ex. 9a)

Proceeding in this manner, using Eq. (k′) for factoring factor no. k, we deduce

$$P(j_1, j_2, \ldots, j_m) = P(n, 0, \ldots, 0)\, A_2 A_3 \cdots A_m, \qquad (*)$$

where, for $k = 2, 3, \ldots, m$,

$$A_k = \begin{cases} 1 & \left(\sum_{i=k}^{m} j_i = 0\right), \\ \dfrac{\mu_{k-1}\left(\sum_{i=k-1}^{m} j_i\right)}{\mu_k(1)}\, \dfrac{\mu_{k-1}\left(\sum_{i=k-1}^{m} j_i - 1\right)}{\mu_k(2)} \cdots \dfrac{\mu_{k-1}(j_{k-1}+1)}{\mu_k\left(\sum_{i=k}^{m} j_i\right)} & \left(\sum_{i=k}^{m} j_i \geq 1\right). \end{cases}$$

Cancellation of factors in (*) results in

$$P(j_1, j_2, \ldots, j_m) = \left[P(n, 0, \ldots, 0) \prod_{r=1}^{n} \mu_1(r)\right] \prod_{i=1}^{m} Q_i(j_i) \qquad \left(\begin{matrix} 0 \leq j_i \leq n, \\ \Sigma j_i = n \end{matrix}\right),$$

where $Q_i(j_i) = 1$ if $j_i = 0$, $Q_i(j_i) = [\mu_i(1)\mu_i(2) \cdots \mu_i(j_i)]^{-1}$ if $j_i \geq 1$. Finally, using $\Sigma P(j_1, j_2, \ldots, j_m) = 1$ and substituting the expression for $\mu_i(j)$, we obtain

$$P(j_1, j_2, \ldots, j_m) = \frac{\prod_{i=1}^{m} Q_i(j_i)}{\Sigma_S \prod_{i=1}^{m} Q_i(r_i)} \qquad \left(\begin{matrix} 0 \leq j_i \leq n, \\ \Sigma j_i = n \end{matrix}\right), \qquad (1)$$

where $S = \{(r_1, r_2, \ldots, r_m) : 0 \leq r_i \leq n, \Sigma r_i = n\}$ and

$$Q_i(j_i) = \begin{cases} \dfrac{(1/\mu_i)^{j_i}}{j_i!} & (j_i < s_i), \\ \dfrac{(1/\mu_i)^{j_i}}{s_i!\, s_i^{\,j_i - s_i}} & (j_i \geq s_i). \end{cases}$$

The probability $P(j_1, j_2, \ldots, j_m)$ has been written as a product of factors dependent on $j_1, j_2, \ldots, j_m$, respectively, but the random variables $N_1, N_2, \ldots, N_m$ are not independent. The reason is that the set S for which Eq. (1) applies is not a product space $J_1 \times J_2 \times \cdots \times J_m$. If, for instance, $N_1 = n$, then $N_2 = 0$, so that, obviously, N_1 and N_2 cannot be independent variables.

(Chap. 4, Ex. 9 b)

$\boxed{\text{b}}$ In the general model, where a departure from Q_i with probability p_{ij} goes to Q_j $(i,j = 1,2,\dots,m)$, the equilibrium state equations are

$$\sum_{i=1}^{m} \mu_i(j_i)\, P(j_1,\dots,j_m) = \sum_{i=1}^{m} \sum_{\substack{k=1\\k\neq i}}^{m} \mu_k(j_k+1)\, p_{ki}\, P(j_1,\dots,j_i-1,\dots,j_k+1,\dots,j_m) + \sum_{i=1}^{m} \mu_i(j_i)\, p_{ii}\, P(j_1,\dots,j_m). \tag{2}$$

We shall verify that the solution is of the form (1), that is $P(j_1,\dots,j_m) = \prod_{i=1}^{m} Q_i(j_i) / \Sigma_{\mathcal{S}} \prod_{i=1}^{m} Q_i(r_i)$, but where, for some $\{x_i\}$,

$$Q_i(j_i) = \begin{cases} \dfrac{(1/x_i)^{j_i}}{j_i!} & (j_i < s_i), \\[2ex] \dfrac{(1/x_i)^{j_i}}{s_i!\, s_i^{\,j_i - s_i}} & (j_i \geq s_i). \end{cases} \tag{3}$$

Substitution of (1) into (2) gives

$$\Big(\prod_{i=1}^{m} Q_i(j_i)\Big) \sum_{i=1}^{m} \mu_i(j_i) = \Big(\prod_{i=1}^{m} Q_i(j_i)\Big) \sum_{i=1}^{m} \sum_{\substack{k=1\\k\neq i}}^{m} \mu_k(j_k+1)\, p_{ki}\, \frac{Q_i(j_i-1)}{Q_i(j_i)}\, \frac{Q_k(j_k+1)}{Q_k(j_k)} + \Big(\prod_{i=1}^{m} Q_i(j_i)\Big) \sum_{i=1}^{m} \mu_i(j_i)\, p_{ii}.$$

Cancellation of the common factor $\prod_{i=1}^{m} Q_i(j_i)$ and the use of (3) results in

$$\sum_{i=1}^{m} \mu_i(j_i) = \sum_{i=1}^{m} \sum_{\substack{k=1\\k\neq i}}^{m} \mu_k(j_k+1)\, p_{ki}\, \frac{\mu_i(j_i)\, x_i}{\mu_i}\, \frac{\mu_k}{\mu_k(j_k+1)\, x_k} + \sum_{i=1}^{m} \mu_i(j_i)\, p_{ii}.$$

Hence,

$$\sum_{i=1}^{m} \mu_i(j_i) \Big[1 - \mu_i^{-1} x_i \sum_{\substack{k=1\\k\neq i}}^{m} \mu_k x_k^{-1} p_{ki} - p_{ii} \Big] = 0,$$

whereby

$$\sum_{i=1}^{m} \mu_i(j_i) \Big[1 - \mu_i^{-1} x_i \sum_{k=1}^{m} \mu_k x_k^{-1} p_{ki} \Big] = 0. \tag{4}$$

This leads to the requirement that

$$\sum_{k=1}^{m} p_{ki} (\mu_k x_k^{-1}) = \mu_i x_i^{-1} \qquad (i = 1,2,\dots,m). \tag{5}$$

A solution $\{\mu_i x_i^{-1}\}$ $(\{x_i\})$ exists provided all queues "communicate." $\square$

Chapter 4, Exercise 10

'The following model can be used ...'

OBS! The server groups have been renamed : $G_1 \to H_2$, $G_2 \to H_1$.

For group H_1 (digit trunks for dialing) let s_1 be the number of (exponential) servers with service rate μ_1 , and for group H_2 (time slots for talking) let s_2 be the number of (exponential) servers with service rate μ_2. The possibility that all servers in H_1 are busy, while a server in H_2 is idle, implies $s_1 < s_2$, since a call holding a server in H_1 will at the same time hold a server in H_2. Let j_1 = calls in dialing phase or in waiting position, and let j_2 = calls in talking phase. Let $\mu_1(j_1) = j_1\mu_1$ if $j_1 < s_1$, $\mu_1(j_1) = s_1\mu_1$ if $j_1 \geq s_1$.

The equilibrium state equations are, for $j_1, j_2 \geq 0$,

$$(\lambda + \mu_1(j_1) + j_2\mu_2)P(j_1,j_2) = \qquad (0 \leq j_1+j_2 < s_2)$$
$$\lambda P(j_1-1,j_2) + \mu_1(j_1+1)P(j_1+1,j_2-1) + (j_2+1)\mu_2 P(j_1,j_2+1),$$

$$(\mu_1(j_1) + j_2\mu_2)P(j_1,j_2) = \qquad (j_1+j_2 = s_2)$$
$$\lambda P(j_1-1,j_2) + \mu_1(j_1+1)P(j_1+1,j_2-1).$$

Hence the equations

$$\lambda P(j_1,j_2) = (j_2+1)\mu_2 P(j_1,j_2+1) \qquad (0 \leq j_1+j_2 < s_2),$$
$$\mu_1(j_1)P(j_1,j_2) = \lambda P(j_1-1,j_2) \qquad (0 \leq j_1+j_2 \leq s_2),$$
$$j_2\mu_2 P(j_1,j_2) = \mu_1(j_1+1)P(j_1+1,j_2-1) \qquad (0 \leq j_1+j_2 \leq s_2),$$

a solution to which will also solve the equilibrium state equations. Disregarding the last, redundant equation, solving the other two recursively starting with $P(0,0)$, we find

$$P(j_1,j_2) = c\,P_1(j_1)\,P_2(j_2) \qquad (0 \leq j_1+j_2 \leq s_2),$$

where

$$P_1(j_1) = \begin{cases} \dfrac{(\lambda/\mu_1)^{j_1}}{j_1!} & (j_1 < s_1), \\ \dfrac{(\lambda/\mu_1)^{j_1}}{s_1!\, s_1^{\,j_1-s_1}} & (s_1 \leq j_1 \leq s_2); \end{cases} \qquad P_2(j_2) = \frac{(\lambda/\mu_2)^{j_2}}{j_2!} \quad (0 \leq j_2 \leq s_2).$$

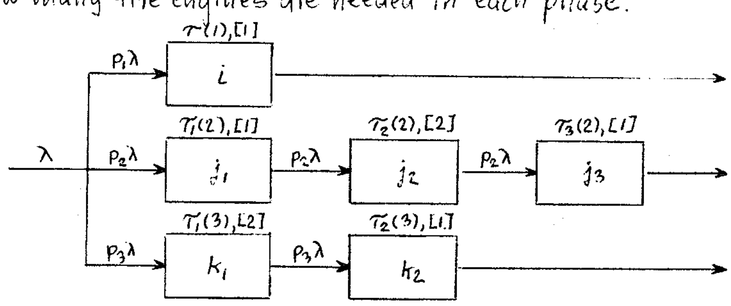
The constant c is determined by $\sum_{0 \leq j_1+j_2 \leq s_2} P(j_1,j_2) = 1.$ $\square$

Chapter 4, Exercise 11

'The following is a simplified version of a model...'

In the present model, the customers are the fires of any of three types, going through phases whose duration has an exponential distribution. The overall arrival rate is λ, and with probability p_r $(r = 1,2,3)$ a fire is of type r. A fire of type 1 is characterized by a single phase with mean $\tau(1)$, requiring the use of 1 fire engine. A fire of type 2 goes through three phases with means $\tau_1(2), \tau_2(2), \tau_3(2)$, requiring 1, 2 and 1 fire engine(s), respectively. A fire of type 3 has two phases with means $\tau_1(3)$ and $\tau_2(3)$, requiring 2 and 1 fire engines respectively.

The fires pass through a network of infinite-server queues. In the figure, the variables $i, j_1, j_2, j_3, k_1, k_2$ denote number of fires in progress, and the numbers in square brackets indicate how many fire engines are needed in each phase.



In reality, the network is composed of three independent queueing systems with arrival rates $p_1\lambda$, $p_2\lambda$, $p_3\lambda$, respectively. The first queueing system is an infinite-server queue, whose equilibrium distribution, by Eq. (4.27) of Chapter 2, is $P_i = ([p_1\lambda\tau(1)]^i/i!)\,e^{-p_1\lambda\tau(1)}$. The other two queueing systems are tandem queues. By Burke's theorem, the input to each constituent queue is Poisson, so that also here the state variables follow a Poisson distribution, and, furthermore, the equilibrium states of the queues are independent. Hence,

$$P(i; j_1, j_2, j_3; k_1, k_2) = \frac{[p_1\lambda\tau(1)]^i}{i!}\,\frac{[p_2\lambda\tau_1(2)]^{j_1}}{j_1!}\,\frac{[p_2\lambda\tau_2(2)]^{j_2}}{j_2!}\,\frac{[p_2\lambda\tau_3(2)]^{j_3}}{j_3!}\,\frac{[p_3\lambda\tau_1(3)]^{k_1}}{k_1!}\,\frac{[p_3\lambda\tau_2(3)]^{k_2}}{k_2!}\cdot c\,,$$

where,

$$c = \exp\{-[p_1\lambda\tau(1) + p_2\lambda\tau_1(2) + p_2\lambda\tau_2(2) + p_2\lambda\tau_3(2) + p_3\lambda\tau_1(3) + p_3\lambda\tau_2(3)]\}.$$

The distribution of $m = i + j_1 + 2j_2 + j_3 + 2k_1 + k_2$ is found by convoluting $i, j_1, 2j_2$ etc. $\square$

Chapter 4, Exercise 12

'Apportioning the moments of the overflow distribution.'

$\boxed{a}$ The equilibrium state equations for the distribution $\{h(j,k_1,k_2)\}$ are

$$(a+j+k_1+k_2)\,h(j,k_1,k_2) = a\,h(j-1,k_1,k_2) + (j+1)\,h(j+1,k_1,k_2) + (k_1+1)\,h(j,k_1+1,k_2) + (k_2+1)\,h(j,k_1,k_2+1) \qquad \begin{pmatrix} j=0,1,\dots,s-1;\\ k_1,k_2=0,1,\dots \end{pmatrix}$$

$$(a+s+k_1+k_2)\,h(s,k_1,k_2) = a\,h(s-1,k_1,k_2) + a_1 h(s,k_1-1,k_2) + a_2 h(s,k_1,k_2-1) + (k_1+1)\,h(s,k_1+1,k_2) + (k_2+1)\,h(s,k_1,k_2+1) \qquad \begin{pmatrix} j=s;\\ k_1,k_2=0,1,\dots \end{pmatrix}$$

We shall verify that

$$h(j,k_1,k_2) = P(j,k)\binom{k}{k_1}\left(\frac{a_1}{a}\right)^{k_1}\left(\frac{a_2}{a}\right)^{k_2}, \qquad (*)$$

where $k = k_1+k_2$. We begin by showing that the suggested solution satisfies the equilibrium state equations. Insertion of the above expression for $h(j,k_1,k_2)$ into the two sets of equilibrium state equations, and a straightforward reduction, result in Eqs. (3.3) and (3.4), respectively, which are always satisfied. Hence, the suggested solution is indeed the solution, at least up to a factor. Now, it is easily shown that $\Sigma_{k_1+k_2=k} h(j,k_1,k_2) = P(j,k)$ by the given expression, just as it should, so the expression does give the correct value.

$\boxed{b}$ By definition of a conditional probability, $h(j,k_1,k_2)$ may be expressed as $h(j,k_1,k_2) = P(j,k)\,P\{N_1=k_1, N_2=k_2 \mid N=k\}$. By comparison with (*), therefore

$$P\{N_1=k_1, N_2=k_2 \mid N=k\} = \binom{k}{k_1}\left(\frac{a_1}{a}\right)^{k_1}\left(\frac{a_2}{a}\right)^{k_2}. \qquad (4)$$

$\boxed{c}$ A binomial variable X with parameters (n,p) has mean $E(X) = np$ and variance $V(X) = np(1-p)$. Hence, $E(X^2) = V(X) + E^2(X) = n(n-1)p^2 + np$. By (4), for $N=k$, N_i is a binomial variable with $(n,p) = (k, a_i/a)$. It follows that

$$E(N_i \mid N=k) = k\,\frac{a_i}{a} \qquad (5)$$

$$E(N_i^2 \mid N=k) = k(k-1)\left(\frac{a_i}{a}\right)^2 + k\,\frac{a_i}{a} \qquad (6)$$

(Chap. 4, Ex. 12 d)

$\boxed{d}$ By (5), $E(N_i) = E_N(E(N_i|N=k)) = E_N(N\frac{a_i}{a})$. Hence,

$$E(N_i) = \frac{a_i}{a}E(N). \tag{1}$$

$\boxed{e}$ By (6), $E(N_i^2) = E_N(E(N_i^2|N=k)) = E_N(N(N-1)(\frac{a_i}{a})^2 + N\frac{a_i}{a})$. Hence

$$E(N_i^2) = (\tfrac{a_i}{a})^2 E(N^2) + \tfrac{a_i}{a}(1-\tfrac{a_i}{a})E(N). \tag{7}$$

$\boxed{f}$ By (1) and (7), $V(N_i) = E(N_i^2) - E^2(N_i) = (\frac{a_i}{a})^2[E(N^2) - E^2(N)] + \frac{a_i}{a}(1-\frac{a_i}{a})E(N)$. Hence,

$$V(N_i) = (\tfrac{a_i}{a})^2 V(N) + \tfrac{a_i}{a}(1-\tfrac{a_i}{a})E(N). \tag{2}$$

Dividing Eq. (1) into Eq. (2), we get

$$\frac{V(N_i)}{E(N_i)} = \frac{a_i}{a}\frac{V(N)}{E(N)} + (1-\frac{a_i}{a}).$$

Letting $z = V(N)/E(N)$ and $z_i = V(N_i)/E(N_i)$, we find, for $p_i = a_i/a$,

$$z_i - 1 = p_i(z-1). \tag{8}$$

$\boxed{g}$ By (5) and (6), when $n=2$,

$$\begin{aligned} E(N_1N_2|N=k) &= E(N_1(N-N_1)|N=k) \\ &= k\,E(N_1|N=k) - E(N_1^2|N=k) \\ &= k^2\frac{a_1}{a} - k(k-1)(\frac{a_1}{a})^2 - k\frac{a_1}{a} \\ &= \frac{a_1}{a}(1-\frac{a_1}{a})k(k-1). \end{aligned}$$

Hence, $E(N_1N_2) = E_N(E(N_1N_2|N=k)) = \frac{a_1}{a}(1-\frac{a_1}{a})E(N(N-1))$, or,

$$E(N_1N_2) = \frac{a_1}{a}\frac{a_2}{a}[E(N^2) - E(N)].$$

By (1),

$$E(N_1)E(N_2) = \frac{a_1}{a}\frac{a_2}{a}E^2(N).$$

Thus,

$$Cov(N_1,N_2) = E(N_1N_2) - E(N_1)E(N_2) = \frac{a_1}{a}\frac{a_2}{a}[V(N) - E(N)]. \tag{3}$$ □

Chapter 4, Exercise 13

'Show that when $a_1 = a_2 = \cdots = a_{s-1} = 0$ and $a_s = a$, then ...'

When $\lambda_1 = \lambda_2 = \cdots = \lambda_{s-1} = 0$ and $\lambda_s = \lambda$, then Eq. (4.6) becomes

$$\lambda_s P(j-1) = j\mu P(j) \qquad (j = 1, 2, \ldots, s-1),$$
$$\lambda_s P(s-1) + \lambda_s P(s) = s\mu P(s) \qquad (j = s),$$

or,

$$a P(j-1) = j P(j) \qquad (j = 1, 2, \ldots, s-1),$$
$$a P(s-1) = (s-a) P(s) \qquad (j = s).$$

The solution, in terms of $P(0)$, is

$$P(j) = \frac{a^j}{j!} P(0) \qquad (j = 0, 1, \ldots, s-1),$$
$$P(s) = \frac{a^s}{s!(1-a/s)} P(0).$$

Hence, and by Eq. (4.8) of Chapter 3, if $a < s$,

$$P(s) = \frac{\dfrac{a^s}{s!(1-a/s)}}{\sum_{k=0}^{s-1} \dfrac{a^k}{k!} + \dfrac{a^s}{s!(1-a/s)}} = C(s,a).$$

Chapter 4, Exercise 14

'Consider again the premise of Exercise 7...'

For a BCD queue with heterogeneous, exponential servers and random selection of server, let $P(x_1, \ldots, x_s; k) = P\{X_1 = x_1, \ldots, X_s = x_s; Q = k\}$.

When $\Sigma_{i=1}^{s} x_i < s$, the equilibrium state equations are precisely as in the BCC queue of Exercise 7, with $P(x_1, \ldots, x_s; 0)$ replacing $\tilde{P}(x_1, \ldots, x_s)$.

When $\Sigma_{i=1}^{s} x_i = s$ ($x_i = 1$ for all i) and $k = 0$, we now have

$$(\lambda + \sum_{i=1}^{s} x_i \mu_i) P(x_1, \ldots, x_s; 0) = \qquad (\sum_{i=1}^{s} x_i = s)$$
$$(\sum_{i=1}^{s} \mu_i) P(x_1, \ldots, x_s; 1) \qquad (*)$$
$$+ \lambda (P(x_1 - 1, x_2, \ldots, x_s; 0) + P(x_1, x_2 - 1, x_3, \ldots, x_s; 0) + \cdots + P(x_1, \ldots, x_{s-1}, x_s - 1; 0)).$$

(Chap. 4, Ex. 14)

In addition we have the rate up = rate down equations

$$\lambda P(1,\dots,1;k) = \Big(\sum_{i=1}^{s}\mu_i\Big)P(1,\dots,1;k+1) \qquad (k=0,1,\dots). \qquad (**)$$

By subtraction of Eq. (**), for $k=0$, from Eq. (*), we derive

$$\Big(\sum_{i=1}^{s}x_i\mu_i\Big)P(x_1,\dots,x_s;0) = \qquad \Big(\sum_{l=1}^{s}x_l = s\Big)$$
$$\lambda\big(P(x_1-1,x_2,\dots,x_s;0)+P(x_1,x_2-1,x_3,\dots,x_s;0)+P(x_1,\dots,x_{s-1};x_s-1;0)\big). \qquad (***)$$

Observe that (***) is the remaining equilibrium state equation in the BCC case of Exercise 7, with $P(x_1,\dots,x_s;0)$ instead of $\tilde{P}(x_1,\dots,x_s)$.
We conclude that for $k=0$ the solution is the same as in the BCC case except for a proportionality constant,

$$P(x_1,\dots,x_s;0) = c\,\tilde{P}(x_1,\dots,x_s) \qquad (x_i=0,1;\ l=1,\dots,s). \qquad (1)$$

By (**),

$$P(1,\dots,1;k) = \Big(\frac{\lambda}{\sum_{i=1}^{s}\mu_i}\Big)^k P(1,\dots,1;0) \qquad (k=1,2,\dots).$$

Utilization of (1) and the definition $\rho = \lambda/\sum_{i=1}^{s}\mu_i$ (utilization factor) give

$$P(1,\dots,1;k) = c\rho^k\tilde{P}(1,\dots,1) \qquad (k=1,2,\dots). \qquad (2)$$

Substitution into the normalization equation leads to

$$\begin{aligned}1 &= \sum_{0\le\Sigma x_i\le s}P(x_1,\dots,x_s;0)+\sum_{k=1}^{\infty}P(1,\dots,1;k)\\ &= c\sum_{0\le\Sigma x_i\le s}\tilde{P}(x_1,\dots,x_s)+c\,\tilde{P}(1,\dots,1)\sum_{k=1}^{\infty}\rho^k\\ &= c + c\,\tilde{P}(1,\dots,1)\frac{\rho}{1-\rho}\end{aligned}$$

Hence,

$$c = \Big(1+\frac{\rho}{1-\rho}\tilde{P}(1,\dots,1)\Big)^{-1} \qquad (3)$$

Thus, the equilibrium probabilities for the BCD queue are given by (1), (2), and (3), where $\{\tilde{P}(x_1,\dots,x_s)\}$ are the equilibrium probabilities of the corresponding BCC queue of Exercise 7. □

Chapter 4, Exercise 15

'Show that, if $a > 0$, then $\vartheta = \alpha$ when $s = 0$, $\vartheta > \alpha$ when $s \geq 1$...'

$$\alpha = a\,B(s,a), \tag{3.1}$$

$$\vartheta = \alpha\left(1 - \alpha + \frac{a}{s+1+\alpha-a}\right). \tag{3.2}$$

$s = 0$. Clearly, $B(0,a) = 1$. By (3.1) and (3.2) then, $\vartheta = \alpha = a$. This should come as no surprise, as the equilibrium state of an infinite-server system with Poisson input has the Poisson distribution with parameter (mean and variance) equal to a.

$s \geq 1$. By Exercise 6 of Chapter 3,

$$B(s,a) = \frac{a\,B(s-1,a)}{s + a\,B(s-1,a)} \qquad (s \geq 1).$$

Hence follow the two equivalent equations

$$s + 1 + a\,B(s,a) = \frac{a\,B(s,a)}{B(s+1,a)} \qquad (s \geq 0), \qquad (*)$$

$$\frac{B(s+1,a)}{B(s,a)} = \frac{a\,(1 - B(s+1,a))}{s+1} \qquad (s \geq 0). \qquad (**)$$

Now, for $s \geq 1$,

$$\begin{aligned}
\vartheta > \alpha \iff& \frac{a}{s+1+\alpha-a} > \alpha && [\text{by } (3.2)]\\
\iff& \frac{a}{s+1+aB(s,a)-a} > a\,B(s,a) && [\text{by } (3.1)]\\
\iff& \frac{B(s+1,a)}{B(s,a)} > a[B(s,a) - B(s+1,a)] && [\text{by } (*)]\\
\iff& \frac{a[1 - B(s+1,a)]}{s+1} > a[B(s,a) - B(s+1,a)]. && [\text{by } (**)]
\end{aligned}$$

As usual, let $\tilde{p}_j$ denote the load on the j'th ordered server in an Erlang loss system. By Eq. (3.18) of Chapter 3, $\tilde{p}_{s+1} = a[B(s,a) - B(s+1,a)]$ and $\sum_{j=1}^{s+1} \tilde{p}_j = a[1 - B(s+1,a)]$ (= carried load with $s+1$ servers). Thus, for $s \geq 1$,

$$\vartheta > \alpha \iff \sum_{j=1}^{s+1} \tilde{p}_j/(s+1) > \tilde{p}_{s+1}.$$

According to Messerli [1972], cf. Sec. 3 of Chap. 3, $\tilde{p}_1 > \tilde{p}_2 > \tilde{p}_3 \cdots$. Hence, $s \geq 1 \Rightarrow \sum_{j=1}^{s+1} \tilde{p}_j/(s+1) > \tilde{p}_s$. The conclusion then is

$$s \geq 1 \Rightarrow \vartheta > \alpha. \qquad \square$$

Chapter 4, Exercise 16

'$a_1 = 10$ erl of Poisson traffic is offered to a group of 10 servers...'

The exercise requires the use of the Equivalent Random Method. The case is desribed by Figure 4-8 for $n = 2$. Offered loads and server group sizes are:

Parts (a) and (b): $(s_1, a_1) = (10, 10)$, $(s_2, a_2) = (5, 5)$, c to be calculated.
Part (c): $(s_1, a_1^*) = (10, 15)$, $(s_2, a_2^*) = (5, 7.5)$, c as in (a) & (b).

$\boxed{a}$ The values of $B(s_1, a_1)$ and $B(s_2, a_2)$ may be read off Figure A-1 in the Appendix. Exact values obtained from tables of the Poisson distribution or the Erlang B-formula are

$$B(s_1, a_1) = B(10, 10) = 0.215\_ ,$$
$$B(s_2, a_2) = B(5, 5) = 0.285\_ .$$

For the two primary groups, the mean and the variance of the equilibrium state of an infinite-server backup group are, by Eqs. (3.1) and (3.2),

$$\alpha_1 = 2.15, \qquad \vartheta_1 = 4.35,$$
$$\alpha_2 = 1.42, \qquad \vartheta_2 = 2.34.$$

Hence, the total overflow is characterized by the parameters

$$\alpha = \alpha_1 + \alpha_2 = 3.57,$$
$$\vartheta = \vartheta_1 + \vartheta_2 = 6.69,$$
$$z = \vartheta/\alpha = 1.87,$$

whereby

$$\tilde{a} = \vartheta + 3z(z-1) = 11.6 \qquad [\text{by } (7.15)],$$
$$\tilde{s} = \frac{\tilde{a}(\alpha+z)}{\alpha+z-1} - \alpha - 1 = 9.6 \qquad [\text{by } (7.16)].$$

Hence,

$$s = [\tilde{s}] = 9,$$
$$a = \frac{(s+\alpha+1)(\alpha+z-1)}{\alpha+z} = 11.1 \qquad [\text{by } (7.17)].$$

Thus, $(s, a) = (9, 11.1)$ define the equivalent random system having the approximate overflow characteristics α and ϑ of the total overflow from the two primary groups.

(Chap. 4, Ex. 16 a)

When the overflow group has size c, the loss among the overflow customers (from the equivalent random group) is, by (7.13),

$$\Pi_c = \frac{B(s+c,a)}{B(s,a)} = \frac{B(9+c,11.1)}{B(9,11.1)}.$$

By Figure A-1, $B(9,11.1) = 0.323$. Thus the requirement $\Pi_c < 0.10$ translates into the condition

$$B(9+c,11.1) < 0.0323 < B(9+c-1,11.1).$$

From Figure A-1 we obtain

$$B(17,11.1) = 0.0259, \qquad B(16,11.1) = 0.0408.$$

Hence, our estimate of the necessary capacity, i.e. size, of the overflow group becomes

$$c = 17 - s = \underline{8}, \tag{1}$$

corresponding to an estimated loss on the overflow customers equal to

$$\Pi_c = \frac{B(17,11.1)}{B(9,11.1)} = \frac{0.0259}{0.323} = \underline{0.080}. \tag{2}$$

$\boxed{b}$ The estimate of the loss for the system as a whole is, by (7.14),

$$\Pi = \frac{a B(s+c,a)}{a_1 + a_2} = \frac{11.1\, B(17,11.1)}{10+5} = \frac{0.287}{15} = \underline{0.019}. \tag{3}$$

$\boxed{c}$ After increasing the loads on the primary groups by 50% the new offered loads will be $a_1^* = 15$ and $a_2^* = 7.5$. We shall estimate the losses on the overflow group as well as the system as a whole assuming the old server group sizes, $s_1 = 10$, $s_2 = 5$, and $c = 8$. To begin, the losses on the primary groups are, by Fig. A-1,

$$B(s_1,a_1^*) = B(10,15) = 0.410,$$
$$B(s_2,a_2^*) = B(5,7.5) = 0.453.$$

(Chap. 4, Ex. 16 c)

The calculation of an approximate equivalent random system proceeds along the same lines as in part (a). The results are as follows,

$$\alpha_1^* = 6.15, \qquad v_1^* = 11.23,$$
$$\alpha_2^* = 3.40, \qquad v_2^* = 5.26,$$

$$\alpha^* = \alpha_1^* + \alpha_2^* = 9.55,$$
$$v^* = v_1^* + v_2^* = 16.49,$$
$$z^* = v^*/\alpha^* = 1.73,$$

$$\tilde{a}^* = v^* + 3z^*(z^*-1) = 20.3,$$
$$\tilde{s}^* = \frac{\tilde{a}^*(\alpha^*+z^*)}{\alpha^*+z^*-1} - \alpha^* - 1 = 11.7,$$

$$s^* = [\tilde{s}^*] = 11,$$
$$a^* = \frac{(s^*+\alpha^*+1)(\alpha^*+z^*-1)}{\alpha^*+z^*} = 19.6.$$

Thus, $(s^*, a^*) = (11, 19.6)$ define the equivalent random system having the approximate overflow characteristics α^* and v^* of the total overflow from the two primary groups, after the 50% increase in loads.

The estimated loss on the overflow group is now calculated to, by the use of Figure A-2 in the Appendix,

$$\Pi_c^* = \frac{B(s^*+c, a^*)}{B(s^*, a^*)} = \frac{B(19, 19.6)}{B(11, 19.6)}$$
$$= \frac{0.179}{0.485}$$
$$= \underline{0.369}, \tag{4}$$

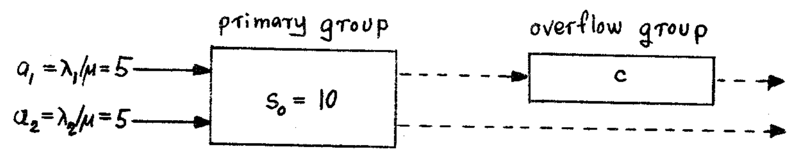
and the estimated loss for the system as a whole is

$$\Pi^* = \frac{a^* B(s^*+c, a^*)}{a_1^*+a_2^*} = \frac{19.6\, B(19, 19.6)}{15+7.5}$$
$$= \frac{3.51}{22.5}$$
$$= \underline{0.156}. \tag{5}$$

$\square$

Chapter 4, Exercise 17

'Poisson traffic totaling 10 erl is offered to a group of 10 servers.'



Total overflow

As in a similar case in Exercise 16, the total overflow from the primary group is characterized by the mean and the variance of the equilibrium state of a fictitious infinite-server back-up group, equal to

$$\alpha = (a_1 + a_2)\, B(s_0, a_1 + a_2) \qquad = 2.15,$$

$$\vartheta = \alpha\left(1 - \alpha + \frac{a_1 + a_2}{s_0 + 1 + \alpha - (a_1 + a_2)}\right) \qquad = 4.35.$$

High-priority overflow

By Exercise 12, mean and variance of the high-priority overflow stream are, respectively,

$$\alpha_1 = \frac{a_1}{a_1 + a_2}\,\alpha \; = \; a_1\, B(s_0, a_1 + a_2) \qquad = 1.07,$$

$$\vartheta_1 = \left(\frac{a_1}{a_1 + a_2}\right)^2 \vartheta + \frac{a_1}{a_1 + a_2}\left(1 - \frac{a_1}{a_1 + a_2}\right)\alpha \qquad = 1.62.$$

Equivalent random system

The decision on c will be based upon a calculation of an equivalent random system (s, a) whose overflow has approximately the mean and the variance of the overflow stream of high-priority customers from the primary group. First, we calculate

$$z_1 = \vartheta_1 / \alpha_1 = 1.51.$$

(Chap. 4, Ex. 17)

As a first approximation we calculate

$$\tilde{a} = \vartheta_1 + 3z_1(z_1 - 1) \qquad = 3.93 \qquad [\text{by } (7.15)],$$

$$\tilde{s} = \frac{\tilde{a}(\alpha_1 + z_1)}{\alpha_1 + z_1 - 1} - \alpha_1 - 1 = 4.35 \qquad [\text{by } (7.16)].$$

Hence, by (7.17),

$$a = \frac{([\tilde{s}] + \alpha_1 + 1)(\alpha_1 + z_1 - 1)}{\alpha_1 + z_1} = 3.72,$$

so that the equivalent random system is described by

$$(s, a) = (4, 3.72).$$

Calculation of c

Our estimate of the loss of high-priority customers on the system as a whole is

$$\Pi = \frac{a B(s+c, a)}{\alpha_1} = \frac{3.72\, B(4+c, 3.72)}{5}.$$

The smallest c meeting the requirement $\Pi < 0.01$ therefore must satisfy the inequalities

$$B(4+c, 3.72) < 0.0134 < B(4+c-1, 3.72)$$

By Figure A-1,

$$B(9, 3.72) = 0.0092, \qquad B(8, 3.72) = 0.0223.$$

It follows that the size of the overflow group should be

$$c = 9 - s = 5,$$

for which

$$\Pi = 0.0068.$$

□

Chapter 4, Exercise 18

'Consider the Erlang loss system with hyperexponential service times.'

Let j_1 be the number of customers whose service time is exponential with mean μ_1^{-1} (with probability p_1), and let j_2 be the number of customers whose service time is exponential with mean μ_2^{-1} (with probability p_2). Let $P(j_1, j_2)$ denote the equilibrium probability that the state of the system is (j_1, j_2). The conservation-of-flow equations are

$$(\lambda p_1 + \lambda p_2 + j_1\mu_1 + j_2\mu_2) P(j_1, j_2) = \lambda p_1 P(j_1-1, j_2) + \lambda p_2 P(j_1, j_2-1) + (j_1+1)\mu_1 P(j_1+1, j_2) + (j_2+1)\mu_2 P(j_1, j_2+1) \qquad (0 \le j_1+j_2 < s)$$

$$(j_1\mu_1 + j_2\mu_2) P(j_1, j_2) = \lambda p_1 P(j_1-1, j_2) + \lambda p_2 P(j_1, j_2-1) \qquad (j_1+j_2 = s).$$

From these equations we extract the following two sets of equations,

$$\lambda p_1 P(j_1, j_2) = (j_1+1)\mu_1 P(j_1+1, j_2) \qquad (0 \le j_1+j_2 < s),$$

$$\lambda p_2 P(j_1, j_2) = (j_2+1)\mu_2 P(j_1, j_2+1) \qquad (0 \le j_1+j_2 < s).$$

The solution of the above equations, which also is a solution of the equilibrium state equations, is

$$P(j_1, j_2) = \frac{(\lambda p_1/\mu_1)^{j_1}}{j_1!} \frac{(\lambda p_2/\mu_2)^{j_2}}{j_2!} c \qquad (0 \le j_1+j_2 \le s).$$

It follows that the equilibrium probability that altogether j $(= j_1+j_2)$ customers will be in service equals

$$P_j = \sum_{j_1+j_2=j} P(j_1, j_2) = \frac{[(\lambda p_1/\mu_1) + (\lambda p_2/\mu_2)]^j}{j!} c \qquad (0 \le j \le s).$$

Introducing the unconditional mean service time by

$$\frac{1}{\mu} = \frac{p_1}{\mu_1} + \frac{p_2}{\mu_2},$$

it is readily verified that

$$P_j = \frac{(\lambda/\mu)^j/j!}{\sum_{k=0}^{s} (\lambda/\mu)^k/k!} \qquad (0 \le j \le s).$$

$\square$

Chapter 4, Exercise 19

'Show that if a random variable X ...'

We need the fact that, if X_i is an exponentially distributed variable with parameter μ_i, that is $P\{X_i \le t\} = 1 - e^{-\mu_i t}$, then

$$E(X_i) = \mu_i^{-1},$$
$$E(X_i^2) = 2\mu_i^{-2},$$
$$V(X_i) = E(X_i^2) - E^2(X_i) = \mu_i^{-2}.$$

Case 1: X is a sum of independent, exponential variables

Clearly,

$$E(X) = \Sigma_i E(X_i) = \Sigma_i \mu_i^{-1},$$
$$V(X) = \Sigma_i V(X_i) = \Sigma_i \mu_i^{-2}.$$

Hence,

$$E^2(X) = (\Sigma_i \mu_i^{-1})^2 = \Sigma_i \mu_i^{-2} + \sum_i \sum_{j \ne i} \mu_i^{-1}\mu_j^{-1} > \Sigma_i \mu_i^{-2} = V(X)$$

Since $E(X) > 0$, we can conclude that

$$E(X) > \sqrt{V(X)}. \qquad (1)$$

Case 2: X is a mixture of independent, exponential variables

Clearly,

$$E(X) = \Sigma_i p_i E(X_i) = \Sigma_i p_i \mu_i^{-1},$$
$$E(X^2) = \Sigma_i p_i E(X_i^2) = 2\,\Sigma_i p_i \mu_i^{-2},$$
$$V(X) = E(X^2) - E^2(X) = 2\,\Sigma_i p_i \mu_i^{-2} - (\Sigma_i p_i \mu_i^{-1})^2.$$

By Schwarz' inequality, $(\Sigma_i a_i b_i)^2 < \Sigma_i a_i^2\, \Sigma_i b_i^2$. Hence,

$$(\Sigma_i p_i \mu_i^{-1})^2 = (\Sigma_i \sqrt{p_i}\sqrt{p_i}\mu_i^{-1})^2 < \Sigma_i p_i \mu_i^{-2},$$

whereby the inequality

$$V(X) - E^2(X) = 2\,[\Sigma_i p_i \mu_i^{-2} - (\Sigma_i p_i \mu_i^{-1})^2] > 0.$$

Thus, in this case,

$$E(X) < \sqrt{V(X)}. \qquad (2)$$

□

Chapter 5, Exercise 1

'(P. J. Burke [1968, unpublished].) Consider a birth-and-death process...'

$\boxed{a}$ Consider the Markov chain of states immediately following events, where an event is either an arrival (causing a change of state) or a departure. Denote by $E_{i_1}, E_{i_2}, E_{i_3}$ three such consecutive states in statistical equilibrium.

For each $j \geq 0$, by the Markov property,

$$\begin{aligned} P\{E_{i_2}=E_{j+1}, E_{i_3}=E_j\} &= P\{E_{i_1}=E_j \ , E_{i_2}=E_{j+1}, E_{i_3}=E_j\} \\ &\quad + P\{E_{i_1}=E_{j+2}, E_{i_2}=E_{j+1}, E_{i_3}=E_j\} \\ &= P\{E_{i_1}=E_j \ , E_{i_2}=E_{j+1}\} P\{E_{i_3}=E_j \mid E_{i_2}=E_{j+1}\} \\ &\quad + P\{E_{i_1}=E_{j+2}, E_{i_2}=E_{j+1}\} P\{E_{i_3}=E_j \mid E_{i_2}=E_{j+1}\}. \end{aligned}$$

Now,

$$\begin{aligned} P\{E_{i_2}=E_{j+1}, E_{i_3}=E_j\} &= \tfrac{1}{2}\pi_j^*, \\ P\{E_{i_1}=E_j \ , E_{i_2}=E_{j+1}\} &= \tfrac{1}{2}\pi_j, \\ P\{E_{i_1}=E_{j+2}, E_{i_2}=E_{j+1}\} &= \tfrac{1}{2}\pi_{j+1}^*. \end{aligned}$$

The first equation, for instance, holds because with probability 1/2 an event is a departure, and the conditional probability of departure state E_j, given a departure, equals π_j^*. Inserting these expressions and writing $P\{E_{i_3}=E_j \mid E_{i_2}=E_{j+1}\} = P\{E_{j+1} \to E_j\}$ we obtain, for each $j \geq 0$,

$$\pi_j^* = \pi_j P\{E_{j+1} \to E_j\} + \pi_{j+1}^* P\{E_{j+1} \to E_j\}. \qquad (*)$$

Clearly,

$$P\{E_{j+1} \to E_j\} = \frac{\mu_{j+1}}{\lambda_{j+1}+\mu_{j+1}},$$

and, by (3.3),

$$\pi_j^* = \pi_j$$

for $j = 0, 1, \ldots$. By substitution of these expressions into (*), and reduction, we find

$$\lambda_{j+1}\pi_j = \mu_{j+1}\pi_{j+1} \quad (j = 0, 1, \ldots). \qquad (1)$$

$\boxed{b}$ For a birth-and-death process with n sources, suppose the arrival rate in state j (E_j), $j = 0, 1, \ldots, n-1$, depends on only the difference $n-j$; i.e. $\lambda_j[n] = f(n-j)$, where $f(\cdot) > 0$ is any function. By (1), $\lambda_{j+1}[n]\Pi_j[n] = \mu_{j+1}\Pi_{j+1}[n]$ ($j = 0, 1, \ldots, n-2$), and as $\lambda_{j+1}[n] = f(n-j-1) = \lambda_j[n-1]$,

$$\lambda_j[n-1]\,\Pi_j[n] = \mu_{j+1}\Pi_{j+1}[n] \qquad (j = 0, 1, \ldots, n-2). \qquad (2)$$

By Eq. (3.15) of Chapter 2, the outside observer's distribution in a system with $n-1$ sources will satisfy

$$\lambda_j[n-1]\,P_j[n-1] = \mu_{j+1}P_{j+1}[n-1] \qquad (j = 0, 1, \ldots, n-2). \qquad (3)$$

A comparison of (2) and (3) leads to the conclusion that

$$\Pi_j[n] = P_j[n-1] \qquad (j = 0, 1, \ldots, n-1) \qquad (4)$$

for any finite-source birth-and-death process with $\lambda_j[n] = f(n-j)$ and $\mu_j > 0$ for $j > 0$.

Chapter 5, Exercise 2

'Burke's theorem'.

"For the M/M/s queue in equilibrium, the sequence of service completion epochs follows a Poisson process (with the same parameter as the input process); that is, the output process is statistically the same as the input process."

Let T_1 and T_2 be two arbitrary consecutive service completion epochs. Define $F_j(t)$ as the probability that simultaneously $T_2 > T_1 + t$ and the number of customers in system at $T_1 + t$ equals j. Let $\mu(j) = j\mu$ if $j \leq s$, $\mu(j) = s\mu$ if $j > s$.

$\boxed{a}$

$$F_0(t+h) = F_0(t)[1-\lambda h] + o(h),$$

$$F_j(t+h) = F_j(t)[1-(\lambda+\mu(j))h] + F_{j-1}(t)\lambda h + o(h) \qquad (j = 1, 2, \ldots).$$

(Chap. 5, Ex. 2 a)

Hence,

$$\frac{dF_0(t)}{dt} = -\lambda F_0(t),$$

$$\frac{dF_j(t)}{dt} = -(\lambda+\mu(j))F_j(t) + \lambda F_{j-1}(t) \qquad (j=1,2,\ldots).$$

with initial condition $F_j(0) = \pi_j^*$ for $j=0,1,2,\ldots$.

It is easily found that $F_0(t) = \pi_0^* e^{-\lambda t}$. We shall verify that the complete solution is

$$F_j(t) = \pi_j^* e^{-\lambda t} \qquad (t \geqq 0,\ j=0,1,2,\ldots). \tag{1}$$

(1) has been shown to produce the right answer for $j=0$, and it clearly satisfies the initial condition for $j=1,2,\ldots$. Thus it remains to demonstrate that the equation satisfies the differential-difference equations above for $j=1,2,\ldots$. Substitution of (1) into the appropriate differential-difference equation and some simple calculation and reduction yield

$$\lambda \pi_{j-1}^* = \mu(j)\pi_j^* \qquad (j=1,2,\ldots). \tag{2}$$

That this equation holds can be seen by making the substitution $\pi_j^* = \pi_j$, which results in a special case of Eq. (1) of Ex. (1), or making the substitution $\pi_j^* = P_j$, which results in the conservation-of-flow equation $\lambda P_{j-1} = \mu(j)P_j$, $j=1,2,\ldots$. We can therefore conclude that our Equation (1) indeed gives the desired probability $F_j(t)$, for all t and j.

$\boxed{b}$ Let $F(t)$ denote the probability that $T_2 > T_1 + t$; that is, $F(t) = P\{T_2 - T_1 > t\}$. By (1), and the definition of $F_j(t)$,

$$\begin{aligned} F(t) &= \sum_{j=0}^{\infty} F_j(t) \\ &= \sum_{j=0}^{\infty} \pi_j^* e^{-\lambda t} \\ &= e^{-\lambda t}. \end{aligned} \tag{3}$$

Thus, the time separating two successive departures is exponentially distributed with the same mean as the interarrival times.

(Chap. 5, Ex. 2c)

$\boxed{c}$ Let $x = T_2 - T_1$ and let $\hat{j}_2$ be the number of customers left behind by the departure at T_2. In equilibrium,

$$\begin{aligned} P\{\hat{j}_2 = j,\ x > t\} &= \int_t^\infty F_{j+1}(x)\,\mu(j+1)\,dx \\ &= \mu(j+1)\,\Pi_{j+1}^* \int_t^\infty e^{-\lambda x}\,dx \qquad [\text{by } (1)] \\ &= \frac{\mu(j+1)}{\lambda}\,\Pi_{j+1}^*\, e^{-\lambda t} \\ &= \Pi_j^*\, e^{-\lambda t}. \qquad [\text{by } (2)] \end{aligned}$$

Thus,

$$P\{\hat{j}_2 = j,\ x > t\} = P\{\hat{j}_2 = j\}\,P\{x > t\}, \tag{4}$$

which means that the length of the interdeparture interval x and the number of customers in the system at the start of the next interval are independent variables.

Denote by T_3 the departure epoch subsequent to T_2. By the Markov property

$$P\{T_2 - T_1 > t \mid \hat{j}_2 = j,\ T_3 - T_2 = s\} = P\{T_2 - T_1 > t \mid \hat{j}_2 = j\}. \tag{5}$$

By (5) $T_2 - T_1$ may depend on $T_3 - T_2$ only through $\hat{j}_2$. But, by (4), $T_2 - T_1$ is independent of $\hat{j}_2$. Hence, $T_2 - T_1$ is independent of $T_3 - T_2$. An extension of this argument leads to the conclusion that all the interdeparture interval lengths are independent variables

Remark. Nowhere has the particular form of the function $\mu(j)$ been used. It is worth noting that the whole line of proof applies to any birth-and-death process with birth rate $\lambda_j = \lambda$ for all $j \geq 0$ and death rates $\mu_0 = 0$ and $\mu_j > 0$ for $j \geq 1$, where the λ and μ_j's only meet the condition for the existence of an equilibrium distribution. That is, the output process is a Poisson process with rate λ also in the general case, not just for an M/M/s queue.

$\square$

Chapter 5, Exercise 3

'Solve Exercise 18 of Chapter 2 by evaluating the integral...'

$$F_{I_t}(y) = \int F_{R_t}(y-x)\,dF_{A_t}(x).$$

It is understood that $F_{R_t}(z) = P\{R_t \leq z\}$, $F_{A_t}(z) = P\{A_t \leq z\}$ and $F_{I_t}(z) = P\{I_t \leq z\}$. The random variables R_t and A_t are forward and backward recurrence time at t, respectively; R_t and A_t are independent variables; and $I_t = R_t + A_t$. Recall that

$$F_{R_t}(z) = 1 - e^{-\lambda z} \quad (z \geq 0), \qquad \text{[by Eq. (5.30), Chap. 2]}$$

$$F_{A_t}(z) = \begin{cases} 1 - e^{-\lambda z} & (0 \leq z < t), \\ 1 & (z \geq t). \end{cases} \qquad \text{[by Eq. (5.33), Chap. 2]}$$

Hence,
$$F_{I_t}(y) = \int_{x=0}^{y} F_{R_t}(y-x)\,dF_{A_t}(x) = \int_{x=0}^{y} [1-e^{-\lambda(y-x)}]\,dF_{A_t}(x).$$

<u>$y < t$:</u>
$$F_{I_t}(y) = \int_0^y [1-e^{-\lambda(y-x)}]\,\lambda e^{-\lambda x}dx = 1 - e^{-\lambda y} - \lambda y e^{-\lambda y}.$$

<u>$y \geq t$:</u>
$$F_{I_t}(y) = \int_0^t [1-e^{-\lambda(y-x)}]\,\lambda e^{-\lambda x}dx + [1-e^{-\lambda(y-t)}]\,e^{-\lambda t} = 1 - e^{-\lambda y} - \lambda t e^{-\lambda y}.$$

Thus, for all y,
$$F_{I_t}(y) = 1 - e^{-\lambda y} - \lambda \min(y,t) e^{-\lambda y}.$$

Chapter 5, Exercise 4

'Let N be a nonnegative, integer-valued random variable ...'

$$\phi(s) = \int_0^\infty e^{-st}dF(t) = \sum_{j=0}^{\infty} e^{-sj} P\{N=j\} = \sum_{j=0}^{\infty} P\{N=j\}(e^{-s})^j = g(e^{-s}).$$

□

Chapter 5, Exercise 5

'Consider again the premise of Exercise 4 of Chapter 2, but...'

X_i, $i=1,2,\ldots$, is a sequence of independent, identically distributed nonnegative random variables with distribution function $F(t) = P\{X \le t\}$ and Laplace-Stieltjes transform $\phi(s) = \int_0^\infty e^{-st} dF(t)$. N is a nonnegative, integer-valued random variable with generating function $g(z) = \sum_{n=0}^\infty P\{N=n\} z^n$. $\{X_i\}$ and N are independent.

[a] Let $S_0 = 0$ if $N=0$, and $S_N = \sum_{i=1}^N X_i$ if $N \ge 1$. Clearly,

$$P\{S_N \le t\} = \sum_{n=0}^\infty P\{N=n\} P\{S_n \le t\} = \sum_{n=0}^\infty P\{N=n\} F_n(t),$$

where $F_0(t) \equiv 1$, and $F_n(t)$, $n \ge 1$, is the n-fold convolution of $F(t)$. Thus, letting $\psi(s)$ denote the Laplace-Stieltjes transform of S_N,

$$\begin{aligned}\psi(s) &= \int_0^\infty e^{-st} d P\{S_N \le t\}\\ &= \sum_{n=0}^\infty P\{N=n\} \int_0^\infty e^{-st} d F_n(t)\\ &= \sum_{n=0}^\infty P\{N=n\} [\phi(s)]^n \qquad \text{[by (6.5)]}\\ &= g(\phi(s)).\end{aligned}$$

[b] By differentiating $\psi(s)$ twice we obtain

$$\psi'(s) = g'(\phi(s))\,\phi'(s),$$
$$\psi''(s) = g'(\phi(s))\,\phi''(s) + g''(\phi(s))[\phi'(s)]^2.$$

As $\phi(0)=1$,

$$\psi'(0) = g'(1)\,\phi'(0),$$
$$\psi''(0) = g'(1)\,\phi''(0) + g''(1)[\phi'(0)]^2.$$

Now, $g'(1) = E(N)$, $g''(1) = E(N^2) - E(N)$, $\phi'(0) = -E(X)$, $\phi''(0) = E(X^2)$. Hence,

$$\psi'(0) = -E(N)E(X), \qquad (1)$$

and $\psi''(0) = E(N)E(X^2) + [E(N^2) - E(N)]E^2(X)$, whereby

$$\psi''(0) = E(N)V(X) + E(N^2)E^2(X). \qquad (2)$$

(Chap. 5, Ex. 5)

Mean and variance of S_N may be derived from the Laplace-Stieltjes transform as follows

$$E(S_N) = -\psi'(0),$$

$$V(S_N) = E(S_N^2) - E^2(S_N) = \psi''(0) - [\psi'(0)]^2.$$

By (1) and (2), then,

$$E(S_N) = E(N)\,E(X),$$

$$V(S_N) = E(N)\,V(X) + V(N)\,E^2(X).$$

Chapter 5, Exercise 6

'We shall show in Section 5.8 that...'

"W(t) in the M/G/1 queue with service in order of arrival has Laplace-Stieltjes transform $\omega(s) = \int_0^\infty e^{-st}dW(t)$ given by

$$\omega(s) = \frac{s(1-\rho)}{s-\lambda[1-\eta(s)]}, \qquad (1)$$

where $\eta(s) = \int_0^\infty e^{-st}dH(t)$ is the Laplace-Stieltjes transform of the service-time distribution function $H(t)$, with mean $\tau = \int_0^\infty t\,dH(t)$, and $\rho = \lambda\tau < 1$, where λ is the arrival rate."

$\boxed{a}$ We shall determine the mean wait from the relation $E(W) = -\omega'(0)$. Differentiation of (1) results in

$$\omega'(s) = -\frac{f(s)}{g(s)},$$

where

$$f(s) = \lambda(1-\rho)[1-\eta(s) + s\eta'(s)],$$

$$g(s) = (s-\lambda[1-\eta(s)])^2.$$

Observe that $f(0) = 0$ and $g(0) = 0$. However, $\omega'(0) = \lim_{s\to 0}\omega'(s)$ can be evaluated by a double application of l'Hospitals rule.

(Chap. 5, Ex. 6a)

First we calculate

$$f'(s) = \lambda(1-\rho)s\,\eta''(s),$$
$$g'(s) = 2(1+\lambda\eta'(s))(s-\lambda[1-\eta(s)]),$$

and, since $f'(0)=0$ and $g'(0)=0$, we differentiate again, obtaining

$$f''(s) = \lambda(1-\rho)\eta''(s) + \lambda(1-\rho)s\,\eta'''(s),$$
$$g''(s) = 2(1+\lambda\eta'(s))^2 + 2\lambda\eta''(s)(s-\lambda[1-\eta(s)]).$$

Hence,

$$f''(0) = \lambda(1-\rho)(\tau^2+\sigma^2),$$
$$g''(0) = 2(1-\rho)^2,$$

where σ^2 is the service-time variance, and we have used that $\lambda\eta'(0) = \lambda(-\tau) = -\rho$. Finally, from $E(W) = -\omega'(0) = f(0)/g(0) = f''(0)/g''(0)$,

$$E(W) = \frac{\rho\tau}{2(1-\rho)}\left(1+\frac{\sigma^2}{\tau^2}\right). \tag{2}$$

$\boxed{b}$ When service-times are exponentially distributed with mean μ^{-1}, then the waiting-time distribution function $W(t)$ is

$$W(t) = \begin{cases} 0 & (t<0), \\ 1-\rho e^{-(1-\rho)\mu t} & (t \geq 0). \end{cases} \tag{3}$$

Thus,

$$\begin{aligned}\omega(s) &= \int_{0-}^{\infty} e^{-st}\,dW(t) \\ &= e^{-s0}P\{W=0\} + \int_{0+}^{\infty} e^{-st}\,dW(t) \\ &= (1-\rho) + (1-\rho)\lambda\int_0^{\infty} e^{-(s+\mu-\lambda)t}\,dt \\ &= (1-\rho)\frac{s+\mu}{s+\mu-\lambda}.\end{aligned}$$

This result is in agreement with Equation (1), since in the case of an exponential service time distribution, we have $\eta(s) = \mu/(\mu+s)$, so that (1) becomes

$$\omega(s) = \frac{s(1-\rho)}{s-\lambda[1-\mu/(\mu+s)]} = (1-\rho)\frac{s+\mu}{s+\mu-\lambda}.$$

$\square$

Chapter 5, Exercise 7

'Show that if $G(\xi) = 1 - e^{-\lambda\xi}$ in (7.9), then $F(x) = 1 - e^{-\lambda x}$;...'

Since the interevent times have the exponential distribution with parameter λ, $G(\xi) = 1 - e^{-\lambda\xi}$, the mean of the interevent interval is

$$\beta = \int_0^\infty x\,dG(x) = \lambda^{-1}.$$

By (7.9) the equilibrium forward recurrence time has the distribution

$$\begin{aligned} F(x) &= \frac{1}{\beta}\int_0^x [1 - G(\xi)]\,d\xi \\ &= \lambda \int_0^x e^{-\lambda\xi}\,d\xi \\ &= 1 - e^{-\lambda x}, \end{aligned}$$

which is the same as the distribution of the interevent times.

Chapter 5, Exercise 8

'Verify Equation (7.13).'

The equilibrium forward recurrence distribution $F(x)$ is given by (7.9) and has the Laplace-Stieltjes transform

$$\phi(s) = \frac{1}{\beta}\,\frac{1-\gamma(s)}{s}. \tag{7.10}$$

Hence,

$$\phi'(s) = -\frac{1}{\beta}\,\frac{s\gamma'(s) + 1 - \gamma(s)}{s^2}.$$

Since both numerator and denominator equal zero for $s = 0$, $\phi'(0)$ is evaluated by L'Hospital's rule.

$$\phi'(0) = -\frac{1}{\beta}\lim_{s\to 0}\frac{s\gamma''(s) + \gamma'(s) - \gamma'(s)}{2s} = -\frac{1}{\beta}\,\frac{\gamma''(0)}{2} = -\frac{1}{\beta}\,\frac{\beta^2+\sigma^2}{2}.$$

As $\beta^* = \int_0^\infty x\,dF(x)$ is determined by $\beta^* = -\phi'(0)$, we have

$$\beta^* = \frac{\beta}{2} + \frac{\sigma^2}{2\beta}. \tag{7.13}$$

□

Chapter 5, Exercise 9

'a. Show that...'

$\boxed{a}$ Define $R_t = T_{j+1} - t$; $I_t = T_{j+1} - T_j$, where $T_j \leq t < T_{j+1}$ for some j. Assume $0 \leq x \leq y < t$.
For given j ($j = 1, 2, \ldots$) and t we have

$$P\{T_j \leq t < T_{j+1}, R_t \leq x, I_t \leq y\} = \int_{t-y}^{t-y+x} [G(y) - G(t-\xi)]\, dP\{T_j \leq \xi\}$$

$$+ \int_{(t-y+x)+}^{t} [G(t-\xi+x) - G(t-\xi)]\, dP\{T_j \leq \xi\}$$

The formula is a simple consequence of the following observations: (i) If $T_j \leq t-y$ or $T_j > t$, then the event $\{T_j \leq t < T_{j+1}, R_t \leq x, I_t \leq y\}$ cannot occur; (ii) If $t-y < T_j \leq t-y+x$, then the event will occur if and only if $t - T_j < I_t \leq y$; (iii) If $t-y+x < T_j \leq t$, then the event will occur if and only if $t - T_j < I_t \leq t - T_j + x$. For $j=0$ we have $T_0 = 0 < t-y$, so that, by (i), the probability of the event is zero.
Obviously, $P\{R_t \leq x, I_t \leq y\} = \sum_{j=0}^{\infty} P\{T_j \leq t < T_{j+1}, R_t \leq x, I_t \leq y\}$. Thus

$$P\{R_t \leq x, I_t \leq y\} = \sum_{j=1}^{\infty} \int_{t-y}^{t-y+x} [G(y) - G(t-\xi)]\, dP\{T_j \leq \xi\}$$

$$+ \sum_{j=1}^{\infty} \int_{(t-y+x)+}^{t} [G(t-\xi+x) - G(t-\xi)]\, dP\{T_j \leq \xi\}$$

$$(0 \leq x \leq y < t)$$

$\boxed{b}$ Since $m(\xi) = \sum_{j=1}^{\infty} P\{T_j \leq \xi\}$,

$$P\{R_t \leq x, I_t \leq y\} = \int_{t-y}^{t-y+x} [G(y) - G(t-\xi)]\, dm(\xi) + \int_{(t-y+x)+}^{t} [G(t-\xi+x) - G(t-\xi)]\, dm(\xi)$$

Letting $t \to \infty$ and using $\lim_{\xi\to\infty} \frac{dm(\xi)}{d\xi} = \frac{1}{\beta}$, we obtain

$$\lim_{t\to\infty} P\{R_t \leq x, I_t \leq y\} = \frac{1}{\beta} \int_{t-y}^{t-y+x} [G(y) - G(t-\xi)]\, d\xi$$

$$+ \frac{1}{\beta} \int_{(t-y+x)+}^{t} [G(t-\xi+x) - G(t-\xi)]\, d\xi .$$

$\square$

(Chap. 5, Ex. 9 b)

By the substitution $t-\xi \to \xi$,

$$\lim_{t\to\infty} P\{R_t \leq x, I_t \leq y\} = \frac{1}{\beta}\int_{y-x}^{y} [G(y)-G(\xi)]\,d\xi + \frac{1}{\beta}\int_0^{y-x} [G(\xi+x)-G(\xi)]\,d\xi,$$

which by further rewriting becomes

$$\lim_{t\to\infty} P\{R_t \leq x, I_t \leq y\} = \frac{1}{\beta}\int_0^{x} G(y)\,d\xi - \frac{1}{\beta}\int_{y-x}^{y} G(\xi)\,d\xi$$
$$+ \frac{1}{\beta}\int_x^{y} G(\xi)\,d\xi - \frac{1}{\beta}\int_0^{y-x} G(\xi)\,d\xi .$$

Hence,

$$\lim_{t\to\infty} P\{R_t \leq x, I_t \leq y\} = \frac{1}{\beta}\int_0^{x} [G(y)-G(\xi)]\,d\xi \quad (0 \leq x \leq y). \qquad (7.20)$$

$\boxed{c}$ By setting $x = y$ in (7.20) we find

$$\lim_{t\to\infty} P\{I_t \leq y\} = \frac{1}{\beta}\int_0^{y}[G(y)-G(\xi)]\,d\xi$$
$$= \frac{1}{\beta}[G(y)-G(\xi)]\,\xi \Big|_0^y$$
$$- \frac{1}{\beta}\int_0^{y} \xi\, d(G(y)-G(\xi))$$

whereby

$$\lim_{t\to\infty} P\{I_t \leq y\} = \frac{1}{\beta}\int_0^{y} \xi\, dG(\xi) .$$

By differentiation with respect to y and the subsequent substitution $y = x$, we derive

$$\lim_{t\to\infty} dP\{I_t \leq x\} = \frac{1}{\beta}\, x\, dG(x) \qquad (7.19)$$

which gives the equilibrium probability density of the covering interval.

It is also worth noting that by setting $y = \infty$ in (7.20) we find again

$$F(x) = \lim_{t\to\infty} P\{R_t \leq x\} = \frac{1}{\beta}\int_0^{x} [1-G(\xi)]\,d\xi \qquad (7.9)$$

(Chap. 5, Ex. 9 d)

$\boxed{d}$ We shall prove

$$\lim_{t\to\infty} P\{R_t \leq x \mid I_t = y\} = \frac{x}{y} \qquad (0 \leq x \leq y). \qquad (7.21)$$

First (7.21) is proven under the assumption that $G(\xi)$ has a discontinuity point at $\xi = y$.

Obviously, in this case,

$$\lim_{t\to\infty} P\{R_t \leq x \mid I_t = y\} = \frac{\lim_{t\to\infty} P\{R_t \leq x, I_t = y\}}{\lim_{t\to\infty} P\{I_t = y\}}. \qquad (*)$$

By (7.20),

$$\lim_{t\to\infty} P\{R_t \leq x, I_t = y\} = \frac{1}{\beta}\int_0^x dG(y)\, d\xi = \frac{x\, dG(y)}{\beta},$$

and setting $x = y$ in the above equation we derive

$$\lim_{t\to\infty} P\{I_t = y\} = \frac{y\, dG(y)}{\beta}.$$

Substitution of the last two expressions into $(*)$ proves (7.21) in the discontinuous case.

Next we prove (7.21) under the assumption that $G(\xi)$ is continuous and differentiable at $\xi = y$. Then

$$\lim_{t\to\infty} P\{R_t \leq x \mid I_t = y\} = \lim_{t\to\infty} P\{R_t \leq x \mid y \leq I_t < y + dy\}$$

$$= \frac{\lim_{t\to\infty} P\{R_t \leq x, y \leq I_t < y + dy\}}{\lim_{t\to\infty} P\{y \leq I_t < y + dy\}} \qquad (**)$$

By (7.20),

$$\lim_{t\to\infty} P\{R_t \leq x, y \leq I_t < y + dy\} = \left(\frac{1}{\beta}\int_0^x \frac{dG(y)}{dy}\, d\xi\right) dy = \frac{x}{\beta}\,\frac{dG(y)}{dy}\, dy,$$

and by setting $x = y$ in this equation we find

$$\lim_{t\to\infty} P\{y \leq I_t < y + dy\} = \frac{y}{\beta}\,\frac{dG(y)}{dy}\, dy.$$

Substitution into $(**)$ once more results in (7.21).

(Chap. 5, Ex. 9e)

$\boxed{e}$ The equivalence of (7.20) and (7.22) is established by the following sequence of pairwise equivalent equations leading from (7.20) to (7.22):

$$\lim_{t\to\infty} P\{R_t \le x, I_t \le y\} = \frac{1}{\beta}\int_0^x [G(y)-G(\xi)]\,d\xi \qquad (0\le x\le y) \qquad (7.20)$$

$$\lim_{t\to\infty} d\,P\{R_t \le x, I_t \le y\} = \frac{1}{\beta}\,d\,G(y)\,dx \qquad (0\le x\le y)$$

$$\lim_{t\to\infty} d\,P\{R_t \le x, A_t \le y\} = \frac{1}{\beta}\,d\,G(x+y)\,dx$$

$$\lim_{t\to\infty} P\{R_t > x, A_t > y\} = \int_x^\infty \int_{y+}^\infty \frac{1}{\beta}\,d\,G(\xi+\eta)\,d\xi$$

$$\lim_{t\to\infty} P\{R_t > x, A_t > y\} = \frac{1}{\beta}\int_x^\infty [1-G(\xi+y)]\,d\xi$$

$$\lim_{t\to\infty} P\{R_t > x, A_t > y\} = \frac{1}{\beta}\int_{x+y}^\infty [1-G(\xi)]\,d\xi \qquad (7.22)$$

$\boxed{f}$ Observe that, by (7.22), the probability that either $R_t = 0$ or $A_t = 0$ is zero, since $\lim_{t\to\infty} P\{R_t > 0, A_t > 0\} = \frac{1}{\beta}\int_0^\infty [1-G(\xi)]\,d\xi = 1$.

Suppose now the renewal process is a Poisson process, that is, $G(\xi) = 1 - e^{-\beta^{-1}\xi}$. Substituting $G(\xi)$ in (7.22) it is found that

$$\lim_{t\to\infty} P\{R_t > x, A_t > y\} = e^{-\beta^{-1}(x+y)} \qquad (*)$$

Now,

$$\lim_{t\to\infty} P\{R_t > x\} = \lim_{t\to\infty} P\{R_t > x, A_t \ge 0\}$$
$$= \lim_{t\to\infty} P\{R_t > x, A_t > 0\}.$$

By (*),

$$\lim_{t\to\infty} P\{R_t > x\} = e^{-\beta^{-1}x}.$$

Similarly,

$$\lim_{t\to\infty} P\{A_t > y\} = e^{-\beta^{-1}y}.$$

As $\lim_{t\to\infty} P\{R_t > x, A_t > y\} = \lim_{t\to\infty} P\{R_t > x\} \lim_{t\to\infty} P\{A_t > y\}$, the conclusion is that R_t and A_t are, in the limit, independent exponential variables. □

Chapter 5, Exercise 10

'Customers arrive at a single server...'

Blocked customers cleared.
$G(t)$ = interarrival time distribution function
$H(t)$ = service-time distribution function
$F(x)$ = cycle-time distribution function

$\boxed{a}$
$$F(x) = \int_0^x P\{R_t \leq x-t\}\, dH(t),$$
where $t=0$ is the time service starts. By Eq. (7.14),
$$F(x) = \int_0^x \Big(\sum_{j=1}^{\infty} \int_t^x [1-G(x-y)]\, dG^{*j}(y)\Big)\, dH(t).$$
Interchanging the order of integration and summation we find
$$F(x) = \sum_{j=1}^{\infty} \int_0^x \int_t^x [1-G(x-y)]\, dG^{*j}(y)\, dH(t). \qquad (1)$$
Henceforth we assume $H(t) = 1 - e^{-\mu t}$ $(t \geq 0)$.

$\boxed{b}$ Substitution of $dH(t) = \mu e^{-\mu t} dt$ and change of the order of integration give
$$F(x) = \sum_{j=1}^{\infty} \int_{y=0}^x [1-G(x-y)] \int_{t=0}^y \mu e^{-\mu t} dt\, dG^{*j}(y).$$
Hence,
$$F(x) = \sum_{j=1}^{\infty} \int_0^x [1-G(x-y)][1-e^{-\mu y}]\, dG^{*j}(y).$$
By differentiation w.r.t. x we find
$$dF(x) = \sum_{j=1}^{\infty} [1-G(0)][1-e^{-\mu x}]\, dG^{*j}(x) - \sum_{j=1}^{\infty} \int_0^x dG(x-y)[1-e^{-\mu y}]\, dG^{*j}(y).$$
With $G(0)=0$, this expression can be written
$$dF(x) = \sum_{j=1}^{\infty} dG^{*j}(x) - \sum_{j=1}^{\infty} e^{-\mu x} dG^{*j}(x) - \sum_{j=1}^{\infty} dG^{*(j+1)}(x) + \sum_{j=1}^{\infty} d\tilde{G}_j(x),$$
where
$$d\tilde{G}_j(x) = \int_0^x dG(x-y)\cdot e^{-\mu y}\, dG^{*j}(y).$$

(Chap. 5, Ex. 10 b)

Hence,

$$\begin{aligned}\phi(s) &= \int_0^\infty e^{-sx}\,dF(x)\\ &= \sum_{j=1}^\infty \int_0^\infty e^{-sx}\,dG^{*j}(x) - \sum_{j=1}^\infty \int_0^\infty e^{-(s+\mu)x}\,dG^{*j}(x)\\ &\quad - \sum_{j=1}^\infty \int_0^\infty e^{-sx}\,dG^{*(j+1)}(x) + \sum_{j=1}^\infty \int_0^\infty e^{-sx}\,d\tilde{G}_j(x)\end{aligned}$$

Introducing $\gamma(s) = \int_0^\infty e^{-st}\,dG(t)$ we derive

$$\phi(s) = \sum_{j=1}^\infty [\gamma(s)]^j - \sum_{j=1}^\infty [\gamma(s+\mu)]^j - \sum_{j=1}^\infty [\gamma(s)]^{j+1} + \sum_{j=1}^\infty \gamma(s)[\gamma(s+\mu)]^j,$$

which reduces to

$$\phi(s) = \frac{\gamma(s)-\gamma(s+\mu)}{1-\gamma(s+\mu)}. \tag{2}$$

c
$$\phi'(s) = \frac{\gamma'(s)}{1-\gamma(s+\mu)} - (1-\gamma(s))\frac{\gamma'(s+\mu)}{(1-\gamma(s+\mu))^2}.$$

The mean cycle time α is given by $\alpha = -\phi'(0)$. Thus,

$$\alpha = \frac{-\gamma'(0)}{1-\gamma(\mu)}. \tag{3}$$

d Evidently, the equilibrium probability P that the server is busy equals the ratio of mean service-time to mean cycle-time. That is $P = \mu^{-1}/\alpha$. By (3),

$$P = [-\gamma'(0)\mu]^{-1}[1-\gamma(\mu)]. \tag{4}$$

e By Eq. (1.4) of Chapter 2, the blocking probability is $\Pi = E(N)/(1+E(N))$, where N is the number of arrivals during a random service initiated by an arrival at an idle server. N is the number of failures in a sequence of Bernoulli trials. A failure is the occurrence of another arrival before service completion and has probability $q = 1-p = \int_0^\infty e^{-\mu t}\,dG(t) = \gamma(\mu)$. Thus N has the geometric distribution and $E(N) = q/p = \gamma(\mu)/(1-\gamma(\mu))$. Hence,

$$\Pi = \gamma(\mu). \tag{5}$$

(Chap. 5, Ex. 10 f)

$\boxed{f}$ Equations (4) and (5) imply

$$P = [-\gamma'(0)\mu]^{-1}[1-\Pi]. \qquad (6)$$

It is easy to see that $P\,(=P_1)$ equals the carried load. Now, $-\gamma'(0)$ is the mean interarrival time. Hence, letting λ = mean arrival rate, $\lambda = [-\gamma'(0)]^{-1}$. Thus, $P = \frac{\lambda}{\mu}(1-\Pi)$. That is, carried load (P) equals offered load ($\frac{\lambda}{\mu}$) times acceptance probability $(1-\Pi)$.

$\boxed{g}$ Suppose $G(t) = 1 - e^{-\lambda t}$ Then $\gamma(s) = \lambda/(\lambda+s)$.
By (5) and (6),

$$\Pi = \frac{\lambda}{\lambda+\mu},$$

$$P = \frac{\lambda}{\mu}(1-\Pi) = \frac{\lambda}{\mu}\left(1-\frac{\lambda}{\lambda+\mu}\right) = \frac{\lambda}{\lambda+\mu}.$$

In this case, then, (i) $P = \Pi$, as anticipated.
By (2),

$$\phi(s) = \frac{\gamma(s)-\gamma(s+\mu)}{1-\gamma(s+\mu)} = \frac{\frac{\lambda}{\lambda+s} - \frac{\lambda}{\lambda+s+\mu}}{1-\frac{\lambda}{\lambda+s+\mu}} = \frac{\lambda}{\lambda+s}\cdot\frac{\mu}{\mu+s}.$$

Since $\phi(s)$ is the product of two Laplace transforms of exponential distributions we can conclude that statement (ii) is true.

$\boxed{h}$ Suppose interarrival times are of constant length τ. Then

$$F(x) = 1 - e^{-\mu j\tau} \qquad (j\tau \leq x < (j+1)\tau;\ j = 0,1,2,\dots), \qquad (7)$$

and

$$\phi(s) = \int_0^\infty e^{-sx}dF(x) = \sum_{j=1}^\infty e^{-sj\tau}[e^{-\mu(j-1)\tau} - e^{-\mu j\tau}]$$
$$= (e^{\mu\tau}-1)\sum_{j=1}^\infty [e^{-(s+\mu)\tau}]^j = \frac{e^{-s\tau} - e^{-(s+\mu)\tau}}{1 - e^{-(s+\mu)\tau}}.$$

We would get the same result from an application of Eq. (2). In the present case, $\gamma(s) = \int_0^\infty e^{-st}dG(t) = e^{-s\tau}$. Substitution into (2) yields the formula above.

$\square$

Chapter 5, Exercise 11

'Customers arrive according to a renewal process...'

$\boxed{a}$ Suppose the arrival stream at a server is a renewal process, and that service times are exponential, and blocked customers are cleared. Let $\{T_j\}$ $(j=1,2,\ldots)$ denote the resultant sequence of overflow epochs.

$T_{j+1}-T_j$ is completely determined by (i) remaining service time at T_j, (ii) the sequence of future service times, (iii) the sequence of future interarrival intervals. All these variables are independent of the process up until T_j, and also independent of j. It follows that the sequence of interevent intervals $\{T_{j+1}-T_j\}$ $(j=1,2,\ldots)$ are independent, identically distributed random variables. That is, the overflow stream is a renewal process.

We conclude that, under the assumptions of this exercise, the overflow stream from the i'th ordered server $(i=1,2,\ldots)$ is a renewal process.

$\boxed{b}$ Let $G_i(t)$ be the distribution function of times between successive overflows from the i'th server. Choose an arbitrary overflow epoch of the i'th server. Let X be the time until next arrival (i.e. overflow from the (i-1)th server), and let Y be the time until next overflow from the i'th server. Then, for $x \leq t$,

$$P\{Y \leq t \mid X=x\} = e^{-\mu x} + (1-e^{-\mu x})G_i(t-x),$$

with $G_0(0)=0$, which implies $G_i(0)=0$. This is so, since for $X=x$ the event $Y \leq t$ will occur if (i) service is completed after next arrival (x time units later), having probability $e^{-\mu x}$, or if (ii) service is completed before next arrival and an overflow from the i'th server takes place during the time interval $(x,t]$, having probability $(1-e^{-\mu x})G_i(t-x)$. Observe that the time from start of service until next overflow from the i'th server has the same distribution as the inter-overflow times of the server.

Clearly,

$$P\{Y \leq t\} = \int_0^t P\{Y \leq t \mid X=x\}\, dP\{X \leq x\}.$$

(Chap. 5, Ex. 11 b)

Now, $P\{Y \leqq t\} = G_i(t)$ and $P\{X \leqq x\} = G_{i-1}(x)$. Hence,

$$G_i(t) = \int_0^t \left[e^{-\mu x} + (1 - e^{-\mu x})\, G_i(t-x)\right] dG_{i-1}(x) \qquad (i = 1, 2, \dots) \qquad (1)$$

as asserted.

$\boxed{c}$ Differentiation of Eq. (1) leads to

$$dG_i(t) = e^{-\mu t} dG_{i-1}(t) + \int_0^t (1 - e^{-\mu x}) dG_i(t-x)\, dG_{i-1}(x).$$

It follows that

$$\begin{aligned}
\gamma_i(s) &= \int_0^\infty e^{-st} dG_i(t) \\
&= \int_0^\infty e^{-(s+\mu)t} dG_{i-1}(t) \\
&\quad + \int_0^\infty e^{-st} \int_0^t dG_i(t-x) dG_{i-1}(x) \\
&\quad - \int_0^\infty e^{-st} \int_0^t dG_i(t-x)\, e^{-\mu x} dG_{i-1}(x) \\
&= \int_0^\infty e^{-(s+\mu)t} dG_{i-1}(t) \\
&\quad + \int_0^\infty e^{-st} dG_i(t) \cdot \int_0^\infty e^{-st} dG_{i-1}(t) \\
&\quad - \int_0^\infty e^{-st} dG_i(t) \cdot \int_0^\infty e^{-st} e^{-\mu t} dG_{i-1}(t)
\end{aligned}$$

Hence,

$$\gamma_i(s) = \gamma_{i-1}(s+\mu) + \gamma_i(s)\gamma_{i-1}(s) - \gamma_i(s)\gamma_{i-1}(s+\mu),$$

from which is obtained the recurrence equation

$$\gamma_i(s) = \frac{\gamma_{i-1}(s+\mu)}{1 - \gamma_{i-1}(s) + \gamma_{i-1}(s+\mu)}. \qquad (i = 1, 2, \dots) \qquad (2)$$

$\square$

Chapter 5, Exercise 12

'The M/G/1 queue with server vacation times.'

Let $P(j)$ be the probability that j customers arrive during a single, arbitrary vacation, and define

$$f(z) = \sum_{j=0}^{\infty} P(j)\, z^j .$$

Let X denote the number of customers ($X \geq 1$) who arrive during the vacation(s). Clearly, $P\{X=0\}=0$ and, for $j \geq 1$, $P\{X=j\} = P(j)/[1-P(0)]$. Letting $\hat{f}(z) = \sum_{j=0}^{\infty} P\{X=j\}\, z^j$ be the probability generating function for X, we derive

$$\hat{f}(z) = \frac{f(z) - P(0)}{1 - P(0)} . \qquad (*)$$

$\boxed{a}$ Proceeding as in the analysis of the M/G/1 queue without vacation, we find the following system of equations:

$$\hat{\Pi}_j^* = r_{j+1} \hat{\Pi}_0^* + \sum_{i=1}^{j+1} p_{j-i+1} \hat{\Pi}_i^* \quad (j = 0, 1, \ldots) . \qquad (8.20)'$$

Here, $r_{j+1} = P\{X+Y = j+1\}$, where X is the number of customers arriving during the vacation period and Y is the number of customers arriving during the initial service after vacation(s).

Consider for a moment the p.g.f. of $X+Y$. Since X and Y are independent variables

$$\sum_{j=0}^{\infty} r_j z^j = \hat{f}(z) \cdot h(z) , \qquad (**)$$

with $h(z) = \sum_{j=0}^{\infty} p_j z^j$ being the p.g.f. of the number of arrivals during an arbitrary service time. Note, $r_0 = 0$, as $X \geq 1$.

Substitution of Eq. (8.20)′ into the generating function

$$\hat{g}(z) = \sum_{j=0}^{\infty} \hat{\Pi}_j^* z^j$$

results in

$$\hat{g}(z) = \hat{\Pi}_0^* \sum_{j=0}^{\infty} r_{j+1} z^j + \sum_{j=0}^{\infty} \sum_{i=1}^{j+1} p_{j-i+1} \hat{\Pi}_i^* z^j . \qquad (8.23)'$$

(Chap. 5, Ex. 12 a)

Now, by (**) and the fact that $r_0 = 0$,

$$\sum_{j=0}^{\infty} r_{j+1} z^j = z^{-1} \sum_{j=0}^{\infty} r_j z^j = z^{-1} \hat{f}(z) h(z),$$

and, furthermore,

$$\sum_{j=0}^{\infty} \sum_{i=1}^{j+1} P_{j-i+1} \hat{\pi}_i^* z^j = z^{-1} (\hat{g}(z) - \hat{\pi}_0^*) h(z). \qquad (8.28)'$$

Hence,

$$\hat{g}(z) = \hat{\pi}_0^* z^{-1} \hat{f}(z) h(z) + z^{-1} (\hat{g}(z) - \hat{\pi}_0^*) h(z). \qquad (8.29)'$$

By (*), $\hat{f}(z) = 1 + (f(z)-1)/(1-P(0))$. Substituting this into (8.29)′ and simplifying, we obtain

$$\hat{g}(z) = \hat{\pi}_0^* z^{-1} \frac{f(z)-1}{1-P(0)} h(z) + z^{-1} \hat{g}(z) h(z).$$

Solving for $\hat{g}(z)$, we find

$$\hat{g}(z) = \frac{(f(z)-1)\, h(z)}{z - h(z)} \cdot \frac{\hat{\pi}_0^*}{1 - P_0}. \qquad (8.30)'$$

A utilization of the condition $\hat{g}(1) = 1$ gives

$$\hat{\pi}_0^* = (1-\rho) \frac{1-P(0)}{f'(1)} \qquad (8.32)'$$

with $\rho = \lambda\tau = h'(1)$. When this expression and $h(z) = \eta(\lambda - \lambda z)$ are substituted into (8.30)′ we get

$$\hat{g}(z) = \frac{[f(z)-1]\, \eta(\lambda-\lambda z)}{z - \eta(\lambda - \lambda z)} \; \frac{1-\rho}{f'(1)}. \qquad (1)$$

A comparison of (1) and (8.12) shows that

$$\hat{g}(z) = \frac{f(z)-1}{f'(1)(z-1)} \times g(z) \qquad (2)$$

where $g(z)$ is the probability generating function of the number of customers left behind by an arbitrary departing customer in the corresponding equilibrium M/G/1 system, in which the server never goes on vacation.

(Chap. 5, Ex. 12 b)

b) We shall prove that $\hat{g}(z) = g(z) \Leftrightarrow f(z) = P(0) + P(1)z$. This means that $\hat{g}(z) = g(z)$ implies that no more than one customer will ever arrive during a vacation. As the arrival process is Poisson, $\hat{g}(z) = g(z)$ therefore automatically rules out the possibility that vacation length is independent of the arrival process. Our explanation of the condition $\hat{g}(z) = g(z)$ is that a vacation, if not already over, will be interrupted the moment an arrival takes place.

By (2), $\hat{g}(z) = g(z) \Leftrightarrow f(z) - 1 = f'(1)(z-1)$. Hence, it will suffice to show that $f(z) - 1 = f'(1)(z-1) \Leftrightarrow f(z) = P(0) + P(1)z$.

Necessity ($\Rightarrow$). Assume $f(z) - 1 = f'(1)(z-1)$ for all z. Then $f(z) = (1 - f'(1)) + f'(1)z$. Since $f(z) = \sum_{j=0}^{\infty} P(j) z^j$, it follows that $P(0) = 1 - f'(1)$ and $P(1) = f'(1)$. Thus, $f(z) = P(0) + P(1)z$.

Sufficiency ($\Leftarrow$). Assume $f(z) = P(0) + P(1)z$. Then $P(0) = 1 - P(1)$ so that $f(z) - 1 = P(1)(z-1)$. Hence $f'(z) = P(1)$ and, in particular, $f'(1) = P(1)$. Thus, $f(z) - 1 = f'(1)(z-1)$ for all z.

c) By Eq. (3.3), $\hat{\pi}_j = \hat{\pi}_j^*$ for all $j \geq 0$. Consequently,

$$\sum_{j=0}^{\infty} \hat{\pi}_j z^j = \sum_{j=0}^{\infty} \hat{\pi}_j^* z^j = \hat{g}(z),$$

where $\hat{g}(z)$ is given by (1). By (8.32)' and $\hat{\pi}_0 = \hat{\pi}_0^*$,

$$\hat{\pi}_0 = (1-\rho) \frac{1 - P(0)}{f'(1)}.$$

($f'(1)/(1 - P(0))$ is mean number of customers by end of vacation(s).)

d) Differentiation of Eq. (2) gives

$$\hat{g}'(z) = \frac{f(z)-1}{f'(1)(z-1)} g'(z) + \frac{g(z)}{f'(1)} \frac{(z-1)f'(z) - (f(z)-1)}{(z-1)^2},$$

whereby

$$\hat{g}'(1) = \lim_{z \to 1} \frac{f(z)-1}{f'(1)(z-1)} g'(1) + \frac{1}{f'(1)} \lim_{z \to 1} \frac{(z-1)f'(z) - (f(z)-1)}{(z-1)^2}.$$

(Chap. 5, Ex. 12 d)

A single application of l'Hospital's rule yields

$$\hat{g}'(1) = \lim_{z\to 1} \frac{f'(z)}{f'(1)\cdot 1}\, g'(1) + \frac{1}{f'(1)} \lim_{z\to 1} \frac{(z-1)f''(z) + f'(z) - f'(z)}{2(z-1)}.$$

Hence,

$$\hat{g}'(1) = g'(1) + \frac{f''(1)}{2f'(1)}. \tag{3}$$

$\boxed{e}$ Letting $\hat{\phi}(s)$ be the Laplace-Stieltjes transform of the sojourn time, we have

$$\hat{\phi}(s) = \hat{\omega}(s)\,\eta(s), \tag{8.42'}$$

$$\hat{g}(z) = \hat{\phi}(\lambda-\lambda z) \tag{8.43'}$$

Thus,

$$\hat{g}(z) = \hat{\omega}(\lambda-\lambda z)\,\eta(\lambda-\lambda z).$$

Insertion of this expression into (1) leads to

$$\hat{\omega}(\lambda-\lambda z) = \frac{f(z)-1}{z-\eta(\lambda-\lambda z)}\,\frac{1-\rho}{f'(1)}. \tag{8.44'}$$

Setting $s = \lambda - \lambda z$, we derive the analogue of (8.38)

$$\hat{\omega}(s) = \frac{1-f(1-\frac{s}{\lambda})}{s-\lambda[1-\eta(s)]}\,\frac{(1-\rho)\lambda}{f'(1)}. \tag{4}$$

$\boxed{f}$ In the case that vacation lengths are independent of the arrival process, evidently the probability of waiting equals 1.

$\boxed{g}$ The L.-S. transform of the distribution function of the waiting time $\hat{W}$ is $\hat{\omega}(s)$, given by (4). By comparison of (4) and (8.38) we find the relation

$$\hat{\omega}(s) = \frac{1-f(1-\frac{s}{\lambda})}{s}\,\frac{\lambda}{f'(1)}\,\omega(s).$$

Hence,

$$\hat{\omega}'(s) = \frac{1-f(1-\frac{s}{\lambda})}{s}\,\frac{\lambda}{f'(1)}\,\omega'(s) + \frac{\frac{s}{\lambda}f'(1-\frac{s}{\lambda}) - (1-f(1-\frac{s}{\lambda}))}{s^2}\,\frac{\lambda}{f'(1)}\,\omega(s),$$

$$\hat{\omega}'(0) = \lim_{s\to 0} \frac{1-f(1-\frac{s}{\lambda})}{s} \frac{\lambda}{f'(1)} \omega'(0) + \lim_{s\to 0} \frac{\frac{s}{\lambda} f'(1-\frac{s}{\lambda}) - (1-f(1-\frac{s}{\lambda}))}{s^2} \frac{\lambda}{f'(1)}.$$

A single application of l'Hospitals rule produces

$$\hat{\omega}'(0) = \lim_{s\to 0} \frac{\frac{1}{\lambda} f'(1-\frac{s}{\lambda})}{1} \frac{\lambda}{f'(1)} \omega'(0) + \lim_{s\to 0} \frac{\frac{1}{\lambda} f'(1-\frac{s}{\lambda}) - \frac{s}{\lambda^2} f''(1-\frac{s}{\lambda}) - \frac{1}{\lambda} f'(1-\frac{s}{\lambda})}{2s} \frac{\lambda}{f'(1)},$$

by which

$$\hat{\omega}'(0) = \omega'(0) - \frac{f''(1)}{2\lambda f'(1)}.$$

Now, $E(\hat{W}) = -\hat{\omega}'(0)$ and $E(W) = -\omega'(0)$. It follows that

$$E(\hat{W}) = E(W) + \frac{f''(1)}{2\lambda f'(1)}, \tag{5}$$

where $E(W)$ is given by (8.39).

$\boxed{h}$ Suppose $f(z) = z^j$ ($j \geq 1$). That is, with probability 1 exactly j customers arrive during a vacation, and arrival no. j signals the end of the vacation. Obviously, the length of a vacation depends on future arrivals.

Eqs. (1), (2), and (3), hold for $f(z) = z^j$ for all $j \geq 1$, since the equations were derived without a requirement of independence between vacation length and arrival process.

Eq. (4) results from (1), (8.42)′ and (8.43)′. Of these, (1) and (8.42)′ hold whether or not vacation length depends on the arrival process. However, (8.43)′ is valid only if sojourn time is independent of the arrival process. For $f(z) = z^j$ with $j = 1$ this condition will be met for every customer, but if $j \geq 2$, then in the case of arrivals no. $1, 2, \ldots, j-1$, the time until the end of vacation will depend on the future arrival epochs.

We conclude that Eq. (4) is valid for $j = 1$, but not for $j \geq 2$, when $f(z) = z^j$. In the case $f(z) = z$, (4) reduces to the Pollaczek-Khintchine formula (8.38), as it should, when $f(1-\frac{s}{\lambda}) = 1 - \frac{s}{\lambda}$ and $f'(1) = 1$ are inserted.

Eq. (5) follows from (3) after application of $L = \lambda W$; (5) is therefore valid for all $j \geq 1$ when $f(z) = z^j$.

$\square$

Chapter 5, Exercise 13

'Derivation of the P.-K. formula by the method of collective marks'

Note that in this exercise the same notation is used for a time interval and its length. For instance, W_k will denote both the time interval during which the k'th customer waits for service and the length of that time interval, the waiting time. No confusion should arise as the meaning is clear from the context.

a We consider an M/G/1 queue with order-of-arrival service. The k'th arriving customer is here the same as the k'th departing customer, so we may also speak of the k'th customer.

Let $\pi_j^{*(k)}$ be the probability that the k'th customer will leave j customers behind, namely those customers who arrive during his sojourn time, and define the generating function

$$g_k(z) = \sum_{j=0}^{\infty} \pi_j^{*(k)} z^j.$$

Now, imagine that each arriving customer is marked with probability $1-z$ and left unmarked with probability z. Clearly, by the theorem of total probability, $g_k(z)$ may be interpreted as the probability that no marked customers arrive during the sojourn time of the k'th customer.

b T_k = sojourn time of the k'th customer (or corresponding time interval)

W_k = waiting time of the k'th customer (or corresponding time interval)

C_k = { the k'th customer is marked }

C_k' = { the k'th customer is not marked }

$M'(X)$ = {no marked customers arrive during time interval X}

It follows from the above definitions and our assumption of order-of-arrival service that

$$\{M'(T_k), C_{k+1}\} \Leftrightarrow \{W_{k+1} = 0, C_{k+1}\},$$
$$\{M'(T_k), C_{k+1}'\} \Leftrightarrow \{M'(W_{k+1}), C_{k+1}'\}.$$

(Chap. 5, Ex. 13 b)

Also, since a customer's probability of being marked is $1-z$ whatever his waiting time and markings of other customers,

$$P\{W_{k+1}=0,\ C_{k+1}\} = P\{W_{k+1}=0\}\,P\{C_{k+1}\},$$
$$P\{M'(W_{k+1}),\ C'_{k+1}\} = P\{M'(W_{k+1})\}\,P\{C'_{k+1}\}.$$

Since $P\{M'(T_k)\} = P\{M'(T_k),\ C_{k+1}\} + P\{M'(T_k),\ C'_{k+1}\}$, we have

$$P\{M'(T_k)\} = P\{W_{k+1}=0\}(1-z) + P\{M'(W_{k+1})\}\,z.$$

$\boxed{c}$ Denote by $\phi_k(s)$ and $\omega_k(s)$ the Laplace-Stieltjes transforms of the distribution functions of T_k and W_k, respectively. By the definition of $g_k(z)$, the interpretation in part (a), and Equation (6.10),

$$P\{M'(T_k)\} = g_k(z) = \phi_k(\lambda-\lambda z).$$

Similarly,

$$P\{M'(W_{k+1})\} = \ \dots\ = \omega_{k+1}(\lambda-\lambda z).$$

$\boxed{d}$ By parts (b) and (c),

$$\phi_k(\lambda-\lambda z) = P\{W_{k+1}=0\}(1-z) + \omega_{k+1}(\lambda-\lambda z)\,z.$$

The substitution $\lambda-\lambda z = s$ gives

$$\phi_k(s) = P\{W_{k+1}=0\}\frac{s}{\lambda} + \omega_{k+1}(s)\left(1-\frac{s}{\lambda}\right).$$

Introducing $\phi_k(s) = \omega_k(s)\,\eta(s)$, letting $k\to\infty$, assuming that $\lim\limits_{k\to\infty}\omega_k(s) = \omega(s)$ and $\lim\limits_{k\to\infty} P\{W_k=0\} = P\{W=0\}$, and solving for $\omega(s)$, we find

$$\omega(s) = \frac{s}{s-\lambda[1-\eta(s)]}\,P\{W=0\}.$$

$\boxed{e}$ Finally, utilizing $\omega(0)=1$ we derive $P\{W=0\} = 1+\lambda\eta'(0) = 1-\lambda\tau = 1-\rho$. Once more we obtain the Pollaczek-Khintchine formula

$$\omega(s) = \frac{s(1-\rho)}{s-\lambda[1-\eta(s)]} \qquad (8.38).$$

$\square$

Chapter 5, Exercise 14

'The M/G/1 queue from the viewpoint of arrivals.'

Definitions

N = number of customers in the system just prior to an arbitrary arrival epoch T_c

R = remaining service time at T_c

$$\Pi_j = P\{N=j\} \qquad (j=0,1,\ldots) \tag{1}$$

$$\Pi_j(x) = P\{R \leq x,\ N=j\} \qquad (x \geq 0;\ j=1,2,\ldots) \tag{2}$$

$$\psi_j(s) = \int_0^\infty e^{-sx}\, d\Pi_j(x) \qquad (j=1,2,\ldots) \tag{3}$$

$$u(s,z) = \sum_{j=1}^{\infty} \psi_j(s)\, z^j \tag{4}$$

Main results

$$u(s,z) = \frac{\Pi_0 \lambda z(1-z)}{z-\eta(\lambda-\lambda z)} \; \frac{\eta(s)-\eta(\lambda-\lambda z)}{s-\lambda(1-z)} \qquad [\text{see parts e-j}], \tag{5}$$

$$\Pi_0 = 1-\rho \qquad (\rho=\lambda\tau<1) \quad [\text{see part a}], \tag{6}$$

where $\eta(s)$ is the Laplace-Stieltjes transform of the service-time distribution function $H(x)$, τ is the mean service time, and λ is the customer arrival rate. Inversion of (5) gives

$$\sum_{j=1}^{\infty} \Pi_j(x) z^j = \frac{(1-\rho)\lambda z(1-z)}{\eta(\lambda-\lambda z)-z} \int_0^\infty e^{-\lambda(1-z)\xi}\left[H(\xi+x)-H(\xi)\right]d\xi \quad [\text{see part k}]. \tag{7}$$

$\boxed{a}$

$$u(0,1) = \Pi_0 \lambda \lim_{z\to 1} \frac{1-z}{z-\eta(\lambda-\lambda z)} \lim_{s\to 0} \frac{\eta(s)-1}{s}$$

$$= \Pi_0 \lambda \frac{-\eta'(0)}{1+\lambda\eta'(0)} = \Pi_0 \frac{\lambda\tau}{1-\lambda\tau} = \Pi_0 \frac{\rho}{1-\rho}.$$

$$1 = \Pi_0 + \sum_{j=1}^{\infty} \Pi_j = \Pi_0 + \sum_{j=1}^{\infty} \psi_j(0)$$

$$= \Pi_0 + u(0,1) = \Pi_0\left(1+\frac{\rho}{1-\rho}\right).$$

Thus, Eq. (6), $\Pi_0 = 1-\rho$, follows from Eq. (5) and $\sum \Pi_j = 1$.

(Chap. 5, Ex. 14 b)

$\boxed{b}$ By Eq. (7),

$$\sum_{j=1}^{\infty} \Pi_j(x) = \lim_{z\to 1} \sum_{j=1}^{\infty} \Pi_j(x) z^j$$
$$= (1-\rho)\lambda \lim_{z\to 1} \frac{1-z}{\eta(\lambda-\lambda z)-z} \int_0^{\infty} [H(\xi+x)-H(\xi)]d\xi$$
$$= (1-\rho)\lambda \frac{1}{1+\lambda\eta'(0)} \left(\int_0^{\infty} [1-H(\xi)]d\xi - \int_x^{\infty} [1-H(\xi)]d\xi \right).$$

Hence,

$$\sum_{j=1}^{\infty} \Pi_j(x) = \lambda \int_0^x [1-H(\xi)]d\xi,$$

so that

$$P\{R \leq x | N \geq 1\} = \frac{P\{R \leq x, N \geq 1\}}{P\{N \geq 1\}} = \frac{\sum_{j=1}^{\infty} \Pi_j(x)}{\sum_{j=1}^{\infty} \Pi_j} = \frac{\lambda \int_0^x [1-H(\xi)]d\xi}{\lambda \int_0^{\infty} [1-H(\xi)]d\xi}.$$

Since the mean service time may be expressed as $\tau = \int_0^{\infty} [1-H(\xi)]d\xi$,

$$P\{R \leq x | N \geq 1\} = \frac{1}{\tau} \int_0^x [1-H(\xi)]d\xi. \qquad (8)$$

This might have been anticipated by considering (7.9); disregarding idle periods, start-of-service epochs form a renewal process.

$\boxed{c}$ By Eq. (7),

$$\sum_{j=0}^{\infty} \Pi_j z^j = \Pi_0 + \lim_{x\to\infty} \sum_{j=1}^{\infty} \Pi_j(x) z^j$$
$$= \Pi_0 + \frac{(1-\rho)\lambda z(1-z)}{\eta(\lambda-\lambda z)-z} \int_0^{\infty} e^{-\lambda(1-z)\xi} [1-H(\xi)]d\xi$$
$$= \Pi_0 + \frac{(1-\rho)\lambda z(1-z)}{\eta(\lambda-\lambda z)-z} \left(\frac{1}{\lambda(1-z)} - \int_0^{\infty} e^{-\lambda(1-z)\xi} H(\xi)d\xi \right)$$
$$= \Pi_0 + \frac{(1-\rho)z}{\eta(\lambda-\lambda z)-z} \left(1 + \int_0^{\infty} H(\xi)de^{-\lambda(1-z)\xi} \right)$$
$$= \Pi_0 + \frac{(1-\rho)z}{\eta(\lambda-\lambda z)-z} \left(1 - \int_{0-}^{\infty} e^{-\lambda(1-z)\xi} dH(\xi) \right)$$
$$= \Pi_0 + \frac{(1-\rho)z}{\eta(\lambda-\lambda z)-z} \left(1 - \eta(\lambda-\lambda z) \right).$$

Finally, substitution of $\Pi_0 = 1-\rho$, by (6), and simplification give as the result,

(Chap. 5, Ex. 14 c)

$$\sum_{j=0}^{\infty} \Pi_j z^j = \frac{(z-1)\eta(\lambda-\lambda z)}{z-\eta(\lambda-\lambda z)}(1-\rho). \tag{9}$$

Equation (9) is seen to be identical to (8.12) which gives the probability generating function of the departure state. This is the direct consequence of the equality $\Pi_j^* = \Pi_j$.

$\boxed{d}$ Let $W_j(x)$ be the conditional waiting time distribution function, given $N=j$, and let $W(x)$ be the unconditional waiting time distribution function, assuming service in order of arrival. Then

$$W(x) = \sum_{j=0}^{\infty} \Pi_j W_j(x) = \Pi_0 + \sum_{j=1}^{\infty} \Pi_j W_j(x) = \Pi_0 + \sum_{j=1}^{\infty} \Pi_j \int_0^x P\{R \le x-\xi \mid N=j\} d H^{*(j-1)}(\xi),$$

where we have used that $W_j(x)$ is the convolution of the distribution of the remaining service time, given $N=j$, and the distribution of a sum of $j-1$ independent service times. By definition of $\Pi_j(x)$, then,

$$W(x) = \Pi_0 + \sum_{j=1}^{\infty} \int_0^x \Pi_j(x-\xi)\, d H^{*(j-1)}(\xi). \tag{10}$$

Hence,

$$\begin{aligned}
\omega(s) &= \int_0^{\infty} e^{-sx} d W(x) = \Pi_0 + \sum_{j=1}^{\infty} \int_0^{\infty} e^{-sx} d \int_0^x \Pi_j(x-\xi) d H^{*(j-1)}(\xi) \\
&= \Pi_0 + \sum_{j=1}^{\infty} \left(\int_0^{\infty} e^{-sx} d\Pi_j(x) \right) \left(\int_0^{\infty} e^{-sx} d H^{*(j-1)}(x) \right) \\
&= \Pi_0 + \sum_{j=1}^{\infty} \psi_j(s) [\eta(s)]^{j-1} \\
&= \Pi_0 + \frac{1}{\eta(s)} U(s, \eta(s)) \\
&= \Pi_0 + \frac{1}{\eta(s)} \frac{\Pi_0 \lambda \eta(s)(1-\eta(s))}{s-\lambda(1-\eta(s))} \\
&= \Pi_0 \frac{s}{s-\lambda(1-\eta(s))}.
\end{aligned}$$

Thus, the Laplace-Stieltjes transform of the waiting time is

$$\omega(s) = \frac{s(1-\rho)}{s-\lambda[1-\eta(s)]}, \tag{11}$$

which, of course, is the same as Eq. (8.38).

(Chap. 5, Ex. 14 e)

$\boxed{e}$ Let T_1 and T_2 be an arbitrary pair of consecutive arrival epochs, and let the associated arrival states be (N_1, R_1) and (N_2, R_2), respectively. Eqs. (12), (13) and (14) are derived on the assumption of identical state probability distributions at T_1 and T_2. The distribution at T_2, defined by Π_0 $(= P\{N_2 = 0\})$, $\Pi_1(x)$ $(= P\{N_2 = 1, R_2 \leq x\})$ and $\Pi_j(x)$ $(= P\{N_2 = j, R_2 \leq x\})$ for $j = 2, 3, \ldots$ is found by conditioning on (N_1, R_1). The equations reflect the fact that $N_2 = j$ may occur if and only if $N_1 \geq j-1$.

$$\Pi_0 = \Pi_0 \int_0^\infty e^{-\lambda t} dH(t) + \sum_{k=1}^\infty \int_0^\infty \int_0^\infty e^{-\lambda(y+z)} d\Pi_k(y)\, dH^{*k}(z). \qquad (12)$$

Eq. (12) is obtained by rewriting $P\{N_2 = 0\} = P\{N_1 = 0, N_2 = 0\} + \sum_{k=1}^\infty P\{N_1 = k, N_2 = 0\}$ as $\Pi_0 = \Pi_0 P\{N_2 = 0 | N_1 = 0\} + \sum_{k=1}^\infty \int_0^\infty P\{N_2 = 0 | N_1 = k, R_1 = y\}\, d\Pi_k(y)$, and observing that $P\{N_2 = 0 | N_1 = 0\} = \int_0^\infty e^{-\lambda t} dH(t)$ and $P\{N_2 = 0 | N_1 = k, R_1 = y\} = \int_0^\infty e^{-\lambda(y+z)} dH^{*k}(z)$.

$$\Pi_1(x) = \Pi_0 \int_0^\infty [H(t+x) - H(t)] \lambda e^{-\lambda t} dt \qquad (13)$$
$$+ \sum_{k=1}^\infty \int_0^\infty \int_0^\infty e^{-\lambda(y+z)} d\Pi_k(y)\, dH^{*(k-1)}(z) \int_0^\infty [H(t+x) - H(t)] \lambda e^{-\lambda t} dt.$$

Eq. (13) follows from $P\{N_2 = 1, R_2 \leq x\} = P\{N_1 = 0, N_2 = 1, R_2 \leq x\} + \sum_{k=1}^\infty P\{N_1 = k, N_2 = 1, R_2 \leq x\}$, rewritten $\Pi_1(x) = \Pi_0 P\{N_2 = 1, R_2 \leq x | N_1 = 0\} + \sum_{k=1}^\infty \int_0^\infty P\{N_2 = 1, R_2 \leq x | N_1 = k, R_1 = y\}\, d\Pi_k(y)$, as well as the observations that $P\{N_2 = 1, R_2 \leq x | N_1 = 0\} = \int_0^\infty [H(t+x) - H(t)] \lambda e^{-\lambda t} dt$ and $P\{N_2 = 1, R_2 \leq x | N_1 = k, R_1 = y\} = \int_0^\infty e^{-\lambda(y+z)} dH^{*(k-1)}(z) \int_0^\infty [H(t+x) - H(t)] \lambda e^{-\lambda t} dt$.

$$\Pi_j(x) = \int_0^\infty [\Pi_{j-1}(t+x) - \Pi_{j-1}(t)] \lambda e^{-\lambda t} dt \qquad (j = 2, 3, \ldots) \qquad (14)$$
$$+ \sum_{k=j}^\infty \int_0^\infty \int_0^\infty e^{-\lambda(y+z)} d\Pi_k(y)\, dH^{*(k-j)}(z) \int_0^\infty [H(t+x) - H(t)] \lambda e^{-\lambda t} dt.$$

Eq. (14) may be derived by the same kind of arguments that were used for Eqs. (12) and (13).

$\boxed{f}$ The double integral of Eq. (12) equals $\int_0^\infty e^{-\lambda y} d\Pi_k(y) \int_0^\infty e^{-\lambda z} dH^{*k}(z)$. Hence, using the Laplace-Stieltjes transform definitions, Eq. (12) becomes

$$\Pi_0 = \Pi_0\, \eta(\lambda) + \sum_{k=1}^\infty \psi_k(\lambda)\, \eta^k(\lambda). \qquad (15)$$

(Chap. 5, Ex. 14 f)

Eq. (13) may be written

$$\Pi_1(x) = K_1(x)\Big[\Pi_0 + \sum_{k=1}^{\infty} \psi_k(\lambda)\eta^{k-1}(\lambda)\Big],$$

where

$$K_1(x) = \int_0^\infty [H(t+x) - H(t)]\lambda e^{-\lambda t}dt.$$

Hence,

$$\psi_1(s) = \int_0^\infty e^{-sx}d\Pi_1(x) = \int_0^\infty e^{-sx}dK_1(x)\Big[\Pi_0 + \sum_{k=1}^{\infty}\psi_k(\lambda)\eta^{k-1}(\lambda)\Big].$$

Now,

$$\int_0^\infty e^{-sx}dK_1(x) = \int_{x=0}^\infty e^{-sx}\Big(\int_{t=0}^\infty \lambda e^{-\lambda t}dH(t+x)\Big)dx$$

$$= \lambda\int_{x=0}^\infty e^{-(s-\lambda)x}\Big(\int_{t=0}^\infty e^{-\lambda(t+x)}dH(t+x)\Big)dx = \lambda\int_{x=0}^\infty e^{-(s-\lambda)x}\Big(\int_{t=x}^\infty e^{-\lambda t}dH(t)\Big)dx$$

$$= \lambda\int_{t=0}^\infty e^{-\lambda t}\Big(\int_{x=0}^t e^{-(s-\lambda)x}dx\Big)dH(t) = \frac{\lambda}{s-\lambda}\int_{t=0}^\infty e^{-\lambda t}[1 - e^{-(s-\lambda)t}]dH(t),$$

so that

$$\int_0^\infty e^{-sx}dK_1(x) = \frac{\lambda}{s-\lambda}[\eta(\lambda) - \eta(s)].$$

Hence,

$$\psi_1(s) = \frac{\lambda}{s-\lambda}[\eta(\lambda) - \eta(s)]\Big[\Pi_0 + \sum_{k=1}^{\infty}\psi_k(\lambda)\eta^{k-1}(\lambda)\Big]. \qquad (16)$$

Eq. (14) may be written

$$\Pi_j(x) = K_j(x) + K_1(x)\sum_{k=j}^{\infty}\psi_k(\lambda)\eta^{k-j}(\lambda) \qquad (j = 2,3,\ldots),$$

where $K_1(x)$ has been defined above and

$$K_j(x) = \int_0^\infty [\Pi_{j-1}(t+x) - \Pi_{j-1}(t)]\lambda e^{-\lambda t}dt \qquad (j = 2,3,\ldots).$$

Hence,

$$\psi_j(s) = \int_0^\infty e^{-sx}d\Pi_j(x) = \int_0^\infty e^{-sx}dK_j(x) + \int_0^\infty e^{-sx}dK_1(x)\sum_{k=j}^{\infty}\psi_k(\lambda)\eta^{k-j}(\lambda) \quad (j=2,3,\ldots).$$

Proceeding precisely as when $\int_0^\infty e^{-sx}dK_1(x)$ was calculated, we derive

$$\int_0^\infty e^{-sx}dK_j(x) = \frac{\lambda}{s-\lambda}[\psi_{j-1}(\lambda) - \psi_{j-1}(s)] \qquad (j = 2,3,\ldots).$$

Hence, for $j = 2,3,\ldots,$

$$\psi_j(s) = \frac{\lambda}{s-\lambda}[\psi_{j-1}(\lambda) - \psi_{j-1}(s)] + \frac{\lambda}{s-\lambda}[\eta(\lambda) - \eta(s)]\sum_{k=j}^{\infty}\psi_k(\lambda)\eta^{k-j}(\lambda) \qquad (17)$$

(Chap. 5, Ex. 14 g)

$\boxed{g}$ Substitution of (16) and (17) into (4) leads to

$$u(s,z) = \frac{\lambda}{s-\lambda}\sum_{j=2}^{\infty}[\psi_{j-1}(\lambda) - \psi_{j-1}(s)]\,z^j + \frac{\lambda}{s-\lambda}[\eta(\lambda)-\eta(s)][\Pi_0 z + \sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\psi_k(\lambda)\eta^{k-j}(\lambda)\,z^j].$$

Now,

$$\sum_{j=2}^{\infty}[\psi_{j-1}(\lambda) - \psi_{j-1}(s)]\,z^j = z\,[u(\lambda,z) - u(s,z)],$$

and

$$\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\psi_k(\lambda)\,\eta^{k-j}(\lambda)\,z^j = \sum_{k=1}^{\infty}\psi_k(\lambda)\,\eta^k(\lambda)\sum_{j=1}^{k}\left(\frac{z}{\eta(\lambda)}\right)^j$$

$$= \frac{z}{\eta(\lambda)-z}\sum_{k=1}^{\infty}\psi_k(\lambda)\,\eta^k(\lambda)\left[1-\left(\frac{z}{\eta(\lambda)}\right)^k\right]$$

$$= \frac{z}{\eta(\lambda)-z}\left[u(\lambda,\eta(\lambda)) - u(\lambda,z)\right].$$

It follows easily that

$$u(s,z)[s-\lambda(1-z)] = \lambda z\,u(\lambda,z) + \lambda z[\eta(\lambda)-\eta(s)]\left[\Pi_0 + \frac{u(\lambda,\eta(\lambda)) - u(\lambda,z)}{\eta(\lambda)-z}\right]. \quad (18)$$

$\boxed{h}$ For $s = \lambda - \lambda z$, Eq. (18) specializes to

$$0 = \lambda z\,u(\lambda,z) + \lambda z[\eta(\lambda)-\eta(\lambda-\lambda z)]\left[\Pi_0 + \frac{u(\lambda,\eta(\lambda)) - u(\lambda,z)}{\eta(\lambda)-z}\right],$$

whereby

$$\Pi_0 = \frac{u(\lambda,z)}{\eta(\lambda-\lambda z)-\eta(\lambda)} + \frac{u(\lambda,\eta(\lambda)) - u(\lambda,z)}{z-\eta(\lambda)}. \quad (19)$$

Substitution of this expression into (18) and solution w.r.t. $u(s,z)$ give us

$$u(s,z) = \frac{\lambda z[\eta(s)-\eta(\lambda-\lambda z)]\,u(\lambda,z)}{[s-(\lambda-\lambda z)][\eta(\lambda)-\eta(\lambda-\lambda z)]}. \quad (20)$$

$\boxed{i}$ By (15),

$$u(\lambda,\eta(\lambda)) = \Pi_0\,[1-\eta(\lambda)]. \quad (21)$$

Substitution of this expression into (19) and solution w.r.t. $u(\lambda,z)$ give us

$$u(\lambda,z) = \frac{\Pi_0(1-z)[\eta(\lambda)-\eta(\lambda-\lambda z)]}{z-\eta(\lambda-\lambda z)}. \quad (22)$$

(Chap. 5, Ex. 14 j)

$\boxed{j}$ Finally, substitution of (22) into (20) yields

$$u(s,z) = \frac{\Pi_0 \lambda z(1-z)}{z-\eta(\lambda-\lambda z)} \frac{\eta(s)-\eta(\lambda-\lambda z)}{s-\lambda(1-z)}. \qquad (5)$$

$\boxed{k}$ We shall show that inversion of (5) gives

$$\sum_{j=1}^{\infty} \Pi_j(x) z^j = A(z) \int_0^{\infty} e^{-\lambda(1-z)\xi}[H(\xi+x)-H(\xi)]\,d\xi, \qquad (7)$$

where

$$A(z) = \frac{(1-\rho)\lambda z(1-z)}{\eta(\lambda-\lambda z)-z}.$$

To begin, we show that the Laplace-Stieltjes transform of the LHS of (7) equals the LHS of (5):

$$\int_{x=0}^{\infty} e^{-sx} d\Big(\sum_{j=1}^{\infty} \Pi_j(x) z^j\Big) = \sum_{j=1}^{\infty} z^j \int_{x=0}^{\infty} e^{-sx} d\Pi_j(x) = \sum_{j=1}^{\infty} \psi_j(s) z^j = u(s,z).$$

Next we show that the Laplace-Stieltjes transform of the RHS of (7) equals the RHS of (5):

$$\begin{aligned}
\int_{x=0}^{\infty} e^{-sx} d\Big(A(z)\int_{\xi=0}^{\infty} e^{-\lambda(1-z)\xi}[H(\xi+x)-H(\xi)]\,d\xi\Big) &= A(z)\int_{x=0}^{\infty} e^{-sx}\Big(\int_{\xi=0}^{\infty} e^{-\lambda(1-z)\xi} dH(\xi+x)\Big)dx \\
&= A(z)\int_{x=0}^{\infty} e^{-[s-\lambda(1-z)]x}\Big(\int_{\xi=0}^{\infty} e^{-\lambda(1-z)(\xi+x)} dH(\xi+x)\Big)dx \\
&= A(z)\int_{x=0}^{\infty} e^{-[s-\lambda(1-z)]x}\Big(\int_{\xi=x}^{\infty} e^{-\lambda(1-z)\xi} dH(\xi)\Big)dx \\
&= A(z)\int_{\xi=0}^{\infty} e^{-\lambda(1-z)\xi}\Big(\int_{x=0}^{\xi} e^{-[s-\lambda(1-z)]x}dx\Big)dH(\xi) \\
&= A(z)\frac{1}{s-\lambda(1-z)}\int_{\xi=0}^{\infty} e^{-\lambda(1-z)\xi}[1-e^{-[s-\lambda(1-z)]\xi}]\,dH(\xi) \\
&= A(z)\frac{1}{s-\lambda(1-z)}\int_{\xi=0}^{\infty}[e^{-\lambda(1-z)\xi}-e^{-s\xi}]\,dH(\xi) \\
&= A(z)\frac{1}{s-\lambda(1-z)}[\eta(\lambda-\lambda z)-\eta(s)]
\end{aligned}$$

Thus, considering the definition of $A(z)$ and the fact, by (6), that $1-\rho = \Pi_0$,

$$\int_{x=0}^{\infty} e^{-sx} d\Big(A(z)\int_{\xi=0}^{\infty} e^{-\lambda(1-z)\xi}[H(\xi+x)-H(\xi)]\,d\xi\Big) = \frac{\Pi_0 \lambda z(1-z)}{z-\eta(\lambda-\lambda z)} \frac{\eta(s)-\eta(\lambda-\lambda z)}{s-\lambda(1-z)}.$$

□

Chapter 5, Exercise 15

'The integrodifferential equation of Takács.'

Let V_t be the virtual waiting time in the M/G/1 queue with order-of-arrival service, and define the distribution function $V(t,x) = P\{V_t \leq x\}$.

$\boxed{a}$ Let $h>0$, and let K be the number of arrivals in the time interval $[t,t+h)$. We shall show that

$$P\{V_{t+h} \leq x,\ K=0\} = (1-\lambda h)\,V(t,x+h) + o(h), \qquad (i)$$

$$P\{V_{t+h} \leq x,\ K=1\} = \lambda h \int_0^{x+h} H(x+h-y)\,d_yV(t,y) + o(h), \qquad (ii)$$

whereby, as $P\{V_{t+h} \leq x,\ K \geq 2\} = o(h)$ and $V(t+h,x) = \sum_{j=0}^{\infty} P\{V_{t+h} \leq x,\ K=j\}$,

$$V(t+h,x) = (1-\lambda h)\,V(t,x+h) + \lambda h \int_0^{x+h} H(x+h-y)\,d_yV(t,y) + o(h). \qquad (*)$$

Eq. (i). Obviously, $\{V_{t+h} \leq x \mid K=0\} \Leftrightarrow \{V_t \leq x+h \mid K=0\}$. Consequently, $P\{V_{t+h} \leq x \mid K=0\} = P\{V_t \leq x+h \mid K=0\}$. Also, since V_t and K are independent, $P\{V_t \leq x+h \mid K=0\} = P\{V_t \leq x+h\}$. Thus, $P\{V_{t+h} \leq x \mid K=0\} = P\{V_t \leq x+h\}$, and,

$$P\{V_{t+h} \leq x,\ K=0\} = P\{K=0\}P\{V_{t+h} \leq x \mid K=0\} = (1-\lambda h + o(h))\,P\{V_t \leq x+h\},$$

whereby (i) follows.

Eq. (ii). For $K=1$, denote by t^* the time of arrival and by Z the service time of the customer arriving in $[t,t+h)$. Observe that $\{V_{t+h} \leq x \mid K=1\} \Leftrightarrow \{V_t + Z \leq x+h,\ t+h-t^* \geq Z-x \mid K=1\}$. Hence, for all $x \geq 0$,

$$P\{V_{t+h} \leq x \mid K=1\} = P\{V_t + Z \leq x+h \mid K=1\} - P\{V_t+Z \leq x+h,\ t+h-t^* < Z-x \mid K=1\}.$$

Since V_t, Z and K are independent variables,

$$\begin{aligned} P\{V_t + Z \leq x+h \mid K=1\} &= P\{V_t+Z \leq x+h\} \\ &= \int_0^{x+h} H(x+h-y)\,d_yV(t,y). \end{aligned}$$

(Chap. 5, Ex. 15 a)

For the sake of brevity, define

$$F(x,h) = P\{V_t + Z \leq x+h,\ t+h-t^* < Z-x \mid K=1\}.$$

Clearly, t^* is uniformly distributed on $[t, t+h)$, independently of V_t and Z. Using this fact we derive

$$\begin{aligned} F(x,h) &= \int_{y=0}^{h} \int_{z=x+}^{x+h-y} \frac{z-x}{h}\, dH(z)\, d_y V(t,y) \\ &\leq \int_{y=0}^{h} \int_{z=x+}^{x+h-y} dH(z)\, d_y V(t,y) \\ &\leq \int_0^h d_y V(t,y) \int_{x+}^{x+h} dH(z) \\ &= V(t,h)\,[H(x+h) - H(x)]. \end{aligned}$$

Hence,

$$\lim_{h\to 0} F(x,h) = 0 \qquad (x \geq 0).$$

Evidently,

$$P\{V_{t+h} \leq x, K=1\} = P\{K=1\}\, P\{V_{t+h} \leq x \mid K=1\} = (\lambda h + o(h))\, P\{V_{t+h} \leq x \mid K=1\}.$$

Hence, by previous results,

$$\begin{aligned} P\{V_{t+h} \leq x, K=1\} &= (\lambda h + o(h))\left[\int_0^{x+h} H(x+h-y)\, d_y V(t,y) - F(x,h)\right] \\ &= \lambda h \int_0^{x+h} H(x+h-y)\, d_y V(t,y) + o(h). \end{aligned}$$

This concludes the proof of (ii). The proof of Eq. (*) is complete.

$\boxed{b}$ Subtracting $V(t,x)$ on both sides of (*), dividing through by h, and letting $h \to 0$, we obtain

$$\frac{\partial V(t,x)}{\partial t} = \frac{\partial V(t,x)}{\partial x} - \lambda V(t,x) + \lambda \int_0^x H(x-y)\, d_y V(t,y), \qquad (1)$$

which is the integrodifferential equation of Takács.

(Chap. 5, Ex. 15 c)

$\boxed{c}$ Assuming $\lim_{t\to\infty} \frac{\partial V(t,x)}{\partial t} = 0$ and $\lim_{t\to\infty} V(t,x) = V(x)$, Eq. (1) becomes

$$\frac{dV(x)}{dx} = \lambda V(x) - \lambda \int_0^x H(x-y)\,dV(y) \qquad (x \geq 0). \qquad (2)$$

Now, define the Laplace-Stieltjes transform

$$\theta(s) = \int_0^\infty e^{-sx} dV(x).$$

All three terms in Eq. (2) are functions of x. The L.-S. transforms of LHS and RHS of the equation are, respectively,

$$\begin{aligned}\int_0^\infty e^{-sx} d\,\frac{dV(x)}{dx} &= V'(0) + \int_{0+}^\infty e^{-sx} d\,\frac{dV(x)}{dx}\\ &= V'(0) + e^{-sx}\frac{dV(x)}{dx}\Big|_0^\infty + s\int_{0+}^\infty e^{-sx} dV(x)\\ &= V'(0) - V'(0) + s\Big[\int_0^\infty e^{-sx} dV(x) - V(0)\Big]\\ &= s\,[\theta(s) - V(0)],\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty e^{-sx} d\Big(\lambda V(x) - \lambda\int_0^x H(x-y)\,dV(y)\Big) &= \lambda\int_0^\infty e^{-sx} dV(x) - \lambda\int_0^\infty e^{-sx} d\Big(\int_0^x H(x-y)\,dV(y)\Big)\\ &= \lambda\,\theta(s) - \lambda\,\eta(s)\,\theta(s),\end{aligned}$$

since $\int_0^x H(x-y)\,dV(y)$ is the convolution of distribution functions with L.-S. transforms $\eta(s)$ and $\theta(s)$.

Equating LHS and RHS transforms we obtain

$$\theta(s) - V(0) = \frac{\lambda}{s}\theta(s) - \frac{\lambda}{s}\eta(s)\theta(s). \qquad (3)$$

$\boxed{d}$ Solving (3) for $\theta(s)$ gives

$$\theta(s) = \frac{sV(0)}{s - \lambda[1-\eta(s)]}.$$

Inserting $V(0) = 1-\rho$ as usual for the M/G/1 queue, we get (8.38), as we should. $\square$

Chapter 5, Exercise 16

'Verify that (8.72) is the solution of (8.71).'

$$f_n^{(j)} = \begin{cases} e^{-\lambda\tau j} & (n=j), \\ \sum_{k=1}^{n-j} \frac{(\lambda\tau j)^k}{k!} e^{-\lambda\tau j} f_{n-j}^{(k)} & (n \geqq j+1). \end{cases} \tag{8.71}$$

$$f_n^{(j)} = \frac{j}{n}\left[\frac{(\lambda\tau n)^{n-j}}{(n-j)!} e^{-\lambda\tau n}\right] \qquad (n \geqq j). \tag{8.72}$$

The proof is by induction. First, we observe that for all feasible n and j such that $n-j=0$, Eq. (8.72) reduces to $e^{-\lambda\tau n}$, which agrees with (8.71). Next, we assume that (8.72) already has been proved for all n and j such that $n-j \leq k_0$. We shall show that then (8.72) will hold for all n and j such that $n-j=k_0+1$.

Thus, assume values of n and j such that $n-j=k_0+1$. By the induction hypothesis, (8.72) applies to all the factors $f_{n-j}^{(k)}$, $k=1,\ldots,n-j$, of (8.71), since $n-j-k \leq k_0$. Substitution of (8.72) into (8.71) and straightforward reduction produce

$$\begin{aligned} f_n^{(j)} &= \sum_{k=1}^{n-j} \frac{(\lambda\tau j)^k}{k!} e^{-\lambda\tau j} \frac{k}{n-j} \frac{(\lambda\tau(n-j))^{n-j-k}}{(n-j-k)!} e^{-\lambda\tau(n-j)} \\ &= \frac{(\lambda\tau(n-j))^{n-j}}{(n-j)!} e^{-\lambda\tau n} \sum_{k=1}^{n-j} \frac{(n-j-1)!}{(k-1)!(n-j-k)!}\left(\frac{j}{n-j}\right)^k \\ &= \frac{(\lambda\tau(n-j))^{n-j}}{(n-j)!} e^{-\lambda\tau n} \frac{j}{n-j} \sum_{\nu=0}^{n-j-1} \binom{n-j-1}{\nu}\left(\frac{j}{n-j}\right)^{\nu} 1^{n-j-1-\nu} \qquad [\nu=k-1] \\ &= \frac{(\lambda\tau(n-j))^{n-j}}{(n-j)!} e^{-\lambda\tau n} \frac{j}{n-j}\left(1+\frac{j}{n-j}\right)^{n-j-1} \\ &= \frac{j}{n} \frac{(\lambda\tau n)^{n-j}}{(n-j)!} e^{-\lambda\tau n}. \end{aligned}$$

By induction, we conclude that (8.72) holds for all n and j where $j \geqq 1$, $n \geqq j$.

$\square$

Chapter 5, Exercise 17

'a. Let N_k be the number of customers served during a k-busy period.'

$\boxed{a}$ Assume an M/G/1 queue. Starting at t_0 with $i+j$ customers in the system, we imagine that first we serve the i customers plus all later arrivals until the moment t_1, when a departure leaves the original j customers behind. Clearly, $[t_0, t_1)$ is an i-busy period. Let N_i be the number of customers served during $[t_0, t_1)$. Now serve the remaining j customers and all later arrivals until, at t_2, the system is empty. Again, $[t_1, t_2)$ is a j-busy period, and we let N_j be the number of customers served during $[t_1, t_2)$. By the independence of interarrival times and service times as well as the assumption of Poisson arrivals, the realizations of the i-busy period and the j-busy period are independent. In particular, N_i and N_j are independent, and

$$N_{i+j} = N_i + N_j \qquad (1)$$

where N_{i+j} is the number of customers served during the entire $(i+j)$-busy period.

$\boxed{b}$ Extending the arguments behind (1), a k-busy period may be decomposed into k independent 1-busy periods with associated number of services $N_1(\nu)$, $\nu = 1, 2, \ldots, k$, so that $N_k = \sum_{\nu=1}^{k} N_1(\nu)$. Hence,

$$E(N_k) = \sum_{\nu=1}^{k} E(N_1(\nu)) = k\,E(N_1).$$

$E(N_1)$ is most easily derived from the mean busy period b as follows. The mean cycle time equals $b + \lambda^{-1}$. Thus, the average number of busy periods (or cycles) per unit time is $[b + \lambda^{-1}]^{-1}$. Since the number of arrivals per unit time (= number of services per unit time if $\rho < 1$) is λ, the average number of services per busy period will be $E(N_1) = \lambda/[b + \lambda^{-1}]^{-1} = 1 + \lambda b$. According to (8.69), $b = \tau/(1 - \lambda\tau)$. Hence, $E(N_1) = 1/(1 - \lambda\tau) = 1/(1-\rho)$. It follows that, for $\rho < 1$,

$$E(N_k) = \frac{k}{1-\rho}.$$

(Chap. 5, Ex. 17c)

$\boxed{c}$ Let $f_n^{(i)}$ denote the probability that $n \geq i$ customers will be served during an i-busy period. By part a,

$$\sum_{k=i}^{n-j} f_k^{(i)} f_{n-k}^{(j)} = f_n^{(i+j)} \qquad \left(\begin{smallmatrix} i\geq 1,\, j\geq 1, \\ n \geq i+j \end{smallmatrix}\right).$$

For an M/D/1 queue $f_n^{(i)}$ is given by (8.72), whose substitution into the above equation gives

$$\sum_{k=i}^{n-j} \frac{i}{k} \frac{(\lambda\tau k)^{k-i}}{(k-i)!} \frac{j}{n-k} \frac{(\lambda\tau(n-k))^{n-k-j}}{(n-k-j)!} = \frac{(i+j)}{n} \frac{(\lambda\tau n)^{n-i-j}}{(n-i-j)!}.$$

After cancellation of powers of $\lambda\tau$, and a rearrangement, we have

$$\sum_{k=i}^{n-j} \frac{k^{k-i-1}}{(k-i)!} \frac{(n-k)^{n-k-j-1}}{(n-k-j)!} = \frac{(i+j)}{ij} \frac{n^{n-i-j-1}}{(n-i-j)!} \qquad \left(\begin{smallmatrix} i\geq 1,\, j\geq 1, \\ n \geq i+j \end{smallmatrix}\right). \qquad (2)$$

For $i = j = 1$, (2) becomes the identity

$$\sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} = 2(n-1)n^{n-2} \qquad (n \geq 2). \qquad (3)$$

Chapter 5, Exercise 18

'Let N_j be the number of customers served during a j-busy period'

In the present case, the M/M/1 queue, $H^{*n}(\xi) = \sum_{k=n}^{\infty} \frac{(\mu\xi)^k}{k!} e^{-\mu\xi}$, so that

$$\frac{dH^{*n}(\xi)}{d\xi} = \frac{\mu^n}{(n-1)!} \xi^{n-1} e^{-\mu\xi}.$$

Setting $t = \infty$ in (8.73) and replacing $dH^{*n}(\xi)$ by $\frac{dH^{*n}(\xi)}{d\xi} d\xi$, we find

$$P\{N_j = n\} = \frac{j}{n} \int_0^{\infty} \frac{(\lambda\xi)^{n-j}}{(n-j)!} e^{-\lambda\xi} \frac{\mu^n}{(n-1)!} \xi^{n-1} e^{-\mu\xi} d\xi = \frac{j\lambda^{n-j}\mu^n}{n(n-j)!(n-1)!} \int_0^{\infty} \xi^{2n-j-1} e^{-(\lambda+\mu)\xi} d\xi.$$

Since $\int_0^{\infty} x^m e^{-ax} dx = m!/a^{m+1}$ for $a > 0$, $m = 0, 1, 2, \dots$ (see a Table of Integrals),

$$\int_0^{\infty} \xi^{2n-j-1} e^{-(\lambda+\mu)\xi} d\xi = \frac{(2n-j-1)!}{(\lambda+\mu)^{2n-j}},$$

so that

$$P\{N_j = n\} = \frac{j}{n} \binom{2n-j-1}{n-j} \frac{\rho^{n-j}}{(1+\rho)^{2n-j}} \qquad (n \geq j),$$

where $\rho = \lambda/\mu$. $\square$

Chapter 5, Exercise 19

'The "polite" customer.'

Throughout in parts a–d an M/G/1 queue is assumed. By a polite customer is meant a customer who declines to enter service when any other customer is present in the queue. His equilibrium waiting time is denoted by W_p.

$\boxed{a}$ By definition, a polite customer who arrives while the server is busy will not enter service until the very end of the busy period that would have been realized without the appearance of the polite customer.

Given Poisson arrivals for all customers, we assume that also the polite customer will arrive at a random time in equilibrium. In particular, in case he arrives while the server is busy, then the arrival takes place at a randomly selected point in time in the renewal process of busy periods, disregarding the idle periods. Accordingly, the waiting time is a residual busy period, whose distribution in equilibrium is given by Eq. (7.9). It follows that

$$P\{W_p \leq x \mid W_p > 0\} = \frac{1}{b}\int_0^x [1 - B(\xi)]\,d\xi, \qquad (1)$$

where $B(t)$ is the distribution function of the busy period, and b is its mean.

$\boxed{b}$ With probability $1-\rho$ the polite customer arrives at an idle server and has waiting time 0. With probability ρ he arrives at a busy server and his conditional waiting time distribution is given by (1). Using that $b = \tau/(1-\rho)$, by Eq. (8.69), we conclude that

$$P\{W_p \leq x\} = (1-\rho) + \frac{\rho(1-\rho)}{\tau}\int_0^x [1 - B(\xi)]\,d\xi. \qquad (2)$$

$\boxed{c}$ If the polite customer has to wait, his waiting time distribution is, by part a, identical to the residual busy period distribution in equilibrium, and so the mean wait equals the mean of the residual busy period given by (7.13). Thus,

(Chap. 5, Ex. 19c)

$$E(W_p \mid W_p > 0) = \frac{b}{2} + \frac{\sigma_B^2}{2b},$$

where σ_B^2 is the variance of the busy period. As $b = \tau/(1-\rho)$,

$$E(W_p \mid W_p > 0) = \frac{\tau}{2(1-\rho)} + \frac{(1-\rho)\sigma_B^2}{2\tau}. \qquad (3)$$

$\boxed{d}$ By (8.70), $E(B^2) = \eta''(0)/(1-\rho)^3$. Now, $\eta''(0)$ equals the second moment of the service time distribution, so we may write $\eta''(0) = \sigma^2 + \tau^2$, where σ^2 is the variance of the service time. Hence,

$$\sigma_B^2 = E(B^2) - b^2 = \frac{\sigma^2+\tau^2}{(1-\rho)^3} - \frac{\tau^2}{(1-\rho)^2}$$
$$= \frac{\sigma^2+\rho\tau^2}{(1-\rho)^3}.$$

Substitution of this expression for σ_B^2 into (3) yields

$$E(W_p \mid W_p > 0) = \frac{\tau^2+\sigma^2}{2\tau(1-\rho)^2}. \qquad (4)$$

$\boxed{e}$ Suppose the polite customer makes his arrival in an M/M/s queue, with arrival rate λ and mean service time μ^{-1}. There he will wait only if all s servers are busy on arrival.

The key observation is that all-servers-busy periods follow precisely the same distribution as does the busy period in the M/M/1 queue with arrival rate λ and mean service time $(s\mu)^{-1}$.

We can conclude that $E(W_p \mid W_p > 0)$ may be derived by the use of Eq. (4), setting $\tau = (s\mu)^{-1}$, $\sigma^2 = \tau^2 = (s\mu)^{-2}$, and $\rho = \lambda/s\mu$. Hence, for the M/M/s queue,

$$E(W_p \mid W_p > 0) = \frac{(s\mu)^{-2} + (s\mu)^{-2}}{2(s\mu)^{-1}(1-\rho)^2},$$

simplifying to

$$E(W_p \mid W_p > 0) = \frac{1}{(1-\rho)^2 s\mu}. \qquad (5)$$

□

Chapter 5, Exercise 20

'Service in reverse order of arrival'

$\boxed{a}$ We consider an arbitrary customer arriving at an M/G/1 queue at a time T_c when the server is busy. Disregarding idle periods, the arrival epoch is a randomly selected point in time in the renewal process where interevent times have probability distribution $H(t)$. Hence, the remaining service time T_1-T_c has the probability distribution function $\tilde{H}(t)$ given by (8.41), by application of (7.9).

If $T_1-T_c=t$, then the number of new arrivals during $[T_c,T_1)$ will follow the Poisson distribution with mean λt. Thus, the joint probability of $T_1-T_c \le x$ and j new arrivals is

$$\tilde{P}_j(x) = \int_0^x \frac{(\lambda\xi)^j}{j!} e^{-\lambda\xi} d\,\tilde{H}(\xi).$$

$\boxed{b}$ Let, as usual, W be the waiting time of an arbitrary customer (the test customer) and let $W(t)$ be the equilibrium waiting time distribution function, given service in reverse order of arrival. Denoting the arrival state by N, clearly

$$W(t) = P\{W\le t\} = P\{N=0\}P\{W\le t|N=0\} + P\{N\ge 1\}P\{W\le t|N\ge 1\}.$$

Given Poisson arrivals, $P\{N\ge 1\}=\rho$. Hence,

$$W(t) = (1-\rho) + \rho\, P\{W\le t|N\ge 1\}.$$

To find $P\{W\le t|N\ge 1\}$, observe that the waiting time is the sum of the remaining service time T_1-T_c and a j-busy period, where j is the number of arrivals during $[T_c,T_1)$, since the test customer must wait until both these j customers and all later arrivals have been served.

Note that for given j, the conditional remaining service time and the j-busy period are independent. It follows that, given $N\ge 1$, the joint probability of j new arrivals and a total wait of less than t equals $\int_0^t \tilde{P}_j(t-x)\,d B_j(x)$, where $B_j(x)$ is the distribution function of the j-busy period. Hence,

$$P\{W\le t|N\ge 1\} = \sum_{j=0}^{\infty} \int_0^t \tilde{P}_j(t-x)\,d B_j(x),$$

(Chap. 5, Ex. 20 b)

so that

$$W(t) = (1-\rho) + \rho \sum_{j=0}^{\infty} \int_0^t \tilde{P}_j(t-x)\, dB_j(x). \tag{1}$$

$\boxed{c}$ Let $\omega(s)$ denote the Laplace-Stieltjes transform of the waiting time distribution function $W(t)$, and let, as before, $\eta(s)$ and $\beta(s)$ be the Laplace-Stieltjes transforms of service time and busy period distribution functions, respectively.

By (1), and part a,

$$\begin{aligned}
\omega(s) &= \int_0^\infty e^{-st} d\,W(t) \\
&= (1-\rho) + \rho \sum_{j=0}^{\infty} \Big(\int_0^\infty e^{-sx} d\tilde{P}_j(x)\Big)\Big(\int_0^\infty e^{-sx} d\,B_j(x)\Big) \\
&= (1-\rho) + \rho \sum_{j=0}^{\infty} \Big(\int_0^\infty e^{-sx} \frac{(\lambda x)^j}{j!} e^{-\lambda x} d\tilde{H}(x)\Big)\, \beta^j(s) \\
&= (1-\rho) + \rho \sum_{j=0}^{\infty} \int_0^\infty e^{-(s+\lambda)x} \frac{(\lambda x \beta(s))^j}{j!} d\tilde{H}(x) \\
&= (1-\rho) + \rho \int_0^\infty e^{-(s+\lambda)x} \Big(\sum_{j=0}^{\infty} \frac{(\lambda x \beta(s))^j}{j!}\Big) d\tilde{H}(x) \\
&= (1-\rho) + \rho \int_0^\infty e^{-(s+\lambda[1-\beta(s)])x} d\tilde{H}(x).
\end{aligned}$$

The last integral is the Laplace-Stieltjes transform of the remaining service time distribution function, evaluated at $s+\lambda[1-\beta(s)]$. According to Eq. (7.10) this transform equals

$$\int_0^\infty e^{-sx} d\tilde{H}(x) = \frac{1}{\tau}\, \frac{1-\eta(s)}{s},$$

where τ is the mean service time. Hence, since $\rho = \lambda\tau$,

$$\omega(s) = (1-\rho) + \lambda\, \frac{1-\eta(s+\lambda-\lambda\beta(s))}{s+\lambda[1-\beta(s)]}. \tag{2}$$

$\boxed{d}$ By (8.67), $\beta(s) = \eta(s+\lambda-\lambda\beta(s))$, so that (2) reduces to

$$\omega(s) = (1-\rho) + \frac{\lambda[1-\beta(s)]}{s+\lambda[1-\beta(s)]}. \tag{3}$$

$\square$

Chapter 5, Exercise 21

'It is required to calculate the arriving customer's equilibrium distribution $\{\Pi_j\}$ for the M/G/1 queue with n waiting positions.'

$\boxed{a}$ Let $\tilde{\Pi}_j^*$ $(j = 0,1,\dots,n)$ denote any unnormalized Π_j^* calculated from (9.1) starting with an arbitrary positive value of Π_0^*, and set

$$d = \tilde{\Pi}_0^* + \tilde{\Pi}_1^* + \dots + \tilde{\Pi}_n^*.$$

By (9.12) and (9.13),

$$\Pi_j = \frac{\Pi_j^*}{\Pi_0^* + a} = \frac{\Pi_j^* d}{\Pi_0^* d + ad} = \frac{\tilde{\Pi}_j^*}{\tilde{\Pi}_0^* + ad} \quad (j = 0,1,\dots,n),$$

$$\Pi_{n+1} = \frac{\Pi_0^* + (a-1)}{\Pi_0^* + a} = \frac{\Pi_0^* d + (a-1)d}{\Pi_0^* d + ad} = \frac{\tilde{\Pi}_0^* + (a-1)d}{\tilde{\Pi}_0^* + ad}.$$

$\boxed{b}$ Suppose that the state distribution $\{\hat{\Pi}_j\}$ for the corresponding infinite-waiting-room queue has been calculated.

By propoprtionality of $\{\hat{\Pi}_j\}$ and $\{\Pi_j^*\}$ for $j = 0,1,\dots,n$, the starting value $\tilde{\Pi}_0^* = \hat{\Pi}_0$ will lead to $\tilde{\Pi}_j^* = \hat{\Pi}_j$ $(j = 0,1,\dots,n)$ and $d = \sum_{\nu=0}^{n} \hat{\Pi}_\nu$. By part a, then,

$$\Pi_j = \frac{\hat{\Pi}_j}{\hat{\Pi}_0 + a\sum_{\nu=0}^{n} \hat{\Pi}_\nu} \quad (j = 0,1,\dots,n),$$

$$\Pi_{n+1} = \frac{\hat{\Pi}_0 + (a-1)\sum_{\nu=0}^{n} \hat{\Pi}_\nu}{\hat{\Pi}_0 + a\sum_{\nu=0}^{n} \hat{\Pi}_\nu}.$$

$\boxed{c}$ For an M/M/1 queue $\hat{\Pi}_j = (1-a)a^j$ $(j = 0,1,\dots)$, see Ex. 4 of Chapter 1. By substitution into the equations of part b:

$$\Pi_j = \frac{(1-a)a^j}{1-a^{n+2}} \quad (j = 0,1,\dots,n+1).$$

The "rate up = rate down" equations are $\lambda P_j = \mu P_{j+1}$ $(j = 0,1,\dots,n)$, by which $\Pi_j = P_j = (1-a)a^j/(1-a^{n+2})$, where $a = \lambda/\mu$, in agreement with the above result. $\square$

Chapter 5, Exercise 22

'A particle-counting device...'

The system may be modeled as a single-server queue with waiting room of size 2, gross arrival rate 3λ, effective arrival rate in state j equal to $\lambda_j = (3-j)\lambda$, constant service time $\tau = 1$. Let

b_j = mean of a j-busy period $(j = 1,2)$,

$p(i|j)$ = probability that i buffers fill (i particles arrive at idle buffers) during a service time, given state j $(j=1,2)$ at the start of service.

Clearly,

$$b_1 = 1 + p(1|1)\,b_1 + p(2|1)\,b_2, \tag{1}$$
$$b_2 = 1 + p(0|2)\,b_1 + p(1|2)\,b_2. \tag{2}$$

Using $p(0|2) + p(1|2) = 1$, Eq. (2) can be written

$$b_2 = \frac{1 + p(0|2)\,b_1}{p(0|2)}. \tag{3}$$

Substitution of (3) into (1) and use of $p(0|1) + p(1|1) + p(2|1) = 1$ give

$$b_1 = \frac{p(0|2) + p(2|1)}{p(0|1)\,p(0|2)}. \tag{4}$$

The probability that an idle buffer will be filled during a service period equals $1 - e^{-\lambda}$. Hence, $p(0|2) = e^{-\lambda}$, $p(2|1) = (1-e^{-\lambda})^2$, and $p(0|1) = (e^{-\lambda})^2$. Substitution into (4) and simplification yield

$$b_1 = e^{3\lambda} + e^{\lambda} - e^{2\lambda} \tag{5}$$

As $\lambda_0 = 3\lambda$, the mean idle period equals $(3\lambda)^{-1}$. Hence, in analogy with (9.14), the carried load is given by

$$a' = \frac{b_1}{(3\lambda)^{-1} + b_1}. \tag{6}$$

The offered load is

$$a = (3\lambda)\tau = 3\lambda. \tag{7}$$

By (5), (6) and (7),

$$p = 1 - \frac{a'}{a} = 1 - \frac{e^{3\lambda} + e^{\lambda} - e^{2\lambda}}{1 + 3\lambda(e^{3\lambda} + e^{\lambda} - e^{2\lambda})}. \qquad \square$$

Chapter 5, Exercise 23

'Let $B(j,k)$ be the duration of the j-busy period in the M/G/1 queue with $j+k-1$ waiting positions ...'

Observe that the system may hold altogether $j+k$ customers. Let $E[B(j,k)] = b(j,k)$, and let $P(j,k)$ be the probability that throughout the busy period there will always be at least one unoccupied waiting position.

a Suppose service begins when there are j customers in the system. We may assume that customers are served in reverse order-of-arrival. Initially, we decompose the j-busy period into two independent time intervals. The first interval is the time needed to reduce the state from j to $j-1$. This time interval is distributed as $B(1,k)$. The second interval is the time needed to reduce the state from $j-1$ to 0, so this time interval is distributed as $B(j-1,k+1)$ assuming $j \geq 2$. Hence,

$$B(j,k) = B(1,k) + B(j-1,k+1) \quad (j \geq 2), \tag{1}$$

so that

$$B(j,k) = \sum_{i=k}^{j+k-1} B(1,i). \tag{2}$$

b Taking means in (2) we obtain

$$b(j,k) = \sum_{i=k}^{j+k-1} b(1,i). \tag{3}$$

We now decompose the j-busy period in a different way. Imagine that the first customer, C, if any, who fills up the queue will not enter service until there are no other waiting customers. Then the time until C, should he exist, will get served is distributed as $B(j,k-1)$, assuming $k \geq 1$. With probability $1-P(j,k)$, C will arrive during the busy period and thus generate a 1-busy period distributed as $B(1,j+k-1)$.

We conclude that

$$B(j,k) = B(j,k-1) + I \ast B(1,j+k-1) \quad (k \geq 1), \tag{4}$$

(Chap. 5, Ex. 23 b)

where $I = 0$ if C does not arrive and $I = 1$ if he does. Note that I and $B(1, j+k-1)$ are independent variables. Taking means in (4) we derive

$$b(j,k) = b(j,k-1) + [1 - P(j,k)]\, b(1, j+k-1) \qquad (k \geq 1). \qquad (5)$$

$\boxed{c}$ Writing $b(j,k)$ and $b(j,k-1)$ as sums of mean 1-busy periods, by the use of (3), Eq. (5) yields

$$P(j,k) = \frac{b(1,k-1)}{b(1,j+k-1)} \qquad (k \geq 1). \qquad (6)$$

$\boxed{d}$ Now assume exponential service times with mean μ^{-1}, and let $a = \lambda/\mu$. In this case the mean busy periods are:

$$M/M/1/n: \qquad b(1,n) = \frac{1}{\mu}\sum_{i=0}^{n} a^i \qquad (n \geq 0). \qquad (7)$$

This formula might be derived from (9.18). Alternatively, one can use the relation $b(1,n)/\lambda^{-1} = (1-P_0)/P_0$ (compare with Eq. (4.10) of Chapter 3) plus the fact that $P_0 = 1/\sum_{i=0}^{n+1} a^i$. By (6) and (7),

$$M/M/1/j+k-1: \qquad P(j,k) = \frac{\sum_{i=0}^{k-1} a^i}{\sum_{i=0}^{j+k-1} a^i} \qquad (k \geq 1). \qquad (8)$$

$\boxed{e}$ Define $P(j,k,a) = P(j,k)$, and let $P'(j,k,p)$ denote the gamblers ruin probability in a game where he starts with j units, the adversary starts with k units, and the probability of his winning 1 unit is p in each trial (thus the adversary's winning probability is $q = 1-p$). We shall show that

$$P'(j,k,p) = P(j,k,p/q). \qquad (9)$$

Consider an M/M/1 queue with $j+k$ positions (incl. service) where $j \geq 1$, $k \geq 1$, and j is the initial state. In state i, $1 \leq i \leq j+k-1$, the transition probabilities are $P\{i \to i+1\} = \lambda/(\lambda+\mu) = a/(1+a) = p$, $P\{i \to i-1\} = 1-p = q$. One sees that the imbedded Markov chain (arrivals and departures) of the state variable i, exactly simulates the game as described if λ and μ satisfy $\lambda/(\lambda+\mu) = p$. Eq. (9) follows. □

Chapter 5, Exercise 24

'Consider the equilibrium M/G/1 queue with batch arrivals...'

W_1 = time from arrival to start of service of the test customer's batch
W_2 = remaining time until start of service of the test customer

Let $\omega_1(s)$ and $\omega_2(s)$ denote the Laplace-Stieltjes transforms of W_1 and W_2, respectively. As W_1 and W_2 are independent, $W = W_1 + W_2$ has Laplace-Stieltjes transform

$$\hat{\omega}(s) = \omega_1(s)\,\omega_2(s) \tag{1}$$

Derivation of $\omega_1(s)$

Let τ_B be the mean and let $\eta_B(s)$ be the Laplace-Stieltjes transform of the batch service time. By (8.38),

$$\omega_1(s) = \frac{s(1-\lambda\tau_B)}{s-\lambda[1-\eta_B(s)]}\,.$$

By Exercise 5, $\tau_B = m\tau$ and $\eta_B(s) = g(\eta(s))$, where τ and $\eta(s)$ are mean and Laplace-Stieltjes transform of the individual service times, and m and $g(z)$ are mean and probability generating function of the number of customers in a batch.
Thus,

$$\omega_1(s) = \frac{s(1-\lambda m\tau)}{s-\lambda[1-g(\eta(s))]}\,. \tag{2}$$

Derivation of $\omega_2(s)$

Let N and N' be, respectively, the size of an arbitrary batch and a test customer's batch, and let N'' be the number of batch customers served ahead of the test customer. We may assume that the customers in a batch are served in random order, but batches must be served in order of arrival.
Clearly,

$$P\{N''=k\} = \sum_{j=k+1}^{\infty} \frac{1}{j}\,P\{N'=j\} \qquad (k=0,1,\ldots).$$

Substitution therein of $P\{N'=j\} = j\,P\{N=j\}/m$, by (10.6), results in

(Chap. 5, Ex. 24)

$$P\{N''=k\} = \frac{1}{m}\sum_{j=k+1}^{\infty} P\{N=j\} \qquad (k=0,1,\ldots).$$

Denoting by $h(z)$ the probability generating function of N'', we find

$$\begin{aligned} h(z) &= \sum_{k=0}^{\infty} P\{N''=k\} z^k \\ &= \sum_{k=0}^{\infty} \frac{1}{m}\sum_{j=k+1}^{\infty} P\{N=j\} z^k \\ &= \frac{1}{m}\sum_{k=0}^{\infty} z^k - \frac{1}{m}\sum_{k=0}^{\infty}\sum_{j=0}^{k} P\{N=j\} z^k \\ &= \frac{1}{m}\sum_{k=0}^{\infty} z^k - \frac{1}{m}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty} P\{N=j\} z^k \\ &= \frac{1}{m}\sum_{k=0}^{\infty} z^k - \frac{1}{m}\sum_{j=0}^{\infty} P\{N=j\} z^j \sum_{k=0}^{\infty} z^k \\ &= \frac{1}{m}\Big[1-\sum_{j=0}^{\infty} P\{N=j\} z^j\Big]\sum_{k=0}^{\infty} z^k, \end{aligned}$$

whereby

$$h(z) = \frac{1-g(z)}{m[1-z]}. \tag{3}$$

Now, W_2 is the sum of N'' independent service times, each with Laplace-Stieltjes transform $\eta(s)$. By Exercise 5,

$$\omega_2(s) = h(\eta(s)). \tag{4}$$

By (3) and (4),

$$\omega_2(s) = \frac{1-g(\eta(s))}{m[1-\eta(s)]}. \tag{5}$$

Finally, combining (1), (2) and (5), we obtain the Laplace-Stieltjes transform of an arbitrary customer's total waiting time W,

$$\hat{\omega}(s) = \frac{s(1-\lambda m\tau)}{s-\lambda[1-g(\eta(s))]}\,\frac{1-g(\eta(s))}{m[1-\eta(s)]}.$$

$\square$

Chapter 5, Exercise 25

'Let c be the minimum mean operating cost per unit time ...'

We assume an M/G/1 queue and the choice between continuous operation and some N-policy. As a consequence of the preceding analysis we distinguish between two cases as follows.

Case 1: $c_0 < \lambda c_N(n^*)/(1-\rho)$

It is known already that in this case c is minimized by continuous operation. Per unit time, the three cost elements are

$$\begin{aligned} \text{running cost} &= c_0, \\ \text{switching cost} &= 0, \\ \text{holding cost} &= \lambda c_2 E(X). \end{aligned}$$

Now, $E(X) = \tau + E(W)$, and the mean waiting time $E(W)$ is given by the Pollaczek-Khintchine formula (8.39). Thus $c = c_0 + \lambda c_2 E(X)$ becomes

$$c = c_0 + c_2\left[\rho + \frac{\rho^2}{2(1-\rho)}\left(1+\frac{\sigma^2}{\tau^2}\right)\right] \qquad \left(c_0 < \frac{\lambda c_N(n^*)}{1-\rho}\right). \qquad (*)$$

Case 2: $c_0 > \lambda c_N(n^*)/(1-\rho)$

In this case c is minimized by choosing an N-policy with parameter $n = n^*$. The calculation of the corresponding cost $c_N(n^*)$, namely the minimal variable cost per customer, has been described previously. The minimal variable cost per unit time is seen to equal $\lambda c_N(n^*)$. c is obtained by adding those fixed costs (independent of n) that were not taken into consideration in calculating n^*.

First, there is a running cost per unit time equal to $c_0\rho$, since, for any n, the server will be busy (= on) the fraction ρ of the time. Second, there is the holding cost $\lambda c_2 E(X)$ of a system where the server is on whenever a customer is present. Thus, $c = c_0\rho + \lambda c_N(n^*) + \lambda c_2 E(X)$, which becomes

$$c = c_0\rho + \lambda c_N(n^*) + c_2\left[\rho + \frac{\rho^2}{2(1-\rho)}\left(1+\frac{\sigma^2}{\tau^2}\right)\right] \quad \left(c_0 > \frac{\lambda c_N(n^*)}{1-\rho}\right). \quad (**)$$

(Chap. 5, Ex. 25)

Examples

In every case, $\lambda = \frac{1}{2}$, $\tau = 1$ (so that $\rho = \frac{1}{2}$), $c_1 = 12$ and $c_2 = 2$. Hence, $n^* = 2$, $c_N(n^*) = 5$, and the borderline value for c_0 is $\hat{c}_0 = \lambda c_N(n^*)/(1-\rho) = 5$.

(a) $c_0 = 4$. As $c_0 < \hat{c}_0$, c is minimized by a do-nothing policy. Thus, Eq. (*) applies. If service times are exponentially distributed, then $\sigma^2 = \tau^2 = 1$ and, by (*), $c = 6$. If service times are constant, then $\sigma^2 = 0$ and, by (*), $c = 5\frac{1}{2}$.

(b) $c_0 = 6$. As $c_0 > \hat{c}_0$, c is minimized by an N-policy with $n^* = 2$. Thus, Eq. (**) applies. If service times are exponentially distributed, then $\sigma^2 = \tau^2 = 1$ and, by (**), $c = 7\frac{1}{2}$. If service times are constant, then $\sigma^2 = 0$ and, by (**), $c = 7$.

Chapter 5, Exercise 26

'Consider the M/G/1 queue operating under a T-policy, with parameter t'

a) The probability that no customer will arrive during a vacation of length t is $P(0) = e^{-\lambda t}$. Hence, with Y being the consecutive number of vacations with no arrivals, $P\{Y = i\} = (e^{-\lambda t})^n(1-e^{-\lambda t})$, whereby

$$E(Y) = \sum_{i=0}^{\infty} i\, P\{Y = i\} = \frac{e^{-\lambda t}}{1 - e^{-\lambda t}}.$$

b) Given Poisson traffic, the number of arrivals during a vacation has the Poisson distribution with mean λt. Hence,

$$f_T(z) = \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} z^j = e^{-(1-z)\lambda t}.$$

c)

$$f_T'(z) = \lambda t e^{-(1-z)\lambda t}, \quad f_T''(z) = \lambda^2 t^2 e^{-(1-z)\lambda t},$$
$$f_T'(1) = \lambda t \quad , \quad f_T''(1) = \lambda^2 t^2.$$

If in (11.14) we substitute $P(0) = e^{-\lambda t}$ and the above expressions for $E(Y)$, $f_T'(1)$ and $f_T''(1)$, we obtain (11.24). □

Chapter 5, Exercise 27

'The Maclaurin series method for the M/G/1 random service queue.'

$$F(t) \equiv P\{W>t \mid W>0\} = 1 - \tilde{H}(t) + \sum_{j=2}^{\infty} \frac{j-1}{j} \int_0^t \breve{W}_j(t-\xi) Q_j'(\xi)\, d\xi. \qquad (12.9)$$

$$\breve{W}_j(x) = 1 - H(x) + \sum_{i=0}^{\infty} \frac{j+i-2}{j+i-1} \int_0^x p_i(\xi) \breve{W}_{j+i-1}(x-\xi)\, dH(\xi) \quad (j=2,3,\ldots). \qquad (12.10)$$

Assume that $\breve{W}_j(x)$ has the Maclaurin series expansion

$$\breve{W}_j(x) = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!} \breve{W}_j^{(\nu)} \qquad (j = 2,3,\ldots;\ \breve{W}_j^{(0)} = 1). \qquad (1)$$

Suppose $H(x)$ is continuous and differentiable, and set

$$h(x) = \frac{d}{dx} H(x), \qquad (2)$$

and let

$$\breve{b}_i(x) = h(x)\, p_i(x). \qquad (3)$$

Also, for any function $f(x)$, define $f^{(k)} = \left(\frac{d^k}{dx^k} f(x)\right)_{x=0}$.

$\boxed{a}$ By (12.10), (2) and (3),

$$\breve{W}_j(x) = 1 - H(x) + \sum_{i=0}^{\infty} \frac{j+i-2}{j+i-1} \int_0^x \breve{b}_i^{(0)}(\xi) \breve{W}_{j+i-1}^{(0)}(x-\xi)\, d\xi \qquad (j = 2,3,\ldots).$$

Repeated differentiation w.r.t. x yields

$$\breve{W}_j^{(\nu)}(x) = -H^{(\nu)}(x) + \sum_{i=0}^{\infty} \frac{j+i-2}{j+i-1} \left[\sum_{k=0}^{\nu-1} \breve{b}_i^{(k)}(x) \breve{W}_{j+i-1}^{(\nu-1-k)}(0) + \int_0^x \breve{b}_i^{(0)}(\xi) \breve{W}_{j+i-1}^{(\nu)}(x-\xi)\, d\xi \right]$$

for $j = 2,3,\ldots;\ \nu = 1,2,\ldots$.
Hence,

$$\breve{W}_j^{(\nu)} = -H^{(\nu)} + \sum_{i=0}^{\infty} \frac{j+i-2}{j+i-1} \sum_{k=0}^{\nu-1} \breve{b}_i^{(k)} \breve{W}_{j+i-1}^{(\nu-1-k)} \qquad \binom{j=2,3,\ldots;}{\nu=1,2,\ldots}.$$

It is easy to show that $p_i^{(k)} = 0$ for $i > k$. Hence, $\breve{b}_i^{(k)} = 0$ for $i > k$, so that the above equation simplifies to

$$\breve{W}_j^{(\nu)} = -H^{(\nu)} + \sum_{i=0}^{\nu-1} \frac{j+i-2}{j+i-1} \sum_{k=i}^{\nu-1} \breve{b}_i^{(k)} \breve{W}_{j+i-1}^{(\nu-1-k)} \qquad \binom{j=2,3,\ldots;}{\nu=1,2,\ldots}. \qquad (4)$$

(Chap. 5, Ex. 27 b)

$\boxed{b}$ Defining $F(t) = P\{W > t \mid W > 0\}$ we assume the Maclaurin series expansion

$$F(t) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} F^{(\nu)} \qquad (F^{(0)} = 1). \tag{5}$$

Using our present notation, (12.9) is written

$$F(t) = 1 - \tilde{H}(t) + \sum_{j=2}^{\infty} \frac{j-1}{j} \int_0^t \breve{W}_j^{(0)}(t-\xi) Q_j^{(1)}(\xi)\, d\xi .$$

Repeated differentiation w.r.t. t yields

$$F^{(\nu)}(t) = -\tilde{H}^{(\nu)}(t) + \sum_{j=2}^{\infty} \frac{j-1}{j} \left[\sum_{k=0}^{\nu-1} \breve{W}_j^{(\nu-1-k)}(0) Q_j^{(k+1)}(t) + \int_0^t \breve{W}_j^{(\nu)}(t-\xi) Q_j^{(1)}(\xi)\, d\xi \right]$$

for $\nu = 1, 2, \ldots$. Hence,

$$F^{(\nu)} = -\tilde{H}^{(\nu)} + \sum_{j=2}^{\infty} \frac{j-1}{j} \sum_{k=0}^{\nu-1} Q_j^{(k+1)} \breve{W}_j^{(\nu-1-k)} \qquad (\nu = 1, 2, \ldots). \tag{6}$$

$\boxed{c}$ $$Q_j^{(1)}(\xi) = \frac{1}{\tau} \int_{\xi}^{\infty} \left(\Pi_0^* P_{j-1}(x) + \sum_{i=1}^{j} \Pi_i^* P_{j-i}(x) \right) dH(x) \qquad (j = 1, 2, \ldots). \tag{12.7}$$

Hence,

$$Q_j^{(1)} = \frac{1}{\tau} \int_0^{\infty} \left(\Pi_0^* P_{j-1}(x) + \sum_{i=1}^{j} \Pi_i^* P_{j-i}(x) \right) dH(x) \qquad (j = 1, 2, \ldots).$$

The integrand (in parentheses) is the equilibrium probability of departure state $j-1$, given service time x. Thus, the integration results in the unconditional probability of departure state $j-1$, that is, Π_{j-1}^*. We conclude that

$$Q_j^{(1)} = \frac{1}{\tau} \Pi_{j-1}^* \qquad (j = 1, 2, \ldots) \tag{7}$$

Repeated differentiation of (12.7) w.r.t. ξ yields

$$Q_j^{(k+1)}(\xi) = -\frac{1}{\tau} \sum_{m=0}^{k-1} \binom{k-1}{m} H^{(k-m)}(\xi) \left(\Pi_0^* P_{j-1}^{(m)}(\xi) + \sum_{i=1}^{j} \Pi_i^* P_{j-i}^{(m)}(\xi) \right) \quad \left(\begin{matrix} k = 1, 2, \ldots; \\ j = 1, 2, \ldots \end{matrix} \right),$$

whereby

$$Q_j^{(k+1)} = -\frac{1}{\tau} \sum_{m=0}^{k-1} \binom{k-1}{m} H^{(k-m)} \left(\Pi_0^* P_{j-1}^{(m)} + \sum_{i=1}^{j} \Pi_i^* P_{j-i}^{(m)} \right) \quad \left(\begin{matrix} k = 1, 2, \ldots; \\ j = 1, 2, \ldots \end{matrix} \right) \tag{8}$$

□

Chapter 5, Exercise 28

(Carter and Cooper [1972]) – cf. Ex. 32 of Chap. 3

The exercise is an application of the results of Exercise 27 on the M/M/1 random service queue. Thus we assume

$$H(x) = 1 - e^{-x/\tau}. \qquad (1)$$

Clearly, $\tilde{H}(x) = H(x) = 1 - e^{-x/\tau}$, whereby

$$\tilde{H}^{(1)}(x) = \frac{1}{\tau} e^{-x/\tau}, \qquad (*)$$

$$\tilde{H}^{(2)}(x) = -\frac{1}{\tau^2} e^{-x/\tau}. \qquad (**)$$

$\boxed{a}$ By Eq. (6) of Exercise 27,

$$F^{(1)} = -\tilde{H}^{(1)} + \sum_{j=2}^{\infty} \frac{j-1}{j} Q_j^{(1)} \breve{W}_j^{(0)}.$$

By $(*)$ and the fact that $\breve{W}_j^{(0)} = \breve{W}_j(0) = 1$ for $j \geq 2$, then,

$$F^{(1)} = -\frac{1}{\tau} + \sum_{j=2}^{\infty} \frac{j-1}{j} Q_j^{(1)}. \qquad (2)$$

$\boxed{b}$ By Eq. (7) of Exercise 27, $Q_j^{(1)} = \frac{1}{\tau} \Pi_{j-1}^{*}$ $(j = 1, 2, \ldots)$. For an M/M/1 queue, $\Pi_j^{*} = P_j = (1-\rho)\rho^j$, where $\rho = \lambda\tau$. Thus

$$Q_j^{(1)} = \frac{1}{\tau}(1-\rho)\rho^{j-1} \qquad (j = 1, 2, \ldots). \qquad (***)$$

By (2) and $(***)$,

$$\begin{aligned} F^{(1)} &= -\frac{1}{\tau} + \frac{1}{\tau}\frac{1-\rho}{\rho} \sum_{j=1}^{\infty} \frac{j-1}{j} \rho^j \\ &= -\frac{1}{\tau} + \frac{1}{\tau}\frac{1-\rho}{\rho} \sum_{j=1}^{\infty} \rho^j - \frac{1}{\tau}\frac{1-\rho}{\rho} \sum_{j=1}^{\infty} \frac{\rho^j}{j} \\ &= -\frac{1}{\tau}\frac{1-\rho}{\rho} \sum_{j=1}^{\infty} \frac{\rho^j}{j}. \end{aligned}$$

Hence,

$$F^{(1)} = -\frac{1}{\tau}\frac{1-\rho}{\rho} \ln \frac{1}{1-\rho}. \qquad (3)$$

(Chap. 5, Ex. 28 c)

$\boxed{c}$ By Eq. (6) of Exercise 27,

$$F^{(2)} = -\widetilde{H}^{(2)} + \sum_{j=2}^{\infty} \frac{j-1}{j}\left(Q_j^{(1)}\breve{W}_j^{(1)} + Q_j^{(2)}\breve{W}_j^{(0)}\right).$$

By (* *) and the fact that $\breve{W}_j^{(0)} = \breve{W}_j(0) = 1$ for $j \geq 2$, then,

$$F^{(2)} = \frac{1}{\tau^2} + \sum_{j=2}^{\infty} \frac{j-1}{j}\left(Q_j^{(1)}\breve{W}_j^{(1)} + Q_j^{(2)}\right) \tag{4}$$

$\boxed{d}$ By Eq. (8) of Exercise 27, where $\binom{0}{0} = 1$, $p_{j-1}^{(0)} = p_{j-1}(0)$, $p_{j-i}^{(0)} = p_{j-i}(0)$,

$$Q_j^{(2)} = -\frac{1}{\tau} H^{(1)}\left(\Pi_0^* p_{j-1}(0) + \sum_{i=1}^{j} \Pi_i^* p_{j-i}(0)\right) \quad (j=1,2,\ldots). \tag{5}$$

As $p_i(x) = [(\lambda x)^i/i!]e^{-\lambda x}$, we have $p_0(0) = 1$ and $p_i(0) = 0$ for $i \geq 1$. Substitution of these values for $p_{j-1}(0)$ and $p_{j-i}(0)$ as well as $H^{(1)} = \frac{1}{\tau}$ into (5), but only for $j = 2,3,\ldots$, we obtain

$$Q_j^{(2)} = -\frac{1}{\tau^2}\Pi_j^* \qquad (j = 2,3,\ldots).$$

Finally, since $\Pi_j^* = P_j = (1-\rho)\rho^j$,

$$Q_j^{(2)} = -\frac{1}{\tau^2}(1-\rho)\rho^j \qquad (j = 2,3,\ldots). \tag{6}$$

$\boxed{e}$ By Eq. (4) of Exercise 27,

$$\breve{W}_j^{(1)} = -H^{(1)} + \frac{j-2}{j-1}\breve{b}_0^{(0)}\breve{W}_{j-1}^{(0)} \qquad (j = 2,3,\ldots).$$

By definition $\breve{b}_0^{(0)} = \breve{b}_0(0) = H^{(1)}(0)\,p_0(0)$. As $H^{(1)}(0) = H^{(1)} = \frac{1}{\tau}$ and $p_0(0) = 1$, we have $\breve{b}_0^{(0)} = \frac{1}{\tau}$. Also, $\breve{W}_1^{(0)} = \breve{W}_1(0) = 0$, and $\breve{W}_j^{(0)} = \breve{W}_j(0) = 1$ for $j \geq 2$. We conclude that

$$\breve{W}_j^{(1)} = \begin{cases} -\frac{1}{\tau} & (j = 2), \\ -\frac{1}{\tau} + \frac{j-2}{j-1}\frac{1}{\tau} & (j \geq 3), \end{cases}$$

or,

$$\breve{W}_j^{(1)} = -\frac{1}{\tau}\frac{1}{j-1} \qquad (j = 2,3,\ldots). \tag{7}$$

(Chap. 5, Ex. 28 f)

$\boxed{f}$ Substitution into (4) of the expressions that have been derived for $Q_j^{(1)}$, $Q_j^{(2)}$ and $\tilde{W}_j^{(1)}$, and subsequent reduction, give

$$F^{(2)} = \frac{1}{\tau^2} + \sum_{j=2}^{\infty} \frac{j-1}{j}\left([\tfrac{1}{\tau}(1-\rho)\rho^{j-1}][-\tfrac{1}{\tau}\tfrac{1}{j-1}] + [-\tfrac{1}{\tau^2}(1-\rho)\rho^j] \right)$$

$$= \frac{1}{\tau^2}\left[1 - \frac{1-\rho}{\rho}\sum_{j=2}^{\infty}\frac{\rho^j}{j} - (1-\rho)\sum_{j=2}^{\infty}\frac{j-1}{j}\rho^j\right]$$

$$= \frac{1}{\tau^2}\left[1 - \frac{1-\rho}{\rho}\left(\sum_{j=1}^{\infty}\frac{\rho^j}{j} - \rho\right) - (1-\rho)\left(\sum_{j=1}^{\infty}\rho^j - \sum_{j=1}^{\infty}\frac{\rho^j}{j}\right)\right]$$

$$= \frac{1}{\tau^2}\left[1 - \frac{1-\rho}{\rho}\sum_{j=1}^{\infty}\frac{\rho^j}{j} + (1-\rho) - \rho + (1-\rho)\sum_{j=1}^{\infty}\frac{\rho^j}{j}\right]$$

$$= \frac{1}{\tau^2}(1-\rho)\left[2 - \frac{1-\rho}{\rho}\sum_{j=1}^{\infty}\frac{\rho^j}{j}\right].$$

Thus, as expected,

$$F^{(2)} = \frac{1}{\tau^2}(1-\rho)\left[2 - \frac{1-\rho}{\rho}\ln\frac{1}{1-\rho}\right]. \qquad (8)$$

$\boxed{g}$ Also Eq. (6) of Exercise 32 of Chapter 3 expresses the conditional waiting time distribution function in terms of a Maclaurin series expansion, but for an M/M/s random service queue. For s = 1 the formula specializes to

$$P\{W > t \mid W > 0\} = 1 + tF^{(1)} + \frac{t^2}{2!}F^{(2)} + \cdots$$

where

$$F^{(1)} = -\frac{1}{\tau}\frac{1-\rho}{\rho}\ln\frac{1}{1-\rho},$$

$$F^{(2)} = \frac{1}{\tau^2}(1-\rho)\left[2 - \frac{1-\rho}{\rho}\ln\frac{1}{1-\rho}\right],$$

in complete agreement with Eqs. (3) and (8). $\square$

Chapter 5, Exercise 29

'The additional-conditioning-variable method for the M/D/1 random-service-queue (Carter and Cooper [1972].'

We assume a service time equal to the constant τ, i.e.

$$H(x) = \begin{cases} 0 & \text{when } x < \tau, \\ 1 & \text{when } x \geq \tau. \end{cases} \tag{1}$$

$\boxed{a}$ By (1), $dH(\tau) = 1$ and $dH(x) = 0$ for $x \neq \tau$. Inserting this into Eq. (12.7) we find that for $\xi > \tau$ is $Q_j'(\xi) = 0$ whereas for $\xi \leq \tau$,

$$Q_j'(\xi) = \frac{1}{\tau}\int_\xi^\infty \Big(\Pi_0^* p_{j-1}(x) + \sum_{i=1}^{j} \Pi_i^* p_{j-i}(x)\Big)\, dH(x)$$
$$= \frac{1}{\tau}\Big(\Pi_0^* p_{j-1}(\tau) + \sum_{i=1}^{j} \Pi_i^* p_{j-i}(\tau)\Big).$$

We conclude that

$$Q_j'(\xi) = \begin{cases} \frac{1}{\tau}\Pi_{j-1}^* & \text{when } \xi \leq \tau, \\ 0 & \text{when } \xi > \tau. \end{cases} \tag{2}$$

$\boxed{b}$ It is obvious, and follows also from Eqs. (12.2) and (1), that remaining service time has distribution function

$$\tilde{H}(t) = \begin{cases} \frac{t}{\tau} & \text{when } t < \tau, \\ 1 & \text{when } t \geq \tau. \end{cases}$$

According to (12.9),

$$P\{W>t \mid W>0\} = 1 - \tilde{H}(t) + \sum_{j=2}^{\infty} \frac{j-1}{j} \int_0^t \breve{W}_j(t-\xi)\, Q_j'(\xi)\, d\xi\,.$$

Substitution of the above expressions for $\tilde{H}(t)$ and $Q_j'(\xi)$ yields

$$P\{W>t \mid W>0\} = 1 - \frac{t}{\tau} + \frac{1}{\tau}\sum_{j=2}^{\infty} \frac{j-1}{j}\Pi_{j-1}^* \int_0^t \breve{W}_j(t-\xi)\, d\xi \quad (0 \leq t < \tau), \tag{3}$$

$$P\{W>t \mid W>0\} = \frac{1}{\tau}\sum_{j=2}^{\infty} \frac{j-1}{j}\Pi_{j-1}^* \int_0^\tau \breve{W}_j(t-\xi)\, d\xi \quad (\tau \leq t < \infty). \tag{4}$$

(Chap. 5, Ex. 29 c)

$\boxed{c}$ Evidently, the waiting time from T_1 and on is a multiple of τ. Hence, $\breve{W}_j(x)$ is constant on each of the intervals $n\tau \leqq x < (n+1)\tau$, $n = 0,1,2,\dots$. In particular, $\breve{W}_j(x) = 1$ for $0 \leqq x < \tau$. Hence,

$$\int_0^t \breve{W}_j(t-\xi)\,d\xi = t \qquad (0 \leqq t < \tau;\ j = 2,3,\dots).$$

so that (3) becomes

$$P\{W>t \mid W>0\} = 1 - \frac{t}{\tau} + \frac{t}{\tau}\sum_{j=2}^{\infty} \frac{j-1}{j}\,\Pi_{j-1}^{*} \qquad (0 \leqq t < \tau) \qquad (5)$$

For $t \geqq \tau$ we have, for each $j \geqq 2$,

$$\int_0^{\tau} \breve{W}_j(t-\xi)\,d\xi = \int_{[\frac{t}{\tau}]\tau}^{t} \breve{W}_j(x)\,dx + \int_{t-\tau}^{[\frac{t}{\tau}]\tau} \breve{W}_j(x)\,dx.$$

By constancy of $\breve{W}_j(x)$ on the intervals $n\tau \leqq x < (n+1)\tau$,

$$\int_0^{\tau} \breve{W}_j(t-\xi)\,d\xi = \left(t - [\tfrac{t}{\tau}]\tau\right)\breve{W}_j(t) + \left([\tfrac{t}{\tau}]\tau - (t-\tau)\right)\breve{W}_j(t-\tau)$$
$$(t \geqq \tau;\ j = 2,3,\dots).$$

Hence, (4) becomes

$$P\{W>t \mid W>0\} = \frac{1}{\tau}\sum_{j=2}^{\infty} \frac{j-1}{j}\,\Pi_{j-1}^{*}\left\{\left(t-[\tfrac{t}{\tau}]\tau\right)\breve{W}_j(t) + \left([\tfrac{t}{\tau}]\tau - (t-\tau)\right)\breve{W}_j(t-\tau)\right\}$$
$$(\tau \leqq t < \infty). \qquad (6)$$

$\boxed{d}$ Since the service time S_1 is a constant τ, conditional and unconditional distributions of S_1 are identical. By definition of $H(\xi \mid k,x)$ therefore

$$H(\xi \mid k,x) = H(\xi). \qquad (7)$$

$\boxed{e}$ By (1) and (7), $d_\xi H(\xi \mid k,x) = 1$ for $\xi = \tau$, and $d_\xi H(\xi \mid k,x) = 0$ for $\xi \neq \tau$. Consequently, as $k>1$ implies $x \geqq \tau$, Eq. (12.17) reduces to

$$\breve{W}_{j,k}(x) = \sum_{i=0}^{\infty} \frac{j+i-2}{j+i-1}\, p_i(\tau)\, \breve{W}_{j+i-1,k-1}(x-\tau) \qquad \left(\begin{smallmatrix} j = 2,3,\dots; \\ k = 1,2,\dots \end{smallmatrix}\right). \qquad (8)$$

(Chap. 5, Ex. 29 f)

$\boxed{f}$ For constant service time $= \tau$, for all x,

$$\{\check{X}(x) = k\} \Leftrightarrow \{k\tau \leq x < (k+1)\tau\} \Leftrightarrow k = [\tfrac{x}{\tau}].$$

Hence,

$$P\{\check{X}(x) = k\} = \begin{cases} 1 & \text{when } k = [\frac{x}{\tau}], \\ 0 & \text{otherwise.} \end{cases} \qquad (9)$$

$\boxed{g}$ According to (12.13),

$$\check{W}_j(x) = \sum_{k=0}^{\infty} \check{W}_{j,k}(x)\, P\{\check{X}(x) = k\} \qquad (j = 2,3,\ldots).$$

Using (9) we find

$$\check{W}_j(x) = \check{W}_{j,[\frac{x}{\tau}]}(x) \qquad (j = 2,3,\ldots). \qquad (10)$$

$\boxed{h}$ By setting $k = [\frac{x}{\tau}]$ and $k-1 = [\frac{x-\tau}{\tau}]$ in Eq. (8) is obtained

$$\check{W}_{j,[\frac{x}{\tau}]}(x) = \sum_{i=0}^{\infty} \frac{j+i-2}{j+i-1}\, p_i(\tau)\, \check{W}_{j+i-1,[\frac{x-\tau}{\tau}]}(x-\tau) \qquad \left(\begin{smallmatrix} j=2,3,\ldots \\ x \geq \tau \end{smallmatrix}\right).$$

The application of (10) leads to

$$\check{W}_j(x) = \sum_{i=0}^{\infty} \frac{j+i-2}{j+i-1}\, p_i(\tau)\, \check{W}_{j+i-1}(x-\tau) \qquad (j = 2,3,\ldots;\ x \geq \tau). \qquad (11)$$

$\boxed{i}$ It is evident that at the start of service at T_1, any waiting customer will have to wait at least τ time units. That is,

$$\check{W}_j(x) = 1 \qquad (j = 2,3,\ldots;\ x < \tau). \qquad (12)$$

Equations (11) and (12) give $\check{W}_j(x)$ for $j = 2,3,\ldots$ and all x.

$\boxed{j}$ For $x \geq \tau$ we have $H(x) = 1$. Furthermore, $dH(\xi) = 1$ for $\xi = \tau$, and $dH(\xi) = 0$ for $\xi \neq \tau$. Substitution of these values into Eq. (12.10) also results in Eq. (11).

□

Chapter 5, Exercise 30

'The M/G/1 queue with gating.'

We shall not give all the details of the proof since it is precisely as the proof of (13.14) for the cyclic queue except that $H(x)$ replaces $B(x)$ and $\eta(s)$ replaces $\beta(s)$.

The explanation for this analogy is simple enough. In both cases we seek the mean, $\bar{n}$, of the state distribution of an imbedded Markov chain. Let j be the state variable. If $j \geq 1$ let $\hat{j} = j$, and if $j = 0$ let $\hat{j} = 1$. When $j \geq 1$, service begins (is resumed) right away, but when $j = 0$ service will not begin until a customer arrives. Denote by t the time from start of service until next epoch of the imbedded Markov chain. In the case of the cyclic queue, t is a $\hat{j}$-busy period; in the case of the queue with gating, t is the sum of $\hat{j}$ service times. In either case the state at next epoch will be the number of customers arriving during the mentioned interval of length t, either at the other queue (cyclic queue) or at the same queue (queue with gating). For the M/G/1 queue with gating,

$$P(k) = \sum_{j=1}^{\infty} P(j) \int_0^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} dH^{*j}(t) + P(0) \int_0^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} dH(t) \quad (k = 0, 1, \ldots), \quad (13.1a)$$

$$g(x) = \sum_{k=0}^{\infty} P(k) x^k, \quad (13.2a)$$

$$g(\eta(\lambda - \lambda x)) - g(x) = P(0)[1 - \eta(\lambda - \lambda x)], \quad (13.5a)$$

$$z_{\nu+1}(x) = z_{\nu+1} = \eta(\lambda - \lambda z_\nu) \quad (\nu = 0, 1, \ldots; z_0 = x), \quad (13.6a)$$

$$P(0) = \{1 + \sum_{j=1}^{\infty} [1 - z_j(0)]\}^{-1}, \quad (13.11a)$$

$$-\lambda \eta'(0) \bar{n} - \bar{n} = \lambda P(0) \eta'(0). \quad (13.13a)$$

By (13.11a) and the definition $x_j = z_j(0)$, $j = 1, 2, \ldots$,

$$P(0) = \{1 + \sum_{j=1}^{\infty} (1 - x_j)\}^{-1}. \quad (*)$$

The mean number $\bar{n} = g(1)$ of customers in the system when the gate opens is found from (13.13a), (*), $-\eta'(0) = \tau$ and $\lambda\tau = \rho$:

$$\bar{n} = \frac{\rho}{1-\rho} \{1 + \sum_{j=1}^{\infty} (1 - x_j)\}^{-1}. \qquad \square$$

Chapter 5, Exercise 31

'Show that Equation (44) of Cooper [1969] is incorrect.'

As hinted, the error in Eq. (44) is introduced in Eq. (42). It is true as stated immediately after Eq. (42) that $\hat{P}_i(n) = P_{i-1}(n)/(1-P_{i-1}(0))$ is the probability that n $(n=1,2,\dots)$ customers wait in queue i when the gate closes, given $n \geq 1$. However, it is not true, as implied by Eq. (42), that $\hat{P}_i(n)$ also is the probability that an arbitrary customer in queue i, who did not arrive when the system was completely empty, will be a member of a group of n customers at the time the gate closes. The latter probability is proportional to both n and $\hat{P}_i(n)$. Hence, in Eq. (42) one should replace $P_{i-1}(n)/[1-P_{i-1}(0)]$ by

$$\frac{n\hat{P}_i(n)}{\sum_{n=1}^{\infty} n\hat{P}_i(n)} = \frac{n P_{i-1}(n)}{\sum_{n=1}^{\infty} n P_{i-1}(n)}.$$

It follows that the same substitution should take place in Eqs (43) and (44). Equation (44) changes into

$$\omega_i(s) = (1-\rho) + \rho \sum_{n=1}^{\infty} \frac{n P_{i-1}(n)}{\sum_{n=1}^{\infty} n P_{i-1}(n)} \, \frac{1}{n\lambda_i^{n-1}} \, \frac{[\lambda_i \eta_i(s)]^n - (\lambda_i - s)^n}{s - \lambda_i + \lambda_i \eta_i(s)},$$

which may be rewritten as

$$\omega_i(s) = (1-\rho) + \frac{\rho\lambda_i}{(\sum_{n=1}^{\infty} n P_{i-1}(n))(s-\lambda_i+\lambda_i\eta_i(s))} \sum_{n=1}^{\infty} P_{i-1}(n)\left[(\eta_i(s))^n - (1-\tfrac{s}{\lambda_i})^n\right].$$

Now,

$$\overline{m}_{i-1} = \frac{\partial}{\partial x} g_{i-1}(x,1,\dots,1)\Big|_{x=1} = \sum_{n=1}^{\infty} n P_{i-1}(n),$$

$$g_{i-1}(\eta_i(s),1,\dots,1) = P_{i-1}(0) + \sum_{n=1}^{\infty} P_{i-1}(n)(\eta_i(s))^n,$$

$$g_{i-1}(1-\tfrac{s}{\lambda_i},1,\dots,1) = P_{i-1}(0) + \sum_{n=1}^{\infty} P_{i-1}(n)(1-\tfrac{s}{\lambda_i})^n.$$

We conclude that, for $i = 0,1,\dots,N-1$,

$$\omega_i(s) = (1-\rho) + \frac{\rho\lambda_i}{\overline{m}_{i-1}[s-\lambda_i+\lambda_i\eta_i(s)]}\left[g_{i-1}(\eta_i(s),1,\dots,1) - g_{i-1}(1-\tfrac{s}{\lambda_i},1,\dots,1)\right].$$

$\square$

Chapter 5, Exercise 32

'Verify that, for Poisson input, Eq.(14.17) reduces to $P\{W>0\} = C(s,a)$'

With Poisson input at rate λ the Laplace-Stieltjes transform of the interarrival time distribution function is

$$\gamma(z) = \frac{\lambda}{\lambda + z}.$$

Inserting $\lambda/s\mu$ for ω in the right-hand side of (14.12) we get $\gamma((1-\lambda/s\mu)s\mu) = \gamma(s\mu-\lambda) = \lambda/s\mu$, which proves that for the M/M/s queue

$$\omega = \frac{\lambda}{s\mu} = \frac{a}{s}. \tag{1}$$

Also, (14.15) becomes

$$\gamma_j = \gamma(j\mu) = \frac{\lambda}{\lambda + j\mu} \qquad (j = 0, 1, \ldots, s), \tag{2}$$

and (14.16) becomes

$$C_j = \prod_{i=1}^{j} \frac{\gamma_i}{1-\gamma_i} = \frac{a^j}{j!} \qquad (j = 1, 2, \ldots, s). \tag{3}$$

By (14.14), (1), (2) and (3), Eq. (14.17) becomes

$$\begin{aligned} P\{W>0\} &= \frac{A}{1-\omega} \\ &= \left[(1-\omega)\left\{\frac{1}{1-\omega} + \sum_{j=1}^{s} \frac{1}{C_j(1-\gamma_j)}\binom{s}{j}\frac{s(1-\gamma_j)-j}{s(1-\omega)-j}\right\}\right]^{-1} \\ &= \left[1 + \left(1-\frac{a}{s}\right)\sum_{j=1}^{s} \frac{j!(\lambda+j\mu)}{a^j \, j\mu} \frac{s!}{j!(s-j)!} \frac{\frac{sj\mu}{\lambda+j\mu} - j}{s-a-j}\right]^{-1} \\ &= \left[1 + \left(1-\frac{a}{s}\right)\sum_{j=1}^{s} \frac{s!}{(s-j)!a^j}\right]^{-1}. \end{aligned}$$

Further rewriting gives

$$P\{W>0\} = \frac{\dfrac{a^s}{s!(1-a/s)}}{\displaystyle\sum_{j=1}^{s} \frac{a^{s-j}}{(s-j)!} + \frac{a^s}{s!(1-a/s)}} = \frac{\dfrac{a^s}{s!(1-a/s)}}{\displaystyle\sum_{k=0}^{s-1} \frac{a^k}{k!} + \frac{a^s}{s!(1-a/s)}}.$$

The rightmost term is precisely the formula for $C(s,a)$ [Eq.(4.8) of Chap. 3]. Thus,

$$P\{W>0\} = C(s,a). \qquad \square$$

Chapter 5, Exercise 33

'Prove that in a GI/M/s queue ...' - cf. Ex. 29 of Chap. 3

P = equilibrium probability that a blocked customer will still be waiting in the queue when the next customer arrives

r_j = conditional probability that a blocked customer will still be waiting in the queue when the next customer arrives, given arrival state $s+j$ $(j = 0, 1, \ldots)$.

Clearly, with service in order of arrival,

$$r_j = \int_0^\infty \sum_{i=0}^{j} \frac{(s\mu x)^i}{i!} e^{-s\mu x} dG(x).$$

Hence,

$$\begin{aligned}
P &= \sum_{j=0}^{\infty} r_j P\{Q = j \mid W > 0\} \\
&= \sum_{j=0}^{\infty} \left(\int_0^\infty \sum_{i=0}^{j} \frac{(s\mu x)^i}{i!} e^{-s\mu x} dG(x) \right)(1-\omega)\omega^j \qquad \text{[by (14.19)]} \\
&= \int_0^\infty e^{-s\mu x} \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{(s\mu x)^i}{i!} (1-\omega)\omega^j dG(x) \\
&= \int_0^\infty e^{-s\mu x} \sum_{i=0}^{\infty} \frac{(s\mu x)^i}{i!} \sum_{j=i}^{\infty} (1-\omega)\omega^j dG(x) \\
&= \int_0^\infty e^{-s\mu x} \sum_{i=0}^{\infty} \frac{(s\mu x\omega)^i}{i!} dG(x) \\
&= \int_0^\infty e^{-(1-\omega)s\mu x} dG(x) \\
&= \gamma((1-\omega)s\mu).
\end{aligned}$$

But, by (14.11), $\omega = \gamma((1-\omega)s\mu)$. Thus,

$$P = \omega.$$

This generalizes the result of Exercise 29 of Chapter 3. □

Chapter 5, Exercise 34

'Verify Equations (14.7) and (14.8)'

Equation (14.7): $i \leq s-1,\ i+1-j \geq 0$

Given interarrival time x, j has the binomial distribution

$$p_{ij}(x) = \binom{i+1}{j}[e^{-\mu x}]^j[1-e^{-\mu x}]^{i+1-j} \quad (0 \leq j \leq i+1),$$

since $p_{ij}(x)$ is the probability of j successes (noncompletions) in $i+1$ trials, each with probability of success equal to $e^{-\mu x}$. As $P_{ij} = \int_0^\infty p_{ij}(x)\,dG(x)$,

$$P_{ij} = \int_0^\infty \binom{i+1}{j} e^{-j\mu x}(1-e^{-\mu x})^{i+1-j}\,dG(x) \quad \left(\begin{smallmatrix} i \leq s-1, \\ i+1-j \geq 0\end{smallmatrix}\right). \qquad (14.7)$$

Equation (14.8): $i \geq s,\ j < s,\ i+1-j \geq 0$

Given interarrival time x, the next arrival state will be j if and only if (i) at some time Y, $0 < Y < x$ (set $T_1 = 0$), a service completion will leave exactly s customers in the system, and (ii) $s-j$ service completions occur in the time interval (Y, x).

In a queue with departure rate $s\mu$ (in effect as long as all servers are busy) the time Y until the $(i+1-s)$th service completion, which will result in state s, has an Erlangian distribution with the density function, by (5.54) of Chapter 2,

$$\frac{d\,P\{Y \leq y\}}{dy} = f(y) = \frac{(s\mu y)^{i-s}}{(i-s)!}e^{-s\mu y}s\mu.$$

For a given $Y = y < x$, the probability of $s-j$ service completions during the remaining interarrival interval, of length $x-y$, equals

$$g_j(x-y) = \binom{s}{j}[e^{-\mu(x-y)}]^j[1-e^{-\mu(x-y)}]^{s-j}.$$

Hence, since $p_{ij}(x) = \int_0^x g_j(x-y)f(y)\,dy$ and $P_{ij} = \int_0^\infty p_{ij}(x)\,dG(x)$,

$$P_{ij} = \int_0^\infty\int_0^x \binom{s}{j} e^{-j\mu(x-y)}(1-e^{-\mu(x-y)})^{s-j}\frac{(s\mu y)^{i-s}}{(i-s)!}e^{-s\mu y}s\mu\,dy\,dG(x). \qquad (14.8)$$

$(i \geq s,\ j < s,\ i+1-j \geq 0)$ $\square$

Chapter 5, Exercise 35

'Derivation of (14.14) and the probabilities $\Pi_0, \Pi_1, \ldots, \Pi_{s-2}$'

For the GI/M/s queue it has been shown that $\Pi_j = A\omega^{j-s}$ for $j \geq s-1$, see (14.10), and now we must prove that A is given by (14.14). At the same time we derive a formula for Π_j for $j = 0, 1, \ldots, s-2$.

$\boxed{a}$ Let

$$U(z) = \sum_{j=0}^{s-1} \Pi_j z^j. \tag{1}$$

Substitution of $\Pi_j = \sum_{i=0}^{\infty} p_{ij}\Pi_i$, by (14.3), and change of the order of summation give

$$U(z) = \sum_{i=0}^{\infty} \sum_{j=0}^{s-1} p_{ij}\Pi_i z^j,$$

or,

$$U(z) = \sum_{i=0}^{s-1} \sum_{j=0}^{s-1} p_{ij}\Pi_i z^j + \sum_{i=s}^{\infty} \sum_{j=0}^{s-1} p_{ij}\Pi_i z^j. \tag{$*$}$$

Calculation of $S = \sum_{i=0}^{s-1} \sum_{j=0}^{s-1} p_{ij}\Pi_i z^j$

Since $p_{ij} = 0$ for $j > i+1$,

$$S = \sum_{i=0}^{s-1} \sum_{j=0}^{i+1} p_{ij}\Pi_i z^j - p_{s-1,s}\Pi_{s-1} z^s.$$

The p_{ij}'s for $i \leq s-1$ and $j \leq i+1$ are given by (14.7). Substitution of (14.7) and interchange of summations and integration yield

$$S_1 = \sum_{i=0}^{s-1} \sum_{j=0}^{i+1} p_{ij}\Pi_i z^j = \int_0^{\infty} \sum_{i=0}^{s-1} \Pi_i \sum_{j=0}^{i+1} \binom{i+1}{j} [e^{-\mu x}]^j [1-e^{-\mu x}]^{i+1-j} z^j \, dG(x).$$

The inner sum is the probability generating function of a binomial variable and equals $(q+pz)^{i+1}$, where $p = e^{-\mu x}$, $q = 1-e^{-\mu x}$. Thus,

$$S_1 = \int_0^{\infty} \sum_{i=0}^{s-1} \Pi_i (1-e^{-\mu x} + ze^{-\mu x})^{i+1} dG(x).$$

By definition of $U(z)$, then

$$S_1 = \int_0^{\infty} (1-e^{-\mu x} + ze^{-\mu x}) U(1-e^{-\mu x} + ze^{-\mu x}) \, dG(x).$$

(Chap. 5, Ex. 35a)

By (14.6) or (14.7), $p_{s-1,s} = \int_0^\infty e^{-s\mu x} dG(x)$, and by (14.10), $\Pi_{s-1} = A\omega^{-1}$. Hence,

$$S_2 = p_{s-1,s}\Pi_{s-1}z^s = A\omega^{-1}z^s \int_0^\infty e^{-s\mu x} dG(x).$$

As $S = S_1 - S_2$,

$$S = \int_0^\infty (1-e^{-\mu x}+ze^{-\mu x})\,U(1-e^{-\mu x}+ze^{-\mu x})\,dG(x) - A\omega^{-1}z^s\int_0^\infty e^{-s\mu x} dG(x) \quad (**)$$

Calculation of $T = \sum_{i=s}^{\infty}\sum_{j=0}^{s-1} p_{ij}\Pi_i z^j$

The p_{ij}'s for $i \geq s$ and $j < s$ are given by (14.8). Substitution of (14.8) and interchange of summations and integrations yield

$$T = \int_{x=0}^{\infty}\int_{y=0}^{x}\left\{\sum_{i=s}^{\infty}\sum_{j=0}^{s-1}\binom{s}{j}[e^{-\mu(x-y)}]^j[1-e^{-\mu(x-y)}]^{s-j}\frac{(s\mu y)^{i-s}}{(i-s)!}e^{-s\mu y}s\mu\Pi_i z^j\right\}dy\,dG(x).$$

Substitution of $\Pi_i = A\omega^{i-s}$, by (14.10), rewriting and simplification give

$$T = A\int_{x=0}^{\infty}\int_{y=0}^{x}\sum_{i=s}^{\infty}\frac{(s\mu y\omega)^{i-s}}{(i-s)!}e^{-s\mu y}\left\{\sum_{j=0}^{s}\binom{s}{j}[e^{-\mu(x-y)}]^j[1-e^{-\mu(x-y)}]^{s-j}z^j - e^{-s\mu(x-y)}z^s\right\}\cdot s\mu\,dy\,dG(x)$$

$$= A\int_{x=0}^{\infty}\int_{y=0}^{x} e^{s\mu y\omega}e^{-s\mu y}(1-e^{-\mu(x-y)}+ze^{-\mu(x-y)})^s s\mu\,dy\,dG(x)$$
$$- Az^s\int_{x=0}^{\infty}e^{-s\mu x}\int_{y=0}^{x}s\mu e^{s\mu y\omega}dy\,dG(x)$$
$$= A\int_{x=0}^{\infty}\int_{y=0}^{x} e^{s\mu y\omega}(e^{-\mu y}-e^{-\mu x}+ze^{-\mu x})^s s\mu\,dy\,dG(x)$$
$$- Az^s\int_0^\infty e^{-s\mu x}\omega^{-1}(e^{s\mu x\omega}-1)\,dG(x)$$

The last integral on the right-hand side reduces to

$$\int_0^\infty e^{-s\mu x}\omega^{-1}(e^{s\mu x\omega}-1)\,dG(x) = \omega^{-1}\int_0^\infty e^{-(1-\omega)s\mu x}dG(x) - \omega^{-1}\int_0^\infty e^{-s\mu x}dG(x)$$
$$= 1 - \omega^{-1}\int_0^\infty e^{-s\mu x}dG(x), \qquad \text{[by (14.11)]}$$

so that

$$T = A\int_{x=0}^{\infty}\int_{y=0}^{x} e^{s\mu y\omega}(e^{-\mu y}-e^{-\mu x}+ze^{-\mu x})^s s\mu\,dy\,dG(x) - Az^s + A\omega^{-1}z^s\int_0^\infty e^{-s\mu x}dG(x). \quad (***)$$

(Chap. 5, Ex. 35a (cont'd))

By (*), (**), (***), and the definitions of S and T,

$$U(z) = \int_0^\infty (1-e^{-\mu x}+ze^{-\mu x})\, U(1-e^{-\mu x}+ze^{-\mu x})\, dG(x)$$
$$+ A\int_{x=0}^{\infty}\Big[\int_{y=0}^{x} e^{s\mu\omega y}(e^{-\mu y}-e^{-\mu x}+ze^{-\mu x})^{s} s\mu\, dy\Big]\, dG(x)$$
$$- Az^{s}. \qquad (2)$$

$\boxed{b}$ By (1) and (14.10),

$$U(1) = \sum_{j=0}^{s-1}\pi_j = 1 - \sum_{j=s}^{\infty}\pi_j = 1 - \sum_{j=s}^{\infty} A\,\omega^{j-s}.$$

Hence,

$$U(1) = 1 - \frac{A}{1-\omega}. \qquad (3)$$

$\boxed{c}$ For $j = 0, 1, \ldots, s-1$ let

$$U^{(j)}(z) = \frac{d^{j}U(z)}{dz^{j}},$$

with $U^{(0)}(z) = U(z)$, and define

$$U_j = \frac{U^{(j)}(1)}{j!} = \frac{1}{j!}\Big(\frac{d^{j}U(z)}{dz^{j}}\Big)_{z=1} \qquad (j = 0, 1, \ldots, s-1). \qquad (4)$$

By definition, $U_0 = \frac{1}{0!}U(1) = U(1)$. By (3), therefore,

$$U_0 = 1 - \frac{A}{1-\omega}. \qquad (5)$$

Repeated differentiation of (2) gives

$$U^{(j)}(z) = \int_0^\infty (1-e^{-\mu x}+ze^{-\mu x})\, U^{(j)}(1-e^{-\mu x}+ze^{-\mu x})\, e^{-j\mu x}\, dG(x)$$
$$+ j\int_0^\infty U^{(j-1)}(1-e^{-\mu x}+ze^{-\mu x})\, e^{-j\mu x}\, dG(x)$$
$$+ As\mu \int_{x=0}^{\infty} e^{-j\mu x}\int_{y=0}^{x} e^{s\mu\omega y}\frac{s!}{(s-j)!}(e^{-\mu y}-e^{-\mu x}+ze^{-\mu x})^{s-j}\, dy\, dG(x)$$
$$- A\frac{s!}{(s-j)!}z^{s-j} \qquad (j = 1, 2, \ldots, s-1).$$

(Chap. 5, Ex. 35 c)

Hence,

$$U_j = \frac{U^{(j)}(1)}{j!} = U_j\,\gamma(j\mu) + U_{j-1}\gamma(j\mu)$$
$$+ A\binom{s}{j} s\mu \int_{x=0}^{\infty} e^{-j\mu x} \int_{y=0}^{x} e^{(s\mu\omega-(s-j)\mu)y}\,dy\,dG(x)$$
$$- A\binom{s}{j} \qquad (j = 1, 2, \ldots, s-1).$$

Substituting

$$\int_0^x e^{(s\mu\omega-(s-j)\mu)y}\,dy = \frac{e^{(s\mu\omega-(s-j)\mu)x} - 1}{s\mu\omega-(s-j)\mu}$$

and reducing, we obtain

$$U_j = U_j\,\gamma(j\mu) + U_{j-1}\gamma(j\mu)$$
$$- A\binom{s}{j}\frac{s}{s(1-\omega)-j}\left(\int_0^{\infty} e^{-(1-\omega)s\mu x}\,dG(x) - \int_0^{\infty} e^{-j\mu x}\,dG(x)\right)$$
$$- A\binom{s}{j} \qquad (j = 1, 2, \ldots, s-1).$$

The two integrals are equal to, respectively, ω (by (14.11)) and $\gamma(j\mu)$. Substitution of these values, simplification, and replacement of $\gamma(j\mu)$ by γ_j, give

$$U_j = U_j\gamma_j + U_{j-1}\gamma_j - A\binom{s}{j}\frac{s(1-\gamma_j)-j}{s(1-\omega)-j} \qquad (j = 1, 2, \ldots, s-1),$$

from which is obtained the difference equation

$$U_j = \frac{\gamma_j}{1-\gamma_j}U_{j-1} - \frac{A}{1-\gamma_j}\binom{s}{j}\frac{s(1-\gamma_j)-j}{s(1-\omega)-j} \quad (j = 1, 2, \ldots, s-1). \qquad (6)$$

$\boxed{d}$ Next we define $C_0 = 1$ and

$$C_j = \prod_{i=1}^{j}\frac{\gamma_i}{1-\gamma_i} \qquad (j = 1, 2, \ldots, s),$$

and divide by C_j on both sides of (6). The result is

$$\frac{U_j}{C_j} = \frac{U_{j-1}}{C_{j-1}} - \frac{A}{C_j(1-\gamma_j)}\binom{s}{j}\frac{s(1-\gamma_j)-j}{s(1-\omega)-j} \quad (j = 1, 2, \ldots, s-1). \qquad (7)$$

(Chap. 5, Ex. 35d)

Assuming $0 \leq i \leq s-2$ we add equations (7) for $j = i+1, \dots, s-1$, whereby we derive

$$\frac{U_i}{C_i} = \frac{U_{s-1}}{C_{s-1}} + A \sum_{j=i+1}^{s-1} \frac{1}{C_j(1-\gamma_j)} \binom{s}{j} \frac{s(1-\gamma_j)-j}{s(1-\omega)-j} \quad (i = 0, 1, \dots, s-2),$$

$$\frac{U_i}{C_i} = \left(\frac{U_{s-1}}{C_{s-1}} - \frac{A\omega^{-1}}{C_s(1-\gamma_s)/\gamma_s} \right) + A \sum_{j=i+1}^{s} \frac{1}{C_j(1-\gamma_j)} \binom{s}{j} \frac{s(1-\gamma_j)-j}{s(1-\omega)-j} \quad (i = 0, 1, \dots, s-2).$$

By (1) and (4), $U_{s-1} = \Pi_{s-1}$, and, by (14.10), $\Pi_{s-1} = A\omega^{-1}$. By definition, $C_s = C_{s-1}\gamma_s/(1-\gamma_s)$. It follows that the term in parentheses vanishes. Furthermore, as is easily verified, the above equation also holds for $i = s-1$. Our conclusion is that

$$\frac{U_i}{C_i} = A \sum_{j=i+1}^{s} \frac{1}{C_j(1-\gamma_j)} \binom{s}{j} \frac{s(1-\gamma_j)-j}{s(1-\omega)-j} \quad (i = 0, 1, \dots, s-1). \qquad (8)$$

Now set $i = 0$, make the substitutions $U_0 = 1 - \frac{A}{1-\omega}$ and $C_0 = 1$, and solve Eq. (8) for A:

$$A = \left\{ \frac{1}{1-\omega} + \sum_{j=1}^{s} \frac{1}{C_j(1-\gamma_j)} \binom{s}{j} \frac{s(1-\gamma_j)-j}{s(1-\omega)-j} \right\}^{-1}. \qquad (14.14)$$

$\boxed{e}$ By definition, $U(z)$ is a polynomial in z of degree $s-1$. Hence, $U(z)$ will be represented exactly by a Taylor series expansion of degree $s-1$ at an arbitrary point z_0. For $z_0 = 1$ the representation is $U(z) = \sum_{j=0}^{s-1} [U^{(j)}(1)/j!](z-1)^j$. That is,

$$U(z) = \sum_{j=0}^{s-1} U_j (z-1)^j. \qquad (9)$$

By (1), $U^{(j)}(z) = \sum_{k=j}^{s-1} \Pi_k \frac{k!}{(k-j)!} z^{k-j}$. Hence, $U^{(j)}(0) = \Pi_j j!$, or,

$$\Pi_j = \frac{U^{(j)}(0)}{j!} = \frac{1}{j!} \left(\frac{d^j U(z)}{dz^j} \right)_{z=0} \quad (j = 0, 1, \dots, s-1). \qquad (10)$$

Differentiation of (9) $[U(z) = \sum_{i=0}^{s-1} U_i (z-1)^i]$ j times leads to

$$\frac{d^j U(z)}{dz^j} = \sum_{i=j}^{s-1} U_i \frac{i!}{(i-j)!} (z-1)^{i-j} \quad (j = 0, 1, \dots, s-1),$$

whose substitution into (10) yields

$$\Pi_j = \sum_{i=j}^{s-1} (-1)^{i-j} \binom{i}{j} U_i \quad (j = 0, 1, \dots, s-2, s-1). \qquad (11) \quad \square$$

Chapter 5, Exercise 36

'Show that for Poisson input this algorithm yields Equation (6) of Exercise 32 of Chapter 3.'

In the case of the M/M/s queue with service in random order, $\omega = \rho \ (= \lambda/s\mu)$, so that (15.9) becomes

$$P\{W>t \mid W>0\} = 1 + (1-\rho)\sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu!} \sum_{j=0}^{\infty} \rho^j W_j^{(\nu)}, \tag{1}$$

where, according to (15.14),

$$W_j^{(\nu)} = a_{j+1}^{(\nu)} + \sum_{i=1}^{j+1} \frac{i}{j+1} \sum_{k=0}^{\nu-1} b_{j+1-i}^{(k)} W_i^{(\nu-1-k)} \qquad \left(\begin{matrix} j = 0,1,\dots; \\ \nu = 1,2,\dots \end{matrix}\right). \tag{2}$$

Eq. (2) may be solved recursively for $\nu = 1, \nu = 2, \dots$ utilizing $W_j^{(0)} = W_j(0) = 1$ and the definitions of $a_{j+1}(t)$, by (15.12), and $b_{j+1-i}(t)$, by (15.13). In the present case where $G(t) = 1 - e^{-\lambda t}$,

$$a_{j+1}(t) = e^{-(\lambda+s\mu)t} \sum_{i=1}^{j+1} \frac{i}{j+1} \frac{(s\mu t)^{j+1-i}}{(j+1-i)!} \qquad (j = 0,1,\dots), \tag{3}$$

$$b_n(t) = \lambda e^{-(\lambda+s\mu)t} \frac{(s\mu t)^n}{n!} \qquad (n = 0,1,\dots), \tag{4}$$

from which $a_{j+1}^{(\nu)}$ and $b_{j+1-i}^{(k)}$ may be derived.

We shall determine all terms of the expansion (1) to the order of t^2. Thus we must find $W_j^{(1)}$ and $W_j^{(2)}$ for $j = 0,1,\dots$. By (2), this requires calculation of $a_{j+1}^{(\nu)}$ for $\nu = 1,2$, and of $b_{j+1-i}^{(k)}$ for $k = 0,1$.

Calculation of $a_{j+1}^{(1)}$ and $a_{j+1}^{(2)}$

Differentiating (3) once, we obtain

$$a_1^{(1)}(t) = -(\lambda+s\mu)e^{-(\lambda+s\mu)t},$$

$$a_2^{(1)}(t) = -(\lambda+s\mu)e^{-(\lambda+s\mu)t}\left[1 - \frac{1}{1+\rho}\frac{1}{2} + \frac{1}{2}(s\mu t)\right],$$

$$a_{j+1}^{(1)}(t) = -(\lambda+s\mu)e^{-(\lambda+s\mu)t}\left[1 - \frac{1}{1+\rho}\frac{j}{j+1} + \sum_{i=1}^{j} \frac{i}{j+1}\frac{(s\mu t)^{j+1-i}}{(j+1-i)!} - \frac{1}{1+\rho}\sum_{i=1}^{j-1}\frac{i}{j+1}\frac{(s\mu t)^{j-i}}{(j-i)!}\right] \quad (j = 2,3,\dots).$$

Setting $t = 0$, we find $a_{j+1}^{(1)} = -(\lambda+s\mu)\left[1 - \frac{1}{1+\rho}\frac{j}{j+1}\right]$ for $j = 0,1,2$, or,

$$a_{j+1}^{(1)} = -s\mu\left(\rho + \frac{1}{j+1}\right) \qquad (j = 0,1,\dots). \tag{5}$$

(Chap. 5, Ex. 36)

Another differentiation results in

$$a_1^{(2)}(t) = (\lambda+s\mu)^2 e^{-(\lambda+s\mu)t},$$
$$a_2^{(2)}(t) = (\lambda+s\mu)^2 e^{-(\lambda+s\mu)t}\left[1-\frac{1}{1+\rho}+\frac{1}{2}(s\mu t)\right],$$
$$a_{j+1}^{(2)}(t) = (\lambda+s\mu)^2 e^{-(\lambda+s\mu)t}\left[1-\frac{2}{1+\rho}\,\frac{j}{j+1}+\frac{1}{(1+\rho)^2}\,\frac{j-1}{j+1}+tR_{j-1}(t)\right]\ (j=2,3,\ldots),$$

where $R_{j-1}(t)$ is a polynomium of degree $j-1$. It follows that

$$a_1^{(2)} = (\lambda+s\mu)^2,$$
$$a_2^{(2)} = (\lambda+s\mu)^2\left[1-\frac{1}{1+\rho}\right],$$
$$a_{j+1}^{(2)} = (\lambda+s\mu)^2\left[1-\frac{2}{1+\rho}\,\frac{j}{j+1}+\frac{1}{(1+\rho)^2}\,\frac{j-1}{j+1}\right] \quad (j=2,3,\ldots).$$

A rewriting yields

$$a_{j+1}^{(2)} = \begin{cases} (s\mu)^2(\rho^2+2\rho+1) & (j=0), \\ (s\mu)^2\left(\rho^2+\frac{2\rho}{j+1}\right) & (j=1,2,\ldots). \end{cases} \tag{6}$$

Calculation of $b_{j+1-i}^{(0)}$ and $b_{j+1-i}^{(1)}$

By (4), clearly $b_0^{(0)} = \lambda$ and $b_n^{(0)} = 0$ for $n \geq 1$. Thus,

$$b_{j+1-i}^{(0)} = \begin{cases} \lambda & (j+1-i=0), \\ 0 & (j+1-i=1,2,\ldots). \end{cases} \tag{7}$$

Differentiation of (4) results in

$$b_0^{(1)}(t) = -\lambda(\lambda+s\mu)e^{-(\lambda+s\mu)t},$$
$$b_n^{(1)}(t) = \lambda e^{-(\lambda+s\mu)t}\left[\frac{(s\mu t)^{n-1}}{(n-1)!}(s\mu) - \frac{(s\mu t)^n}{n!}(\lambda+s\mu)\right] \quad (n=1,2,\ldots).$$

It follows that $b_0^{(1)} = -\lambda(\lambda+s\mu)$, $b_1^{(1)} = \lambda s\mu$, $b_n^{(1)} = 0$ for $n \geq 2$. Thus,

$$b_{j+1-i}^{(1)} = \begin{cases} -\lambda(\lambda+s\mu) & (j+1-i=0), \\ \lambda s\mu & (j+1-i=1), \\ 0 & (j+1-i=2,3,\ldots). \end{cases} \tag{8}$$

(Chap. 5, Ex. 36 (cont'd))

Calculation of $W_{j-}^{(1)}$

By (2),

$$W_j^{(1)} = a_{j+1}^{(1)} + \sum_{i=1}^{j+1} \frac{i}{j+1} b_{j+1-i}^{(0)} W_i^{(0)} \quad (j = 0,1,\ldots).$$

Substitution of $W_i^{(0)} = 1$, (5), and (7), leads to $W_j^{(1)} = -s\mu[\rho + \frac{1}{j+1}] + \lambda$ for $j = 0,1,\ldots$. Hence,

$$W_j^{(1)} = -\frac{s\mu}{j+1} \quad (j = 0,1,\ldots). \qquad (9)$$

Calculation of $W_{j-}^{(2)}$

By (2),

$$W_j^{(2)} = a_{j+1}^{(2)} + \sum_{i=1}^{j+1} \frac{i}{j+1} b_{j+1-i}^{(0)} W_i^{(1)} + \sum_{i=1}^{j+1} \frac{i}{j+1} b_{j+1-i}^{(1)} W_i^{(0)} \quad (j = 0,1,\ldots),$$

which by substitution of $W_i^{(0)} = 1$, (7) and (9) reduces to

$$W_j^{(2)} = a_{j+1}^{(2)} - \frac{\lambda s\mu}{j+2} + \frac{1}{j+1} \sum_{i=1}^{j+1} i\, b_{j+1-i}^{(1)} \quad (j = 0,1,\ldots).$$

Finally, application of (6) and (8) results in

$$W_0^{(2)} = (s\mu)^2(\rho^2 + 2\rho + 1) - \frac{\lambda s\mu}{2} - \lambda(\lambda + s\mu),$$

$$W_j^{(2)} = (s\mu)^2(\rho^2 + \frac{2\rho}{j+1}) - \frac{\lambda s\mu}{j+2} + \frac{j}{j+1}\lambda s\mu - \lambda(\lambda + s\mu) \quad (j = 1,2,\ldots),$$

or,

$$W_j^{(2)} = \begin{cases} (s\mu)^2(1 + \frac{\rho}{2}) & (j = 0), \\ (s\mu)^2 \frac{\rho}{(j+1)(j+2)} & (j = 1,2,\ldots). \end{cases} \qquad (10)$$

Calculation of $P\{W > t \mid W > 0\}$

Equations (9) and (10) are precisely those derived previously in Exercise 32 of Chapter 3. As in that exercise, the substitution of (9) and (10) into (1) and subsequent reduction therefore yield Equation (6) of Exercise 32 of Chapter 3. □

Chapter 6, Exercise 1

'Show that if U is uniform on $(0,1)$, then so is $1-U$.'

Assume that
$$P\{U \leq u\} = u \qquad (0 \leq u \leq 1).$$
Hence,
$$P\{U < u\} = u \qquad (0 \leq u \leq 1).$$
Now, $\{U < u\} \Leftrightarrow \{1-U > 1-u\}$, so that
$$P\{1-U > 1-u\} = u \qquad (0 \leq u \leq 1),$$
whereby
$$P\{1-U \leq 1-u\} = 1-u \quad (0 \leq u \leq 1).$$
Substituting $u' = 1-u$ we obtain
$$P\{1-U \leq u'\} = u' \qquad (0 \leq u' \leq 1).$$
Thus, $1-U$ is uniformly distributed on $(0,1)$ if U is.

Chapter 6, Exercise 2

'Let X have the Erlangian distribution function of order n,'
$$F_X(x) = 1 - \sum_{j=0}^{n-1} \frac{(\lambda x)^j}{j!} e^{-\lambda x}.$$
X may be interpreted as the sum of n independent exponential variables with parameter λ, that is,
$$X = X_1 + X_2 + \ldots + X_n,$$
where
$$F_{X_i}(x) = 1 - e^{-\lambda x} \quad (i = 1,\ldots,n).$$
But, by (2.7), $X_i = -\frac{1}{\lambda} \ln U_i$, where U_i is uniform on $(0,1)$. Hence,
$$X = \sum_{i=1}^{n} (-\tfrac{1}{\lambda} \ln U_i) = -\tfrac{1}{\lambda} \ln (U_1 U_2 \ldots U_n). \qquad \square$$

Chapter 6, Exercise 3

'Derive Equation (4.18).'

The variable under consideration is

$$\hat{P}(n) = \frac{\overline{X}(n)}{\overline{Y}(n) + \overline{X}(n)}, \tag{1}$$

with $\overline{X}(n) = \Sigma_{i=1}^{n} X_i/n$, $\overline{Y}(n) = \Sigma_{i=1}^{n} Y_i/n$, where all X_i's and Y_i's are independent variables. The service time X_i has a general distribution, $E(X_i) = \tau$, $V(X_i) = \sigma^2$. The idle time Y_i has an exponential distribution, $E(Y_i) = \lambda^{-1}$, $V(Y_i) = \lambda^{-2}$.

Now define the distribution function

$$F_{\hat{P}(n)}(t) = P\{\hat{P}(n) \leq t\}. \tag{2}$$

Substitution of (1) into (2) gives

$$F_{\hat{P}(n)}(t) = P\{\overline{X}(n) - \tfrac{t}{1-t}\overline{Y}(n) \leq 0\}. \tag{3}$$

Defining $Z_i(t) = X_i - \frac{t}{1-t} Y_i$ and $\overline{Z}(n;t) = \Sigma_{i=1}^{n} Z_i(t)/n = \overline{X}(n) - \frac{t}{1-t}\overline{Y}(n)$, (3) may be written

$$F_{\hat{P}(n)}(t) = P\{\overline{Z}(n;t) \leq 0\}. \tag{4}$$

We find easily that mean and variance of $\overline{Z}(n;t)$ are

$$E(\overline{Z}(n;t)) = \tau - \tfrac{t}{1-t}\lambda^{-1}, \quad V(\overline{Z}(n;t)) = \tfrac{1}{n}[\sigma^2 + (\tfrac{t}{1-t})^2\lambda^{-2}]. \tag{5), (6}$$

By the central limit theorem $\overline{Z}(n;t)$ is asymptotically normal distributed:

$$\lim_{n\to\infty} P\{\overline{Z}(n;t) \leq x\} = \Phi\left(\frac{x - E(\overline{Z}(n;t))}{\sqrt{V(\overline{Z}(n;t))}}\right). \tag{7}$$

Setting $x = 0$, and substituting (5) and (6) we derive

$$\lim_{n\to\infty} P\{\overline{Z}(n;t) \leq 0\} = \Phi\left(\frac{\frac{t}{1-t} - a}{\sqrt{\frac{1}{n}[(\frac{t}{1-t})^2 + \lambda^2\sigma^2]}}\right).$$

By (4),

$$\lim_{n\to\infty} F_{\hat{P}(n)}(t) = \Phi\left(\frac{\frac{t}{1-t} - a}{\sqrt{\frac{1}{n}[(\frac{t}{1-t})^2 + \lambda^2\sigma^2]}}\right). \tag{4.18}$$

$\square$

Chapter 6, Exercise 4

'Consider simulation of the single-server Erlang loss model with constant service times.'

Our two estimates $\hat{\Pi}(n)$ and $\hat{P}(n)$ of the loss probability Π, based on a simulation of n cycles, have been shown to be asymptotically normal:

$$F_{\hat{\Pi}(n)}(t) \simeq \Phi\left(\frac{\frac{t}{1-t}-a}{\sqrt{\frac{1}{n}(a+\lambda^2\sigma^2)}}\right), \qquad (4.11)$$

$$F_{\hat{P}(n)}(t) \simeq \Phi\left(\frac{\frac{t}{1-t}-a}{\sqrt{\frac{1}{n}[(\frac{t}{1-t})^2+\lambda^2\sigma^2]}}\right). \qquad (4.18)$$

$\boxed{a}$ First, observe that $\Pi = \frac{\tau}{\lambda^{-1}+\tau} = \frac{a}{1+a}$, whereby $a = \frac{\Pi}{1-\Pi}$. Also, as $n \to \infty$, both $\hat{\Pi}(n)$ and $\hat{P}(n)$ converge in probability to Π. It is therefore obvious, and may be proved rigorously, that for any $\varepsilon > 0$ there exists an n_0 such that

$$\Phi\left(\frac{\frac{t}{1-t}-a}{\sqrt{\frac{1}{n}[(\frac{t}{1-t})^2+\lambda^2\sigma^2]}}\right) - \Phi\left(\frac{\frac{t}{1-t}-a}{\sqrt{\frac{1}{n}(a^2+\lambda^2\sigma^2)}}\right) < \varepsilon$$

for all t, $0<t<1$, and $n > n_0$. This means that asymptotically $\hat{P}(n)$'s distribution function may be also expressed

$$F_{\hat{P}(n)}(t) \simeq \Phi\left(\frac{\frac{t}{1-t}-a}{\sqrt{\frac{1}{n}(a^2+\lambda^2\sigma^2)}}\right). \qquad (4.18a)$$

For $a=1$, a comparison of (4.11) and (4.18a) leads to the conclusion that, asymptotically, the two probability distributions are the same, $F_{\hat{\Pi}(n)}(t) \simeq F_{\hat{P}(n)}(t)$. Hence, for all values of δ,

$$P\{\Pi-\delta < \hat{\Pi}(n) < \Pi+\delta\} \simeq P\{\Pi-\delta < \hat{P}(n) < \Pi+\delta\} \quad (a=1).$$

$\boxed{b}$ Now assume constant service times, that is $\sigma^2=0$, and $n=100$. Substituting these values and $a=\frac{\Pi}{1-\Pi}$ into (4.11) and (4.18) we find

$$F_{\hat{\Pi}(100)}(t) \simeq \Phi\left(10\left(\frac{t}{1-t}-\frac{\Pi}{1-\Pi}\right)\sqrt{\frac{1-\Pi}{\Pi}}\right), \qquad (1)$$

$$F_{\hat{P}(100)}(t) \simeq \Phi\left(10\left(1-\frac{\Pi}{1-\Pi}\,\frac{1-t}{t}\right)\right). \qquad (2)$$

(Chap. 6, Ex. 4)

Hence we obtain the approximation formulas

$$P\{\pi - 0.05 < \hat{\pi}(100) < \pi + 0.05\} = \Phi(u_1) - \Phi(u_2), \qquad (3)$$

$$P\{\pi - 0.05 < \hat{P}(100) < \pi + 0.05\} = \Phi(u_3) - \Phi(u_4), \qquad (4)$$

where

$$u_1 = 10\left[\frac{\pi+0.05}{1-(\pi+0.05)} - \frac{\pi}{1-\pi}\right]\sqrt{\frac{1-\pi}{\pi}},$$

$$u_2 = -10\left[\frac{\pi}{1-\pi} - \frac{\pi-0.05}{1-(\pi-0.05)}\right]\sqrt{\frac{1-\pi}{\pi}},$$

$$u_3 = 10\left[1 - \frac{\pi}{1-\pi}\,\frac{1-(\pi+0.05)}{\pi+0.05}\right],$$

$$u_4 = -10\left[\frac{\pi}{1-\pi}\,\frac{1-(\pi-0.05)}{\pi-0.05} - 1\right].$$

Calculations

Table 1. Arguments of $\Phi(\cdot)$

π	u_1	u_2	u_3	u_4
0.05	2.55	− 2.29	5.26	− ∞
0.10	1.96	− 1.75	3.70	−11.11
0.15	1.75	− 1.56	2.94	− 5.88
0.20	1.67	− 1.47	2.50	− 4.17
0.25	1.65	− 1.44	2.22	− 3.33
0.30	1.68	− 1.45	2.04	− 2.86
0.35	1.75	− 1.50	1.92	− 2.56
0.40	1.86	− 1.57	1.85	− 2.38
0.45	2.01	− 1.68	1.82	− 2.27
0.50	2.22	− 1.82	1.82	− 2.22
0.55	2.51	− 2.01	1.85	− 2.22
0.60	2.92	− 2.27	1.92	− 2.27
0.65	3.44	− 2.62	2.04	− 2.38
0.70	4.36	− 3.12	2.22	− 2.56
0.75	5.77	− 3.85	2.50	− 2.86
0.80	8.33	− 5.00	2.94	− 3.33
0.85	14.00	− 7.00	3.70	− 4.17
0.90	33.33	− 11.11	5.26	− 5.88
0.95	∞	− 22.94	10.00	−11.11

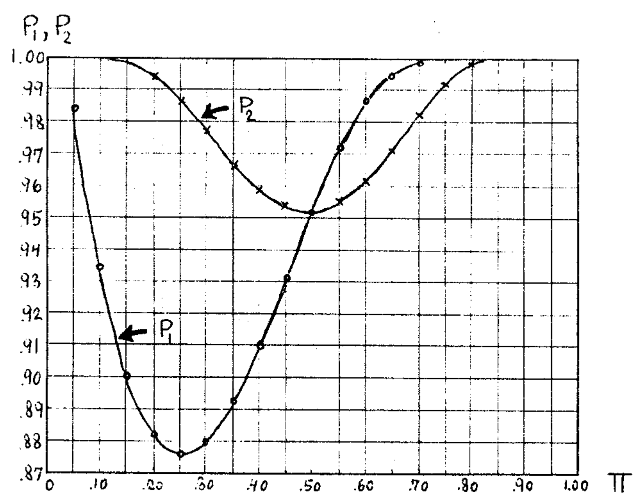
(Chap. 6, Ex. 4 (cont'd))

Table 2. $P_1 = P\{\pi - 0.05 < \hat{\pi}(100) < \pi + 0.05\}$

π	$\Phi(u_1)$	$\Phi(u_2)$	P_1
0.05	.9946	.0110	.984
0.10	.9750	.0401	.935
0.15	.9594	.0594	.900
0.20	.9525	.0708	.882
0.25	.9505	.0749	.876
0.30	.9535	.0735	.880
0.35	.9599	.0668	.893
0.40	.9686	.0582	.910
0.45	.9778	.0465	.931
0.50	.9868	.0344	.952
0.55	.9940	.0222	.972
0.60	.9982	.0116	.987
0.65	.9998	.0044	.995
0.70	1.0000	.0009	.999
0.75	1.0000	.0000	1.000
0.80	1.0000	.0000	1.000
0.85	1.0000	.0000	1.000
0.90	1.0000	.0000	1.000
0.95	1.0000	.0000	1.000

Table 3 $P_2 = P\{\pi - 0.05 < \hat{P}(100) < \pi + 0.05\}$

π	$\Phi(u_3)$	$\Phi(u_4)$	P_2
0.05	1.0000	.0000	1.000
0.10	.9999	.0000	1.000
0.15	.9984	.0000	.998
0.20	.9438	.0000	.994
0.25	.9868	.0004	.986
0.30	.9793	.0021	.977
0.35	.9726	.0052	.967
0.40	.9678	.0087	.959
0.45	.9656	.0116	.954
0.50	.9656	.0132	.952
0.55	.9678	.0132	.955
0.60	.9726	.0116	.961
0.65	.9793	.0087	.971
0.70	.9868	.0052	.982
0.75	.9438	.0021	.992
0.80	.9984	.0004	.998
0.85	.9999	.0000	1.000
0.90	1.0000	.0000	1.000
0.95	1.0000	.0000	1.000



□